5/13/25

HW2 p6



F_D = paths on D, obj's on V(D)

edge iff extend one path to other along edge in D.

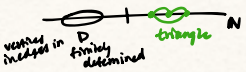
Lemma: $\gamma \in F_D$ has only 1 predecessor = γ' of length −1 as path on V(D)

HW3 p6.1

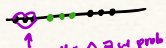
$\{G$: G has no odd cycle$\}$ is open dense.

pf: nbhds $N_D(G) := \{H : H\restriction_D = G\restriction_D\}$, $D \subseteq_{fin} \mathcal{P}(\binom{\mathbb{N}}{2})$

prove dense by "putting △ upstairs"



full mtr.



specific △ ∃ w/ prob $\frac{1}{8}$.

no triangle ∃ w/ prob $\frac{7}{8}$.

$P(\text{all these } \triangle\text{s do not exist}) \leq_{\forall n} P(\text{no } \triangle \text{ on } n \text{ triples of vertices}) \leq (\frac{7}{8})^n$

HW3 p6.2 $\{G$ conn$\}$ — full mtr.

Fix 2 vtxs m, n. Prob no path (even of length 2) connecting them is 0.

$\bigcap_{n,m}$ countable intersection.

dense b/c fixing nbhd $N_D(G_0)$, can connect & disconnect easily



disconnected = $\exists m,n\ \forall$ paths $_{m\text{-}n}$ $\exists$ edge on path that doesn't exist.

connected = $\forall m,n\ \exists$ path $_{m\text{-}n}$

$= \forall n\ \exists$ path $_{n\text{-}n+1}$

$\bigcap_n \bigcup_{\text{paths } \gamma \text{ n-n+1}}$ graphs containing γ (open set)

union of opens hence open.

And dense (indeed, {graphs w/ len 2 path 1-on} is dense)



HW3 p7 saturated graphs are connected & connd

saturated ⇒ connected (by our pf)

$S = 2^{\mathbb{N}}$, $T = \emptyset$

$\exists u$

connected.

HW4 p3 6/8/25 lec10 9/24/25

start w/ Borel N-coloring

$A := \{x \in X : c(x) > c(\Gamma x)\}$

Γ = each of $\aleph_0$ dests.

$X' := \{x \in X$: ∞ recur to $A\}$ — ∞ often go down ↓ opposite

$X'' := \{x \in X$: eventually always ↗$\}$

N: left of local min, set self to 0, run stage 2 (can't keep)

FIX: start w/ old N-coloring, bump all colors by 1 so min color is 1; that way when set I rmvs 0, stage 2 can really just keep exactly old colors.

$G[X']$ Aut(N) colored by $I := \{x \in X' : \begin{matrix} x \in A \\ \Gamma x \notin A \end{matrix}\}$ ⊆ A

Borel indep rest of A

x = left of loc. min.

then 2 1 0 1 0 2 1 and 0 2 1 and 0 2 ...

lots of $\frac{1}{3}$'s 0 's, not good.

my idea, swap 0 and 2 seriously improves the situation!

to I rmvs 0

then sort colored according to $\text{dist}_2 \bmod 2$

and $\text{dist}_I \equiv 1$ rmvs 2

so then have shape 2 3 0 (acceptable)

$\text{dist}_I \equiv 0$ rmvs 1 ✓

Inductive for whole graph G,

Stage 1: 0 rmvs $I := \{x$ left of loc. min$\}$

color x by $\text{dist}_I(x) \bmod 2$ N $\neq \infty$.

Stage 2: x not colored yet ⇒ start at x, $c(x), c(\Gamma x) \ldots$ always ↗.

so actually we're just literally done, don't have to do anything for this stage.

5/13/25 HW5 p1

$X \hookrightarrow \mathcal{L}$

$x \mapsto \mathcal{L}(x)$

$a_n = i$ iff $x \in \mathcal{L}_n$

nested, but pretty low on bond hierarchy

lex order on $\mathcal{L}$

write $x <_{\mathcal{L}} y$ as $\exists V$ and $\forall \mathcal{U}$.

$x <_{\mathcal{L}} y$ iff $\exists n\ \forall m \geq n$, $x \in \mathcal{L}_m \leftrightarrow y \in \mathcal{L}_m$, $x \notin \mathcal{L}_n \leftrightarrow y \in \mathcal{L}_n$ ($\bigcup_n F_\sigma$ is F_σ)

also $x <_{\mathcal{L}} y$ iff $\forall n$ $x_n <_{\text{lex}} y_n$ (truncation to n).

open & closed conditions, F_σ.

≥1 string of each len

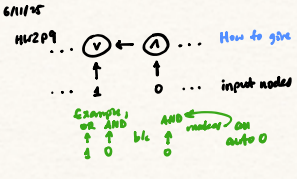


first figure how to connect (induction hyp)

differ in k coords: $x_n \frown 0 \frown t$, $y_n \frown 0 \frown t$, $y_n \frown 1 \frown t$, $x_n \frown 1 \frown t$

6/11/25

HW2 p9



How to give logically consistent assignments to this ∞/cyclic list of AND/OR gates?

(Here, e.g. everything 0 or everything 1 both work ⇒ logical consistency)

6/15/25 HW2

perfect matching

Lin's lemma: $\forall$ vtx deg = ∞ ⇒ connected and connd.

3/31/25 223D Anton Bernshteyn

Descriptive Combo

DST "Graph Theory" in this class

General Q in Math: prove ∃, but how "constructive"?

Goal: seek constructive sols to combo problems

?

Example: Thm in graph theory:

Graph w/ no odd cycles is bipartite. — no edges inside

(⇒) pf sketch: pick starting x in each conn comp.
color x w/ color 1.
color nbrs of x color 2,
their nbrs color 1
no odd cycles ⇒ no conflicts.

Is this constructive? Yes for finite. For infinite graphs... how do I even?
AoC...
Def: $\mathcal{F}$ family of non-$\emptyset$ sets, f fn picking elt $c(A) \in A$

"most non constructive statement in math, non constr by nature".

This pf not constr, maybe ∃ another pf?

Example 1: $T_+ : \mathbb{R} \to \mathbb{R}$, $x \mapsto x+1$ $\quad G_{T_+} :=$ vertex set $= \mathbb{R}$, edges $\{(x, T_+(x)) : x \in \mathbb{R}\}$

no cycles, certainly bipartite

each component is infinite path, one path ∀ pts in $[0,1)$.

Explicit 2-coloring:



"here, can pick 1 vertex from all conn comp w/o AoC!"

Example 2: $T_\theta : S^1 \to S^1$, $x \mapsto$ rotate by θ. (angle) $\quad G_{T_\theta}$ = vertex set S^1, edge set $\{(x, T_\theta(x)) : x \in S^1\}$.

θ = rat'l mult of π ⇔ loop.

θ irrational, again every comp is infinite path, i.e. $G_{T_\theta} \cong G_{T_+}$, bipartite.
continuum many paths.

Explicit 2-coloring ???

we know ∃ 2-coloring....

Thm: no "constructive"/"explicit" 2-coloring of G_{T_θ}.

need formal def.

Logician happy w/ "doesn't use AoC"

But "working mathematician" don't want to worry about axioms.
Said Anton disparagingly

→ Borel (notion of)

Anton defined σ-alg, Borel σ-alg of topo space. Showed: orange pts $\bigcup_{k\in\mathbb{Z}} [2k, 2k+1)$ Borel. $(-\infty, 2k+1) \cap [2k, \infty)$

Idea: dynamic of transformation / shift in $\mathbb{R}$ and shift in $\mathbb{R}/2\pi\mathbb{Z}$
look similar, but interact differently w/ topology

Study dynamics, ergodic theory.

Step1: Borel sets are Leb msble

Leb: $\mathrm{Msb}(S^1) \to [0,1]$,
Leb msble ⇔ symm diff has Leb mes 0.

Step2: Suppose $S^1 = R \sqcup B$ (red, blue) msble.

Then $\mathrm{Leb}(R) = \frac{1}{2}$ bc Leb is invt under rotation, and $B = T_\theta R$.

But $R = T_{2\theta} R$.

R is invt under $T_{2\theta}$.

Step3: prove irrational rotation is ergodic ("indecomposable")
⇔ msble sets invt under / have null or full mes.
0-1 law.

Step4: $\frac{1}{2} \neq 0, 1$. ※

One more example: Banach Tarski. 1930.

pieces not msble so volume need not be preserved.

By now, whole industry of cutting up shapes and rearranging.

we realize now these are all graph theory problems.

only pfs known use graph thy.
Idea: upside down:
fix $T_1 \dots T_n$ transls.

Thm (Laczkovic '90): disc & square unit area in $\mathbb{R}^2$ finite ∧ equidecomposable by translation
Borel (Marks Unger 2016)

$G :=$ vertices = disc ⊔ sq.
edges = disc ∋ $x \sim T_i(x) \in$ sq
want perfect matching in G.

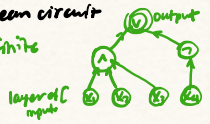
9/2/25 Borel sets, computational perspective on Borel sets

point in ambient space; "effective" way of telling inside or outside set

computation w/ ctble resources

Analogy w/ finite computation tight/explored

Boolean circuit



"acyclic directed graph"

don't define class P ppl familiar w/,

but complexity mostly by Def: circuit-size $f:\{0,1\}^n \to \{0,1\}$

num gates in circuit computing f

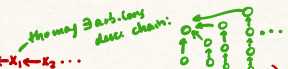
and P/poly := Sequences of fns $f_n:\{0,1\}^n \to \{0,1\}$ whose $CC(f_n) \le$ polyn.

Can choose $P \subseteq P/poly$ polytime algo w/ polytime advice?

Def: Boolean pseudocircuit/precircuit C is

- directed graph G +
  - a set $I \subseteq V(C)$ of input nodes (need to add (#sequential gates = #circuit inputs) or require nonzero in-nbr)
  - one output node $o \in V(C)$
  - each noninput node is called a gate, labelled by $\wedge, \vee, \neg$ (each $\neg$ gate has 1 in-nbr)

circuit is a pseudocircuit that is well-fdd: no infinite seq of nodes $x_0 \leftarrow x_1 \leftarrow x_2 \cdots$ (inputs, x_i could repeat, so this rules out cycles)



For us, replace "n" by "ℕ" and "poly(n)" by "ctble"

Def a (pseudo) circuit C w/ input I, an evaluation of C on input $\vec{x} = (x_i)_{i \in I} \in \{0,1\}^I$ is map $eval: V(C) \to \{0,1\}$ consistent w/ given inputs and logic of gates

① $eval(i) = x_i$ $\forall i \in I$

② $\forall$ gate v, $eval(v)$ = application of Boolean op to values of $eval$ on in-nbrs of v

Exc prop: if C circuit, then $\exists!$ $eval$

Def: for input $\vec{x} = (x_i)_{i \in I}$, $C(\vec{x}) := eval(o)$. (output node)

C computes $f:\{0,1\}^I \to \{0,1\}$ if $\forall \vec{x}$, $C(\vec{x}) = f(\vec{x})$

Exc prop. $\forall f:\{0,1\}^I \to \{0,1\}$, $\exists$ circuit C computing it w/ $|V(C)| \le app(|I|)$ if $|I|$ infinite, $\le 2^{|I|}$.

Def: I ctbl. a set $A \subseteq 2^I$ (usually $I = \mathbb{N}$) (usually 2^I Cantor space) is computed in "poly time", i.e. $|V(C)| \le poly(|I|)$ (i.e. is ctble) instead of exp time.

~~Let me copy from "GSO" Yeaay Fall~~

11/25/24 Anton Bernstein Intro Talk (Descriptive Combinatorics)

"descriptive set theory" "graph theory"

what is DST about?

Many motivations, from set Theory (obviously), e.g. Cantor originally

In Combo talk: "computational complexity for ctble problems"

Inputs - ctble bit strings, as opposed to finite bit string input.

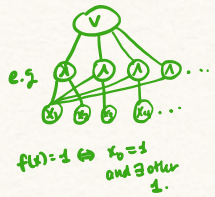
- e.g. real #s,
- ctble sequences of real #s.
- ctble graphs (represented by say $\infty \times \infty$ adjacency matrix)
- ctble groups
- ctbl fns $\mathbb{R} \to \mathbb{R}$

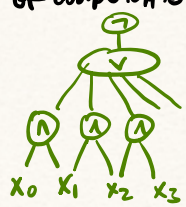
"how hard it is to tell if seq x_n has limit?"

"how hard it is to tell if ctbl fn is diffable?"

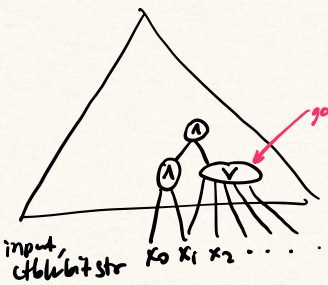
Fairly versatile setting.

Model of computation: ctble Boolean circuits





computes $f(\vec{x})$:= no two consecutive 1's in $\vec{x}$.



compute $f: \{0,1\}^{\mathbb{N}} \to \{0,1\}$.

ctbl

- ∞ directed graph, nodes labelled by $\wedge$ $\vee$ $\neg$
- no cycles
- has no infinite descending chains.

[more rigorous def:

defines well-defined fn.

Key Def: $A \subseteq \{0,1\}^{\mathbb{N}}$ is Borel if its characteristic fn is computed by a ctbl Boolean circuit.

in DST, this is perspective for "Borel"

"reasonable procedure to decide if in the set or not"

Ex: which ctble graphs (families of) is Borel? (i.e. vertex set $v_0 \cdots$ adjacency matrix, $f:(\ ) \mapsto \{0,1\}$ "1 is in family, 0 if not in family")

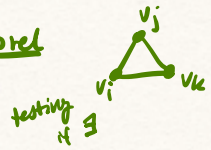
e.g. Cantor space

$F := \{x \in C : x \text{ has finitely many } 1\text{s}\}$

$x \in F \iff \exists n_0 \in \mathbb{N}\ \forall n \geq n_0\ (x_n = 0)$

i.e. $\bigvee_{n_0 \in \mathbb{N}} \bigwedge_{n \geq n_0} \neg x_n$

$-\{G : G \text{ triangle free}\}$ is Borel



testing if $\exists$

e.g. Fix $\delta \in [0,1)$

$D_\delta := \{x \in C : \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} x_i = \delta\}$

$x \in D_\delta \iff \forall k \geq 1\ \exists n_0 \in \mathbb{N}\ \forall n \geq n_0$

$\left|\frac{1}{n}\sum_{i=0}^{n-1} x_i - \delta\right| < \frac{1}{k}$

only depend on $x_0 \dots x_{n-1}$ finite

~~finite computation~~, can be computed by finite circuits (by Exc. Prop)

stupidest method (just by Lookup table)

$-\{G : G \text{ connected}\}$ is Borel — go through all pairs of vertices, all possible lengthless paths.

$-\{G : G \text{ bipartite}\}$ can not go through all possible subsets / partitions — e.g. finite graphs, go through all subgraphs is EXP, not P.

but, another check works:

is Borel (test all possible odd cycles)

$D := \bigcup_{\delta \in [0,1)} D_\delta = \{x \in C : \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} x_i \text{ } \exists\}$

is Borel (Exc.)

Idea: $\lim \exists \iff$ sequence is Cauchy.

That is easily seen to be Borel.

$-\{G : G \text{ tripartite}\}$

analogy in finite case, "definitely in NP"

in ctble setting, still "efficiently solvable",

still Borel.

ctbly many.

i.e. enough to check finite subgraphs

like "odd cycle" thing generalizes: philosophy

"all finite subgraphs 3-colorable

$\Rightarrow$ whole graph is 3-colorable"

Prop: $\exists$ nonBorel sets $A \subseteq C$

Cheapest: $|\mathcal{B}(C)| = 2^{\aleph_0}$

But $|\mathcal{B}(C)| \leq 2^{\aleph_0}$

by bounding # of ctble circuits (up to isom) not too redundantly

or L_{ω_1} fmlas?

Pf: example of nonBorel set.

enough to work w/ circuits C

s.t. $= V(C)$ fixed, fixed wire names are input $\exists$ $\forall$ $\neg$ p, q, r, n

move all $\neg$ to input vars $\Rightarrow$ only inners at n_i is p_i $\forall i \in \mathbb{N}$.

not well-fold circuits eliminated by deMorgan's law.



Turn circuit determined by edgeset.

Enumerate all pairs $(i,j) \in V \times V$ w/ $v \in \{\text{nil}\}_{n \in \mathbb{N}}$.

as $e_0, e_1, e_2 \dots$

For $x \in C$, let $C_x :=$ pseudocircuit w/ $E(C_x) := \{e_i : x_i = 1\} \cup \{(p_i, n_i) : i \in \mathbb{N}\}$ (thus already $\Rightarrow$ # circuits $\leq 2^{\aleph_0}$)

The following set is not Borel: $A := \{x \in C : C_x \text{ is well-fold and } C_x(x) = 0\}$ "Liar paradox, Halting problem, diagonalization".

Analogy: P $\longleftrightarrow$ Borel

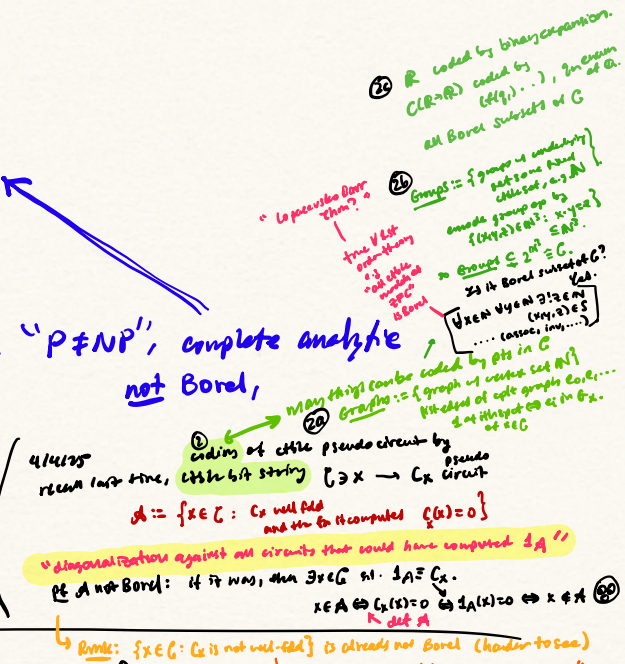
NP $\longleftrightarrow$ analytic

NP-cplt $\longleftrightarrow$ complete analytic

$-\{G : G \text{ has } \infty \text{ clique}\}$ complete subgraph

NOT Borel

Thm (1917): "$P \neq NP$", complete analytic not Borel,



That was DST. What about Descriptive Combo?

Def: a Borel graph G is

vertices

$V :=$ Borel subset of $\{0,1\}^{\mathbb{N}}$ (think: each V encoded as infinite string,

$E :=$ Borel subset of $V \times V$ and $\exists$ "reasonable" / ctble procedure to determine if string is vertex

given 2 ctble strings ctble procedure to determine if they are vertices of G, and there is or not edge b/t them.

Ex: $V = \{0,1\}^{\mathbb{N}}$ (∞ strings that differ by 1 entry)

$x \sim y \iff |x \Delta y| = 1$. is Borel.

- $\exists$ path $x \rightsquigarrow y \iff x, y$ differ in finitely many entries

$\Rightarrow$ not connected, in fact uncbly many components.

- graph has no odd cycles. (flip every coord. an even # of times)

so it's bipartite. $\rightarrow \chi(\mathbb{H}) = 2$

But! $\nexists$ Borel partition of V into I_0, I_2 (Borel) independent sets.

"same pt as Vitali set not mble"

Even worse! $\nexists$ partition $V = \underbrace{I_0 \cup \dots}_{\text{cbly}}$ into cbly Borel indepsets

$\Rightarrow$ χ_B of this graph is uncbly! (Borel chromatic)

Why? (Intuitively)

Fact! Suppose $I \subseteq V$ Borel indep set.

Pick random vertex $x \in \{0,1\}^{\mathbb{N}}$ (fair coin toss for each coord)

Then $\mathbb{P}(x \in I) = 0$.

(This proves

Where are why I is Borel? Well, to even define $\mathbb{P}$, Borel sets mble subset of prob space.

How to prove? "regularity of msr" "Lebesgue density"

if f.sac. $\mathbb{P}(I) > 0 \Rightarrow \exists$ finite str. $x_0 \dots x_{n-1} \in \{0,1\}^n$

s.t. $A := \{x$ w/ initial$\}$

$= \{\vec{x} : x_0 \dots x_{n-1} \sqsubset \vec{x}\}$.

s.t. $\frac{\mathbb{P}[I \cap A]}{\mathbb{P}(A)} = \mathbb{P}(I \mid A) > 99\%$.

(infact $> 50\%$ would already give $\nexists$)

Ex2: shift graph $\mathbb{S}$.

$V := \{\vec{x} \in \{0,1\}^{\mathbb{N}} : \infty$ many 1s$\}$

$T : V \to V$: $T(x)$ replaces the first 1 w/ 0. $\quad T(0010110\dots) = 0000110\dots$

define adjacencies: x adj. to $T(x)$.

Observe:
- $\mathbb{S}$ subgraph of $\mathbb{H}$ (so bipartite) $\rightarrow \chi(\mathbb{S}) = 2$
- $\mathbb{S}$ locally finite
- $\mathbb{S}$ has no cycles (so forest!)

Claim: $\chi_{Borel}(\mathbb{S}) = \chi_0$ Define $I_n := \{ x \in V : x = (\underbrace{0 \dots 0}_{n} 1 \dots) \}$

Borel.

Indep in $\mathbb{S}$ (bc $x, T(x)$ can't be n, or $T(x)$ would have more initial zeroes)

So $\chi_B(\mathbb{S}) \leq \infty$.

If $= \infty$, by infinite dim Ramsey theory

G Borel graph, max deg $d \in \mathbb{N}$, G has no cycles (forest)

d	0	1	2	3		d
max possible χ	1	2	2	2	2 .	.
max possible χ_B	1	2	3	.	.	$d+1$

↖ lowerbound due to Andrew Marks.



$\frac{\theta}{\pi} \notin \mathbb{Q}$. (again by regularity of measure)

	0	1	2	3	4	5	. . .	d
χ_M	1	2	3	3 or 4	?			asymptotically $\sim \frac{d}{\log d}$.

↑ measurable, instead of Borel

↑ natural log

(Had the result due to Anton himself)

Lowerbd $\geq \frac{1}{2} \frac{d}{\log d}$ 2014

Past $\frac{1}{2}$ threshhold 2018

upperbound by Anton "distributive coupling"

4/14/25 ctd.

Borel functions $f : C \to C$ Borel if $\exists$ cttble circuits $(C_i)_{i \in \mathbb{N}}$ s.t. $\forall x \in C$, $C_i(x) = f(x)$ (or $2^I \to 2^I$ for any index set I) ith bit of

"can compute each coordinate is cttble by cttble circuit"

Excl Prop: $f : C \to C$ Borel $\Leftrightarrow$ preimage Borel are Borel

Caution: Image of Borel sets under Borel fns may not be Borel.

"to decide if $x \in$ preimage, easy, compute f which is cttble, compute if $f(x) \in A$, which we can do bc A assumed Borel"

but functions compose forward, not backward.

whereas $x \in f(B) \Leftrightarrow \underbrace{\exists y \in C : f(y) = x}_{\text{not a cttble quantifier!}}$

↘ Going back to above $d := \{x \in G : C_x \text{ will fold } C_x(x) = 0\}$

Exc: A^C is image of Borel set under Borel map.

Thm (Luzin–Suslin): $A \subseteq C$ Borel, $f : A \hookrightarrow C$ Borel injective (or even "almost cttble-to-one") ($\exists$ family of circuits computing f on inputs $\in A$)

then, $f(A)$ is Borel!

Cor: $f : C \to C$ Borel $\Leftrightarrow$ graph(f) is Borel subset $C \times C = 2^{\mathbb{N} \sqcup \mathbb{N}}$

contents checkable

($\Rightarrow$) trivial. Given x Compute $f(x)$, compare $f(x)$ to y.

LHS is given x, compute y

RHS is given x, y, compute yes/no question is $f(x) = y$

($\Leftarrow$) take any Borel $A \subseteq C$, consider map $(C \times A) \cap \text{graph}(C) \to C$ (Borel set, proj)

$(x, y) \mapsto x$

this map is Borel and injective (always computable, using that f is fn.)

image is $f^{-1}(A)$!

Done by Luzin–Suslin

We switch now to topological perspective.

$\{0,1\}$ w/ discrete top

$\mathcal{C} = 2^{\mathbb{N}}$ w/ product top, smallest top that all coordinate proj are cts. $\text{proj}_i : 2^{\mathbb{N}} \to 2$

or in terms of sets & pts: we have actually a metric

Normalized Hamming metric on $2^{\mathbb{N}}$

$d(x,y) := \sum_{n=0}^{\infty} \underbrace{\frac{1}{2^n}}_{\text{arbitrary summable}} |x_n - y_n|$

Def: $D \subseteq \mathbb{N}$ $x \in \mathcal{C}$

D-nbhd $N_D(x) := \{y \in \mathcal{C} : y|_D = x|_D\}$

Lem: $\forall x \in \mathcal{C}$, $\{N_D(x) : D \subseteq \mathbb{N} \text{ finite}\}$ is nbhd basis at x.

Pf: check $\forall$ ball, ball $\supseteq N_D$.
$\forall$ D-nbhd, $\supseteq$ ball. easy.

Say $U \subseteq \mathcal{C}$ determined by $D \subseteq \mathbb{N}$

if $\forall x \in \mathcal{C}$, membership of $x \in U$ only depend on $x|_D$

Lemma: If $U \subseteq \mathcal{C}$ is finitely determined then U is both open and closed

4/7/25

Pf: U det by finite $D \subseteq \mathbb{N}$.

to prove U open, take $x \in U$, $N_D(x) \subseteq U$

to prove closed, U^c also det by D. Apply.

Converse is true. Bad.

for $\mathcal{N} := \mathbb{N}^{\mathbb{N}}$ Baire space, not every clopen is fin det.

Lemma: (exc) normalized Hamming dist on $\mathcal{C}$ is complete. defined last time

Thm: Family of all fin det sets is a basis for topology on $\mathcal{C}$

i.e. U open $\Leftrightarrow$ union of fin det sets.

Pf: fin det sets open, hence unions are open.

conversely, take union of N_x (each N_x is finitely det)

Thm: $\mathcal{B}(\mathcal{C}) = \{\text{sets } 1_A \text{ characteristic cptble by ctbl Boolean circuit}\} = \sigma\langle \text{opens in } \mathcal{C} \rangle$

Summary of Cantor space $\mathcal{C}$

① topology on $\mathcal{C}$ gen by cplt metric

② $\mathcal{C}$ is 2nd ctble.
= all fin det sets.

③ zero dimensional.
i.e. topology has basis consisting of clopen sets

④ $\mathcal{C}$ is cpct.
"combinatorial fact, can prove by hand"

"Polish"

Examples. $\mathcal{C}$
$\mathcal{N} := \mathbb{N}^{\mathbb{N}}$
$\mathbb{R}^n$
in fact $\mathbb{R}^{\mathbb{N}}$
$(0,1)$ (not w/ usual metric) but use $\cong \mathbb{R}$
$\mathbb{R} \setminus \mathbb{Q} \cong \mathcal{N}$ (usual metric is very not complete)

Thm: X Polish $\Rightarrow$
$\exists$ code $X \to \mathcal{C}$ s.t.
$A \subseteq X$ is Borel in X
$\Leftrightarrow \{\text{code}(x) : x \in A\}$ is Borel in $\mathcal{C}$

implies code is Borel.

Pf: construct code:
use 2nd ctble: Let $\mathcal{U} := (U_n)_{n \in \mathbb{N}}$ be ctble basis for topology on X.
$(\text{code}(x))_n := \begin{cases} 1 & \text{if } x \in U_n \\ 0 & \text{o.w.} \end{cases}$

Use deep Borel map $\Leftrightarrow$ Borel graph thm from last time

$X \times \mathcal{C}$
$(x,c) \in \text{graph}(\text{code}) \Leftrightarrow \forall n \in \mathbb{N}$ (ctble $\cap$) $\big((c_n = 0 \wedge x \in U_n) \vee (c_n = 1 \wedge x \in U_n) \big)$
Borel condition. in fact clopen in $\mathcal{C}$ (closed $\wedge$ open)
clopen $\Rightarrow$ Borel

code is Borel map. proving ($\Leftarrow$) in Thm.

use Polish

For other direction, code is injective, so Lusin-Suslin concludes.

Why Polish spaces?

- ones we understand best lol
- philosophically, recall Borel $\Leftrightarrow$ ctbly cpt'bl.
In this context, topology is window dressing.
- Polish somehow fundamentally still about

Lemma: X Polish.
May assume metric (cpat w/ top) is bdd

Pf: min w/ 1.

Lemma: ctble Σ (direct sum) product Π of Polish is Polish.

What about subspaces? $A \subseteq X$.
A closed $\Rightarrow$ A Polish (obvious)
open $\Rightarrow$ Polish (not as obv)

Thm (Alexandroff):
X Polish, $A \subseteq X$
"semantics" "syntactic"
A (w/ subspace top) Polish $\Leftrightarrow$ A is $G_\delta = \bigcap^{\infty}$ open.
Note $F_\sigma = \bigcup^{\infty}$ closed.

generalizes to result, but really, is mathematical logic result.

The real thm is

Thm (Borel Isom): X Polish, $A \subseteq X$ Borel

may not be G_δ because may not be Polish itself

uncntble

"cntble sets trivial from our perspective"

$$(A, \underbrace{\mathcal{B}(A)}_{\{B \in \mathcal{B}(X) : B \subseteq A\}}) \cong (C, \mathcal{B}(C)).$$

"Borel sets of Borel sets in Polish = Borel sets of C"

On level of Borel sets, Polish give us nothing new.

(topology is new, but we don't care about topology)

"really when subject is about Polish it just most common/easy way to show a given concrete example is std Borel (exhibit it as Borel subset of a Polish space)

∃ some rare example that don't live "naturally" in Polish spaces

so indeed Polish is but just a doorway to the room of std Borel.

Def: std Borel space is str $(X, \mathcal{B}(X))$ a σ-alg on X, called "Borel"

s.t. either
- X cntbl & $\mathcal{B}(X) = \mathcal{P}(X)$
- X uncntbl & $(X, \mathcal{B}(X)) \cong (C, \mathcal{B}(C))$.

"seems like stupid def, but it's then to give dramatic thm that says tons of things are std Borel"

though there are connections.

Thm: X std Borel

$A \subseteq X$ Borel $\Rightarrow$ $\exists$ cpt Polish top τ on X s.t. A is τ-clopen.

Note: is true if $A \subseteq X$ is open in some cpt Polish top τ' on X:

Pf: A open $\Rightarrow$ Polish

A^c closed $\Rightarrow$ Polish

take $X = A \sqcup A^c$ disj union of Polish is Polish.

4/9/25

Changing topology

Notate $\langle \mathcal{U} \rangle$:= topo generated by $\mathcal{U}$ (family of subsets of X)

Thm. X std Borel. ($(X, \mathcal{B}(X))$)

Suppose $(\tau_n)_{n \in \mathbb{N}}$ cntbl Polish top on X (i.e. $\sigma(\tau_n) = \mathcal{B}(X)$ too.)

Then $\langle \bigcup_{n \in \mathbb{N}} \tau_n \rangle$ is also cntbl Polish top. (τ_∞)

Pf: τ_∞ topo ✓

cntble ✓

Remain to show Polish. 2nd cntble ✓

completely metrizable?

- Fix cplt metric $d_n \hookrightarrow \tau_n$.

Then define $d_\infty(x,y) := \sum_{n \in \mathbb{N}} \frac{1}{2^{n+1}} d_n(x,y)$

Check details - is cplt metric

- generates τ_∞.

Return to "Making Borel sets clopen" thm

std σ-alg bootstrap.

Pf: fix cpat Polish topo τ' on X.

Let $\mathcal{A} := \{A \subseteq X : \exists$ cpat top τ on X s.t. A is τ-clopen$\}$

① every τ'-open set is in $\mathcal{A}$ (said last time)

② now show $\mathcal{A}$ is σ-alg.

- complements ✓ (trivial)
- cntble union $A_n \in \mathcal{A}$

Let τ_n be cpat Polish top on X s.t. A_n is τ_n-clopen.

Define $\tau := \langle \bigcup_{n \in \mathbb{N}} \tau_n \rangle$

In τ, all A_n clopen $\Rightarrow A$ is τ-open.

Now can make A clopen by ①

Cor: X std Borel

(Y,τ_Y) Polish. If $f: X \to Y$ Borel,

$\exists$ coarser (w/ BOF) Polish top τ_X s.t. $f:(X,\tau_X)\to(Y,\tau_Y)$ is cts.

Pf: fix cbbl basis $(U_n)_{n\in\mathbb{N}}$ for τ_Y

Need to make all $f^{-1}(U_n)$ open.

Let τ_n be coarsest Polish top on X s.t. $f^{-1}(U_n)$ is τ_n-open.

Set $\tau_X := \langle \bigcup_{n\in\mathbb{N}} \tau_n \rangle$

Change gear to sets that arise Borel. "next chapter of DST we're speeding thru lol"

Analytic, co-Anly sets

Analogy

If Borel $\leftrightsquigarrow$ P

Analytic $\leftrightsquigarrow$ NP

Def: X std Borel.

$A \subseteq X$ is analytic if $\exists$ std Borel Y (where witnesses live) and Borel $B \subseteq X\times Y$ s.t.

$A = \text{proj}_X(B) := \{x\in X : \exists y\in Y \text{ s.t. } (x,y)\in B\}$

("witnesses" — not obviously Borel b/c quantifies over uncntbly many Y e.g. if $\subseteq$)

("cputbl" / "Borel relation" — verify x cputably)

/ of course, all ISO, so could take $\subseteq$ but sometimes convenient to work concretely w/ others.

Equiv (more set-theoretic flavor) as opposed to computability

A analytic $\iff$ $\exists$ std Borel Y s.t. Borel fn $f: Y\to X$ s.t. $A = f(Y)$.

Pf: ($\Rightarrow$) $\text{proj}_X : B \to X$ Borel fn w/ A as its range

($\Leftarrow$) $f: Y\to X$, $A = \text{proj}_X(\text{graph}(f))$.

Def: $A\subseteq X$ coanly if A^c is anly.

Equiv (exc!) w/ $\forall$ except $\exists$.

i.e. $A = \{x\in X : \forall y\in Y\ (x,y)\in B)\}$

Ex: Graph $\cong C$

w/ fixed vset $\mathbb{N}$ identified by coding

Then $\text{Clique}_\infty := \{G\in \text{Graphs} : G \text{ has } \infty \text{ clique}\}$

analytic b/c $G\in \text{Clique}_\infty \iff \exists x\in C$ (Borel) (x has ∞ many 1's and $\forall i<j\in\mathbb{N}$ ($x_i=x_j=1 \Rightarrow G_{ij}=1$) ($\{(i,j)\in E(G)\}$))

finitely determined

Ex2: Directed Graph $\cong C$ on $\mathbb{N}$

$WF := \{\text{well-founded } D\in \text{DirGph}\}$ coanalytic.

(i.e. can't have $\cdots \to \cdots \to \cdots$ ∞)

Examples don't have to be combinatorial

Ex 3: $C[0,1] = \{\text{cts } f: [0,1] \to \mathbb{R}\}$.

coded by values on enumerable rationals. $\cong \mathbb{C}$.

Alternately, classical top makes it Polish

$d(f,g) := \|f-g\|_{\sup}$ complete and generates a Polish topology. (Ex C) (underlies)

$PC := \{(f_n)_{n\in\mathbb{N}} : C[0,1]^{\mathbb{N}} : f_n \text{ converges pointwise}\}$

coanalytic

$\forall x \in [0,1],\ f_n(x)$ converges

Borel (from previous example)

(yes there is subtlety that extra slight param x)

Fun Ex: $Diff := \{f \in C[0,1] : f \text{ diffble everywhere}\}$ coanalytic

one-sided on $\partial[0,1]$.

Thm (Suslin 1917): not all analytic sets are Borel (hence co any Borel closed under complement)

Famous story: Suslin corrected mistake Lebesgue made.

DST's P≠NP

Pf sketch: recall our coding circuits pseudo circuits

$C \ni x \mapsto C_x$ (continuous circuit on $\mathbb{N}$)

$A := \{x \in C : C_x \text{ is well-fdd} \ \&\ C_x(x) = 0\}$

but is coanalytic

C_x WF (coany) $\wedge$ $C_x(x)=0$ (by unsuccessful attempt) $\iff \forall$ eval: $V \to [0,1]$, ($2^V \cong 2^{\mathbb{N}}$)

(eval is an evaluation of C_x on x, then eval(output)=0)

(go thru old daily guessing over vertex set)

any & coany closed under $\bigcup_{\mathbb{N}}, \bigcap_{\mathbb{N}}$.

A is a bit artificial, purposely self-referential, ...

But Fact: Clique, WF, PC, Diff are non-Borel

by reduction to A or A^c ← "NP-complete".

4/11/25

X, Y all denote std Borel

Recall analytic: image of Borel fn / $\exists y \in Y (\ldots)$

coanalytic: complement of ↑ / $\forall y \in Y (\ldots)$

Def: $A \subseteq X$, $B \subseteq Y$ **Borel reduction from $A \to B$** is Borel map $f: X \to Y$ s.t. $f^{-1}(B) = A$, i.e. $\forall x \in X$, $x \in A \iff f(x) \in B$.

"membership prob for A $\leq$ (easier) membership for B"

Def: $A \subseteq X$ **complete (ω) analytic** if it is (ω)analytic and every analytic is Borel reducible to A.

Notes: complete coanalytic $\iff$ complement is complete analytic

and complete analytic/coanalytic can't be Borel.

Thm: $WF \subseteq Dir G$ on reduced $\mathbb{N}$

well fdd directed graphs on $\mathbb{N}$, coanalytic as discussed last time, is **cplt coanalytic**

from this, can show $Cbbve_\infty \subseteq Graphs$,

$PC \subseteq (\{0,1\})^{\mathbb{N}}$

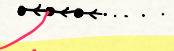
$Diff \subseteq \mathbb{C}(\{0,1\})$

are cplt coanalytic

Pf: Suppose have Borel $f: X \to Y$. Want to find Borel reduction from

$Y \setminus f(X)$ (arb. coanalytic set) to WF

i.e. a Borel assignment $y \longmapsto D_y \in Dir$ s.t. $y \in f(X) \iff D_y$ not well fdd, i.e. $\exists$ infinite chain



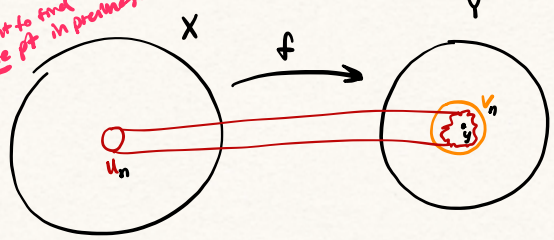
node in digraph, going up in improving approximation, can take conv. so chain $\iff$ limit

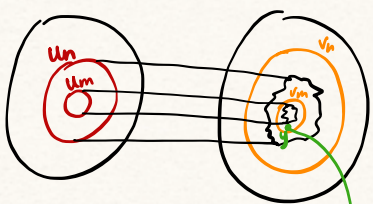
using previous tools we developed,

Fix cpat Polish top on X and Y (ctble bases $\mathcal{U}, \mathcal{V}$) s.t. $f: X \to Y$ cts

Fix cplt metric compat w/ on X, Y

"want to find one pt in preimage of y. Do this by finding smaller preimages of balls of y.





Not depend on y.

- Enumerate as $(U_n, V_n)_{n \in \mathbb{N}}$ all pairs s.t. $f(U_n) \subseteq V_n$
- Define digraph on $\mathbb{N}$, "$n \leftarrow m$" iff "m better approx than n (in a way that ∞ seq $\rightsquigarrow$ conv seq)"

$\overline{U_m} \subseteq U_n$, $\overline{V_m} \subseteq V_n$

diam of $U_m \leq \frac{1}{2}$ diam U_n

$V_m \leq \frac{1}{2} V_n$

Now use y:

Let $D_y :=$ spanning (same vtx set) subgraph of D w/ edges $\{n \leftarrow m : n \leftarrow m \text{ in } D \text{ and } y \in V_m\}$

Must check: $y \mapsto D_y \in \mathrm{Dir} \cong \mathcal{C}$ is Borel (it is in fact cts)

$\mathcal{C}$ cptbl

$n \xleftarrow{D_y} m \iff n \xleftarrow{D} m$ and $y \in V_m$ ✓

($n \xleftarrow{D} m$: Borel condition in y b/c indep on y. $y \in V_m$: open condition)

and D_y ∞ ascending seq $n_0 \leftarrow n_1 \leftarrow n_2 \cdots \Rightarrow U_{n_0} \supseteq \overline{U_{n_1}} \supseteq \overline{U_{n_2}} \supseteq \cdots$

diams going to 0

and Lemma: metric cplt $\iff$ all nested closed sets w/ diams $\to 0$ has intersection = 1 pt

so contains unique pt x

Claim: $f(x) = y$.

w/o $x \in \bigcap_k U_{n_k} \Rightarrow f(x) \in \bigcap_k V_{n_k}$

$\Rightarrow$ uniqueness, $f(x) = y$, and the fact that $y \in V_m$ for D_y

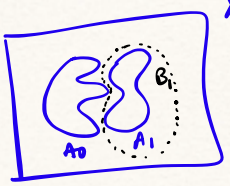
Conversely $y \in f(X) \Rightarrow D_y \notin WF$

QED

---

very powerful Thm (analytic separation): $A_0, A_1 \subseteq X$ analytic disjoint

Then $\exists$ partition $X = B_0 \sqcup B_1$ into Borels s.t. $A_i \subseteq B_i$ $i = 0,1$



(ct. cptbility theory: separation of Π^1_1 sets. Here is sep of Σ^1_1 sets)

Cor: If a set is both anly and coanly, then is Borel (compare to open $P \stackrel{?}{=} NP \cap coNP$)

Pf: can sep A and A^C w/ Borel, but not much room to sep...

Cor: $f: X \to Y$, graph(f) is analytic $\Rightarrow$ f is Borel fn (recall last time, said f Borel $\iff$ graph f Borel. So in fact graph(f) analytic $\Rightarrow$ it's actually Borel — bootstrap)

Pf: $(x,y) \in$ graph$(f) \iff f(x) = y \iff \forall z \in Y\ ((x,z) \in \text{graph}(f) \to z = y)$

Pf: What does above amount to considering trees?

B/c WF cplt anly, (may reduce to working w/ Digraphs.) have Borel $D_0, D_1 : X \to \mathrm{Dir}$ s.t. $A_i = D_i^{-1}(\neg WF)$

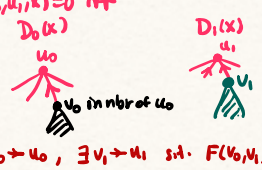
i.e. $\forall x \in X$, have 2 digraphs $D_0(x)$, $D_1(x)$ cptbl in Borel way s.t. we're promised at least one of them is WF (this uses that A_0, A_1 disjoint)

Want Borel fn $F: X \to \{0,1\}$ s.t. if $F(x) = i$ then $D_i(x)$ is WF (the other one could be WF too, but don't care) (we have advantage of knowing at least one of $i = 0,1$ will work)

Idea: use recursion. (dynamic programming, not just compute $F(x)$, compute entire $\mathbb{N} \times \mathbb{N}$ table)

Recursively $\forall u_0, u_1 \in \mathbb{N}$ (vtx set), want $F(u_0, u_1, x)$ to pick either u_0 or u_1 so that in $D_i(x)$ there is no $u_i \leftarrow \cdots$ ∞ chain.

$F(u_0,u_1,x) = 0$ iff "no $u_0 \leftarrow \cdots$ in $D_0(x)$"



$\forall v_0 \to u_0$, $\exists v_1 \to u_1$ s.t. $F(v_0, v_1, x) = 0$

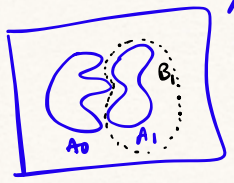
4/14/25

Thm (analytic separation): $A_0, A_1 \subseteq X$ std Borel analytic disjoint

Warning: not true in general for coanalytic.

then $\exists$ partition $X = B_0 \sqcup B_1$ into Borels

s.t. $A_i \subseteq B_i$ $i=0,1$



Continue pf: suffices to compute F w/ "local version, $\exists$ ω chain starting at given pt u"

GOAL: $F(x,u_0,u_1)=i \iff D_i(x)$ has no ω-chain starting at u_i

Def: $F(x,u_0,u_1)=0 \iff \forall v_0 \in \mathbb{N}$ s.t. $v_0 \xrightarrow{D_0(x)} u_0$,

$\exists v_1$ s.t. $v_1 \xrightarrow{D_1(x)} u_1$

s.t. $F(x,v_0,v_1)=0$

Where is base of recursion?

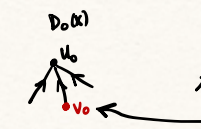
if $\nexists v_0 \xrightarrow{D_0(x)} u_0$, auto true

i.e. u_0 has no in nbr, so can safely conclude u_0 tree well-fdd

if $\nexists v_1$ s.t. $v_1 \xrightarrow{D_1(x)} u_1$

then can conclude u_1 tree well fdd.

Picture:



If can guarantee each sub tree for all v_0 is well-fdd, then tree under u_0 will be well-fdd

Claim1: this is well fdd recursion that does define value of $F(x,u_0,u_1)$ $\forall$ choice of $x \in X$, $u_0, u_1 \in \mathbb{N}$.

i.e. $\nexists$ recursion $F(x, sub_0, sub_1)$ never hits base.

Suppose not. Then $\exists$ ω seq of pairs

$(u_{0i}, v_{0i})_{i \in \mathbb{N}}$ s.t. $u_{00} \leftarrow u_{01} \leftarrow \dots$ in $D_0(x)$

$u_{00} = u_0$

$u_{10} \leftarrow u_{11} \leftarrow u_{12} \dots$ in $D_1(x)$

this can't happen b/c then both $D_0(x)$, $D_1(x)$ not well fdd, $\lightning$

Claim2: F satisfies GOAL, i.e. if $F(x,u_0,u_1)=0$, then $D_0(x)$ has no ω chain starting from u_0.

well, Suppose it did: $u_{00} \leftarrow u_{01} \leftarrow u_{02} \leftarrow \dots$ in $D_0(x)$.

$u_{00} = u_0$, $u_{01} = v_0$

Since $F(x,u_0,u_1)=0$, by def $\exists$ some $v_1 = u_{11} \xrightarrow{D_1(x)} u_{10} = u_1$

s.t. $F(x,u_{01},u_{11})=0$

Iterating this argument, find $u_1 = u_{10} \leftarrow u_{11} \leftarrow u_{12} \leftarrow \dots$ in $D_1(x)$, so get same $\lightning$ as

Claim3: $F(x,u_0,u_1)=1 \Rightarrow D_1(x)$ has no ω chain starting at u_1

Exc. (same arg)

negate quantifiers in def of F, same argument.

Claim 4: F is Borel.

$F(x,u_0,u_1)=0 \iff \forall v_0 \in \mathbb{N}$ s.t. $v_0 \xrightarrow{D_0(x)} u_0$, $\exists v_1$ s.t. $v_1 \xrightarrow{D_1(x)} u_1$ s.t. $F(x,v_0,v_1)=0$

compute these bits ✓

Now take one node $\forall$ pair $(u_0,u_1) \in \mathbb{N}^2$, arrange wires in circuit to compute

Sadly, not well fdd. Above, fixing x, can get well fdd circuit computing $(u_0,u_1) \mapsto F(x,u_0,u_1)$

But need circuit $\forall x$....

Use topological techniques. (why useful?)

May assume D_0, D_1 cts fns on X w/ nice Polish topology

fix cplt metric d for X, ctble basis (U_n)

cts $D_i: X \to \mathrm{Dir} \cong 2^{\mathbb{N}}$ means $\forall x \in X$ s.t. $v \xrightarrow{D_i(x)} u$ true, then in fact $\exists$ an open nbhd $U \ni x$ s.t. $v \xrightarrow{D_i(y)} u$ $\forall y \in U$.

"coordinate = 1 in dir graph, a clopen condition"

terminology: "U guarantees $v \xrightarrow{D_i} u$"

Fix: compute $F(x,u_0,u_1)$ separately on each U_n separately: (to try to make it indep of x as much as possible)

$F_n(x,u_0,u_1)=0 \iff \exists m$ s.t. extra conds we figure out later:
- $x \in U_m$
- $\overline{U_m} \subseteq U_n$
- $\mathrm{diam}\, U_m \le \frac{1}{2}\mathrm{diam}\, U_n$

$\forall v_0 \in \mathbb{N}$ s.t. U_m guarantees $v_0 \xrightarrow{D_0} u_0$

$\exists v_1 \in \mathbb{N}$ s.t. U_m guarantees $v_1 \xrightarrow{D_1} u_1$

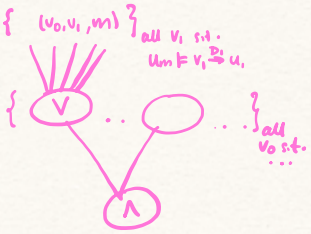
s.t. $F_m(x,v_0,v_1)=0$

*adding extra quantifiers didn't change meaning, by cty!!! i.e. $v_0 \xrightarrow{D_0(x)} u_0$, $v_1 \xrightarrow{D_1(x)} u_1 \Rightarrow$ some U_m guaranteeing! "for free"

"meaning of def didn't change, only syntax"

So, now, build ctble circuit as follows:

have node $\forall$ triple $(u_0,u_1,n) \in \mathbb{N}^3$ computing $F_n(x,u_0,u_1)$ wired according to above def



Claim: is now well fdd!

Pf: Suppose not. Then have ∞ seq $(u_{0i}, u_{1i}, n_i)_{i \in \mathbb{N}}$ s.t.

U_{n_i} guarantees $u_{0i} \xrightarrow{D_0} u_{0(i+1)}$

$u_{1i} \xrightarrow{D_1} u_{1(i+1)}$

and $\overline{U_{n_{i+1}}} \subseteq U_{n_i}$

diam shrinks by $\frac{1}{2}$.

$\Rightarrow \exists!\, x \in \bigcap U_{n_i}$ and we have

$u_{00} \leftarrow \dots$ in $D_0(x)$

$u_{10} \leftarrow \dots$ in $D_1(x)$, ⚠

"only technique the subject" lol

* if someone gave me def of f_n F, this procedure is the one I would take to locate bad/non-well fdd behaviour"

Last of background content. (Borel) Mzr & Baire cat. on Polish spaces or std Borel spaces

3 reasons why mzr useful:

Def: μ null if $\exists$ Borel $\supseteq$ of μ-zero

- things can't do w/ Borel, but Borel up to a.e.

Def: "μ almost Borel" μ msble if $\exists$ Borel $\mu(A \triangle \text{Borel})=0$

- just like our clopen/cts trick: Borel → mzr allow us to replace Borel w/ open

1
2
3
4 A null $\iff$ A msble ...

4/16/25 previously, msr μ: Borels $\to [0,\infty]$ on $(X, \mathcal{B}(X))$ std Borel.

extend μ: μ-msble $\to [0,\infty]$
$\quad\quad\quad

Prop: X Polish, $A \subseteq X$,

A comeager $\iff \exists$ dense open U_n s.t. $A \supseteq \bigcap_N U_n$

$\iff A$ contains G_δ set.

Example: C natural (Polish) top we already discussed.

also natural msr μ, coin flip msr.

Defined on basic opens by: $\forall x \in C$, $D \subseteq N$ finite

$\mu(N_D(x)) = \frac{1}{2^{|D|}}$ agree w/ x on coordinates D

Consider example set

$$L := \left\{ x \in C : \lim_{n\to\infty} \frac{1}{n} \sum_{i<n} x_i = \frac{1}{2} \right\}$$

LLN says L is μ-comeil.

But, Claim: L is meager. "Intuition: distinction btw msr, category — generative "likely vs unlikely" playing in casino for ∞ time to ∞ many events.

msr: look at finite times, take limit. So expect $\frac{1}{2}$. or always expect to lose on avg.

Category: anything that can happen, will. So will always recuperate losses! at some time

$$L^c \supseteq \left\{ x \in C : \forall n_0 \; \exists n > n_0 : \left(\frac{1}{n} \sum_{i<n} x_i > 99\% \right) \right\}$$

$\bigcap_N$ over n_0; the inner condition is open! $=: U_{n_0}$

Claim: $\bigcap_N U_{n_0}$ is dense

Pf: take $x \in C$, finite $D \subseteq N$

want to find pt $y \in N_D(x) \cap U_{n_0}$

set $y_i = \begin{cases} x_i & i \in D \\ 1 & \text{o.w.} \end{cases}$

Def: $A \subseteq X$ is Baire mble if $\exists$ Borel s.t. $A \triangle B$ is meager.

also "has prop of Baire" almost Borel w.r.t. meagerness

Now is σ-alg.

Thm "100% lemma" — as opposed to 99% above

X Polish (actually ult metric subtree)

$A \subseteq X$ is Baire mble $\Rightarrow \exists$ open U s.t. $A \triangle U$ meager.

"half pg. pf"

Main diff: A closed, $A \setminus \mathring{A} = \partial A$ is meager!!!

btw msr, category. whereas ∂A may have huge msr.

Cor (Baire alternative)

X Polish, $A \subseteq X$ Baire mble

Let $\mathcal{U}$ be any weak basis of X (open, $\neq \emptyset$; contains $\exists$ guy in $\mathcal{U}$, $\neq \emptyset$)

Then either ① A is meager

② A comeager in some $\neq \emptyset$ set $U \in \mathcal{U}$

"most of U is in A", "A mostly contains U"

This marks end of background = review of DST.

Ex: graph, vertex set is Polish space.

H graph w/ vertex set C

$\{x,y\} \in E(H) \iff |x \triangle y| = 1$

"∞-dim hypercube"

Note: $\exists$ path (necessarily finite) from x to y in $H \iff x, y$ differ in finite places.

in ptclr, H not connected graph.

w/ uncountably many components.

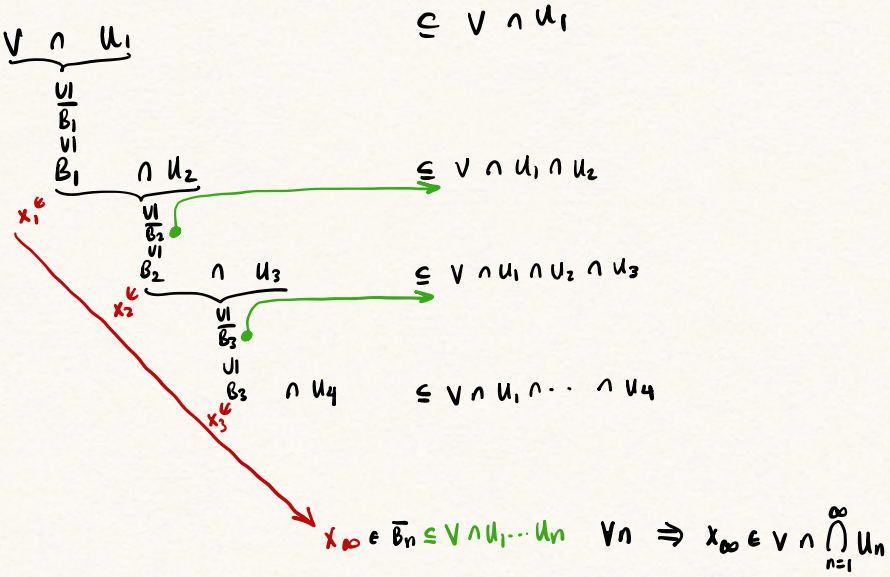
$2^{\aleph_0}$

most graph theory, can always reduce to conn comp; but here, can not do something on each piece bc uncthly many pieces, and we need cthbl / Borel procedures.

But Locally cthble: every vtx has cthly many nbrs. $\iff$ every comp is cthbl

This graph is bipartite (no odd cycles), i.e. $\chi(H) = 2$.

Q: $\exists$ Borel 2-coloring?


$V \cap U_1$
$\subseteq V \cap U_1$
$\overline{B_1}$
B_1
$B_1 \cap U_2$
$\subseteq V \cap U_1 \cap U_2$
$x_1 \in$
$\overline{B_2}$
$B_2 \cap U_3$
$\subseteq V \cap U_1 \cap U_2 \cap U_3$
$x_2 \in$
$\overline{B_3}$
$B_3 \cap U_4$
$\subseteq V \cap U_1 \cap \dots \cap U_4$
$x_3 \in$
$x_\infty \in \overline{B_n} \subseteq V \cap U_1 \dots U_n \quad \forall n \Rightarrow x_\infty \in V \cap \bigcap_{n=1}^{\infty} U_n$

4/18/25 applying DST tools to

∞ hypercube H, vtx set C

$\{x,y\} \in E(H) \Leftrightarrow |x \triangle y| = 1$.

Discussed some properties:

- locally cble
- $\chi(H) = 2$, i.e. $\exists$ partition in 2 H-indep sets

  proper 2-coloring $c: C \to 2$ of H

recall 1st lecture, to find coloring, general pf uses AoC to choose 1 vertex in each conn comp.

we will show indep set decomp $I_0 \sqcup I_1$ ↘ can't be Borel!

Prop: no Borel proper coloring $C \to \mathbb{N}$ of H

($\chi_{Borel}(H)$ is uncountable)

$:=$ min cardinality of a std Borel space C s.t. $\exists$ a Borel proper coloring $V \to C$ (vtx set of G, must be std Borel)

(possible values (by def of std Borel): $1, 2, \dots, \aleph_0, 2^{\aleph_0}$) "as many colors as vertices of graph"

In fact, if τ is usual topo on C, μ is coinflip msr on C, then $\chi_{\mu\text{-msble}}(H)$, $\chi_{\text{Baire msble}}(H)$ also uncble.

(fn agree w/ Borel on comull) (fn agree w/ Borel on comeager.)

often prove lowerbd for msble or BM to get lowerbd on χ_{Borel} (of course $\leq \chi_{Borel}$)

Prop: Every μ-msble (or Baire msble) H-indep set is μ-null (or meager)

(hence can't cover C by cble many H-indep sets)

(of BM; msr is CVC)

Pf: Say $I \subseteq C$ is H-indep, Baire msble.

Suppose f.o.c. I is not meager.



By Baire alternative, $\exists$ a nonempty (basic) open U s.t. I comeager in U (i.e. $U \setminus I$ is meager)

finitely determined ($w_D \leq \mathbb{N}$) set

for μ msble msr, 99% lemma



99%, and intersect 2 99% sets is ≥ 98%.

Idea: "throw away meager set of vertices, can't kill all the edges"

Fleshed out:

Pick any $i_0 \in \mathbb{N} \setminus D$ and consider $f_{i_0}: C \to C$, $x \mapsto$ flip i_0th coord of x.

Observe f_i is homeomorphism of C (in fact isometry under normalized Hamming metric)

$\Rightarrow$ preserves all topo notions, such as Baire category

So I comeager in $U \iff f_{i_0}^{-1}(I)$ comeager in $f_{i_0}^{-1}(U) = U$ (b/c U fin. det. by D, and i_0 not in D.)

$\Rightarrow I \cap f_{i_0}^{-1}(I)$ is comeager in U

$\Rightarrow I \cap f_{i_0}^{-1}(I) \neq \emptyset$

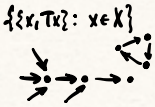
$\forall x$ is s.t. $x \in I$, $f_{i_0}(x) \in I$ $\Rightarrow I$ is not indep.

Graphs generated by a function (see old Anton seminar lecture I copied above, finally got to it)

$T: X \to X$ assume has no fixed pts. Then graph G_T, vtx set X

edge set $\{\{x, Tx\} : x \in X\}$ (undirected, but natural choice of orientation)

*picture:



Fact: $\chi(G_T) \le 3$ (optimal b/c )

Pf: suffice to look at finite graphs

Thm (De Bruijn–Erdős): graph G k-colorable $\iff$ all finite subgraphs of G are k-colorable $(k \in \mathbb{N})$

(a corollary of 1st order cpctness thm, or e.g. Tychonoff
but no matter what, uses Axiom of Choice in some way
"mumbo jumbo")

So, think about G_T. What do finite subgraphs of look like?

Cycles w/ trees feeding into them.

Or, rigorously, cleanly: (that works more generally in situations that can't describe explicitly)

Suppose f.t.o.c. not all finite subgraphs of G_T are 3-colorable.

Let $Y \subseteq X$ be finite set w/ min cardinality s.th. induced subgraph $G' := G_T[Y]$ is not 3-colorable

But handshake lemma $\sum_{y \in Y} \deg_{G'}(y) = 2|E(G')| \le 2|Y|$ (by def of $\{x, Tx\}$ edges)

$\Rightarrow \exists$ vtx $y \in Y$ of degree ≤ 2.

By min card assumption, $G' - y$ is 3-colorable, can extend to

Clever part of pf! I had not recognized how unexpected this is!

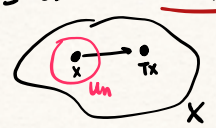
Return to Borel X, X std Borel

$T: X \to X$ no fixed pts

Prop (Kechris–Solecki–Todorcevic 99) $\chi_B(G_T) \le \aleph_0$

Pf: Let $(U_n)_{n \in \mathbb{N}}$ be ctbl basis for a cpat. Polish topology on X

$\Rightarrow \forall x, \exists n \in \mathbb{N}$ s.t. $x \in U_n$, $Tx \notin U_n$



Then $c: X \to \mathbb{N}$ is proper coloring.

$c(x) := \min_{n \in \mathbb{N}} S$?

Is it Borel?

Suffice to show $c^{-1}(n)$ Borel $\forall n \in \mathbb{N}$.

$X_n := \{x \in X : x \in U_n, Tx \notin U_n\} = U_n \setminus T^{-1}(U_n)$, Borel

$c^{-1}(n) = X_n \setminus \bigcup_{i<n} X_i$, Borel

"Ex 2 in old Anton lecture" $X := \{x \in \mathbb{C} : x$ has ∞ many 1s$\}$

$S: X \to X$ "the shift", no fixed pts

$S(x)$:= change 1st 1 into a 0

e.g. $S(\overset{0:1\,2\,3\,9}{0\,0\,1\,0\,1\,1\ldots}) = (\overset{0:1\,2\,3\,9}{0\,0\,0\,0\,1\,1\ldots})$
$= (2,4,5\ldots)$ $= (4,5\ldots)$

Note: G_S is subgraph of H (can only change 1 coord)

$\Rightarrow \chi(G_S) = 2$

- also locally finite (but not bdd degree)
- G_S has no cycles at all (i.e. is forest)

however... **Thm**: $\chi_{Borel}(G_S) = \aleph_0$

infinitary Ramsey theory. A bit complicated.

we will prove first it has to be complicated:

Thm: $T: X \to X$, any finite msr μ on X and any cpat. Polish topology on X

$\chi_{msble}(G_T)$, $\chi_{BM}(G_T) \le 3$.

i.e. Whatever trouble is here is null $\forall \mu$, meager $\forall$ Polish topo

"error w/ prob 0, can color w/ 3"

randomized vs deterministic algo

4/21/25
Lec 10

Last time, **Setup**

X std Borel space

$T: X \to X$ Borel map w/ no fixed pts. (can define things in such a way that we do not need to make this assumption)

$G_T :=$ vtx $= X$, edge $= \{\{x, Tx\} : x \in X\}$ (no loops)

Know Lusin "epctness", which requires some amt of AoC) $\quad \chi(G_T) \leq 3$

$\chi_{\text{Borel}}(G_T) \leq \aleph_0$.

Thm: ① μ finite msr on X, then $\chi_\mu(G_T) \leq 3$ (remains true for μ σ-finite too)

at some pt in pf, will be another msr that shows up, we would have to rescale, yuck.

② if X given compat Polish topology, then $\chi_{BM}(G_T) \leq 3$. "Baire mble"

Need definitions

Def: a set $A \subseteq X$ is (forward) recurrent if $\forall x \in X$, $\exists n \in \mathbb{N}$ s.t. $T^n x \in A$ (no need to disambiguate today)

($\Leftrightarrow$ forward orbit of every x by T hits A, in fact ∞ many times)

$d_A(x) := \min\{n \in \mathbb{N} : T^n x \in A\}$.

Observation: relating back to our graph G_T,

$d_A(Tx) = d_A(x) - 1$ unless $x \in A$

in ptclr, different parity

Lemma: if $\exists$ Borel G_T-independent recurrent set $I \subseteq X$, then $\chi_B(G_T) \leq 3$.

pf: define coloring $c: X \to \{0,1,2\}$ by

$$x \mapsto \begin{cases} 0 & \text{if } x \notin I \text{ and } d_I(x) \text{ even} \\ 1 & \text{if } x \notin I \text{ and } d_I(x) \text{ odd} \\ 2 & \text{if } x \in I \end{cases}$$

Check Borel (exc.)

Thm (Kechris–Solecki–Todorcevic): TFAE

① $\chi_B(G_T) < \infty$

② $\chi_B(G_T) \leq 3$

③ $\exists$ a Borel G_T-indep recurrent set $I \subseteq X$

④ $\exists$ a Borel set $A \subseteq X$ s.t. both A and A^c are recurrent. ("∞ often hit A, ∞ often leave A")

most useful for proving ①⇔②

Pf: Lemma ②⇒② trivial ⇒①

④ ?

①⇒④: suppose $c: N \to \{0,\dots,q-1\}$ a proper Borel coloring.

$\chi_B(G_T) < \infty$

Need to cook up Borel set A s.t. A, A^c recurrent.

wlog, $q=2$, win. Need exactly 2 options.

$A := \{x \in X : c(x) > c(Tx)\}$ ← no = case b/c proper coloring!

$A^c := \{x \in X : c(x) < c(Tx)\}$

A Borel b/c c, T both Borel.

④⇒③: Let $A \subseteq X$ be Borel s.t. both A, A^c recurrent.

Need to cook up recurrent independent set I for G_T

$I := \{x \in X : x \in A, Tx \notin A\} = A \setminus T^{-1}(A)$

Anton sketches alternative pf, skipping ④

Now, pf of $\chi_\mu(G_T) \le 3$. Let μ be finite mgr on X.

By ④ above, and μ-meble ⇔ Borel except on null set.

We'll find a Borel set $A \subseteq X$ s.t. A, A^c both recurrent "on a conull set"

Lemma: ∃ Borel set $A \subseteq X$ s.t. $N_A(x) := \{n \in N : T^n x \in A\}$ and $N_{A^c}(x) := \dots = N \setminus N_A(x)$ are infinite for μ-a.e. $x \in X$. ($\forall^\mu$)

Pf of lemma: try our previous ④⇒① construction but on a cble Borel coloring $c: X \to N$.

i.e. try $A := \{x \in X : c(x) > c(Tx)\}$ (Borel)

What can go wrong?

Complement of A is recurrent b/c coloring must go up ∞ often, but A may not be (b/c coloring may not go down)

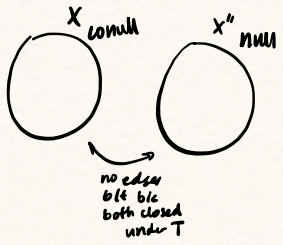
Let $X' := \{x \in X : N_A(x) \text{ is } \infty\}$ "happy pts"

$X'' := \{\dots < \infty\}$. "sad pts"

⇕ sequence of colors is eventually always increasing

Such x put into set X' "happy pts" and $X'' := X \setminus X'$

- both Borel
- also T-invt !

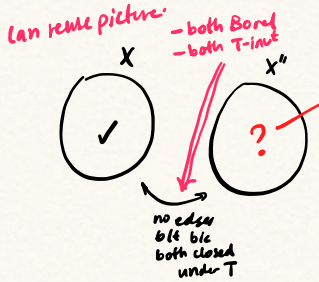


Thus the induced subgraph $G_T[X']$ has $\chi_B \le 3$ by A and Thm ④⇒②

$G_T[X'']$ has $\chi \le 3$, but X'' null.

so union of these 2 colorings witnesses $\chi_\mu(G_T) \le 3$

QED



need to figure out what to do here.

"zoom in" on X'', i.e. assume $X = X''$,

i.e. assume $\forall x \in X$, seq $c(x), c(Tx), \dots$ is eventually ↗.

⇓

Set $C(x) := \{c(T^n x) : n \in N\} \subseteq N$ is infinite.

Key Observation: for set $S \subseteq \mathbb{N}$,

$$\text{Let } A_S := c^{-1}(S) = \{x \in X : c(x) \in S\}$$

If we choose $S \subseteq \mathbb{N}$ randomly by tossing $\frac{1}{2}$ coin (indep) for each $n \in \mathbb{N}$,

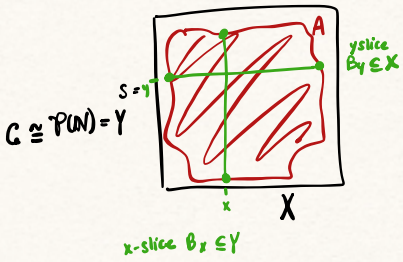
then $\forall$ fixed $x \in X$,

$$\mathbb{P}_S\big(c(x) \cap S, \; c(x) \setminus S \text{ are both } \infty \big) = 1$$

fixed ∞ set $c(x)$, tossing coin randomly, land in ∞ often, and land outside ∞ often!

But this is for single x...

Fubini will save us:



(Special case applied to indicator fns)

Fubini Thm: (X, μ), (Y, ν) fin. msr spaces, $B \subseteq X \times Y$ msble w.r.t. $\mu \times \nu$.

TFAE:

① B conull in $(X \times Y, \mu \times \nu)$

② B_x conull in Y for $\forall^\mu x \in X$

③ B_y conull in X for $\forall^\nu y \in Y$.

$$B := \{(x, S) : \begin{matrix} c(x) \cap S \\ c(x) \setminus S \end{matrix} \text{ are } \infty\}$$

computable: compute x (iterates), check how often $\in S$ or not in S.

We have shown: ($\forall^\nu$ = coin-flip msr on $C \cong \mathcal{P}(\mathbb{N})$)

$$\forall x \in X, \quad \forall^\nu S \subseteq \mathbb{N}, \quad (x, S) \in B.$$

So Fubini quantifier switch:

$$\forall^\nu S \subseteq \mathbb{N}, \quad \forall^\mu x \in X, \quad (x, S) \in B$$

$$\Downarrow$$

$\exists S \subseteq \mathbb{N}$ s.t. $\forall^\mu x \in X$, $N_{A_S}(x)$, $N_{A_S^c}(x)$ are infinite, as desired.

4/23/25 Last time, (X,μ), $T: X \to X$ Borel, no fixed pts,

Thm: $\chi_\mu(G_T) \le 3$

Borel coloring $c: X \to \mathbb{N}$

Want: a set $A \subseteq X$ s.t. both A, A^c are recurrent.

Case 1: $A = \{x \in X : c(x) > c(Tx)\}$ works.

|

Case 2: if doesn't work, then $\forall x \in X$, the set $\{c(T^n x) : n \in \mathbb{N}\}$ is infinite.

Then take $A := \{x \in X : c(x) \in S\}$ where S is random set,

which works for μ-almost all $x \in X$.

used Main Tool: Fubini which allows us to interchange

$$\forall^{a.e.}\, x \;\; \forall^{a.e.}\, y \;\; P(x,y)$$

$$\Leftrightarrow \forall^{a.e.}\, y \;\; \forall^{a.e.}\, x \;\; P(x,y)$$

(at least as long as P is "nice")

To get Thm: $\chi_{BM}(G_T) \le 3$,

follow same arg, but use Baire version of Fubini:

Thm (Kuratowski-Ulam): X, Y (2nd ctble suffices)

$A \subseteq X \times Y$ is Baire nible (i.e. Borel up to meager set)

TFAE:

① A comeager in $X \times Y$

② $\{x \in X : A_x$ (slice/fiber) is comeager in $Y\}$ is comeager in X

②' $\forall^{gen=generic}\, x \in X$, $\forall^{gen}\, y \in Y$, $(x,y) \in A$

③ $\forall^{gen}\, y \in Y$, $\forall^{gen}\, x \in X$, $(x,y) \in A$.

Above thm uses μ on X, so do we have to lose a null set?

Borel Setting

Notation: for set X, $X^{\subseteq \mathbb{N}}$, let $[X]^k := \{$subsets of X of size $k\}$

$[X]^{<\infty}$

$[X]^{\infty}$

Daniel will use $\binom{X}{k}$, $\binom{X}{<\infty}$, $\binom{X}{\infty}$

$S: \binom{\mathbb{N}}{\infty} \to \binom{\mathbb{N}}{\infty}$

$X \longmapsto X \setminus \{\min X\}$

Note $\binom{\mathbb{N}}{\infty} \cong \{x \in C : x$ has ∞ many 1's$\}$

Thm: $\chi_B(G_S) = \aleph_0$

Pf uses ∞-dim Ramsey theory

Proto Ramsey Theory: Pigeonhole principle:

$$f: \mathbb{N} \to q \quad \Rightarrow \quad \exists X \in \binom{\mathbb{N}}{\infty} \text{ s.t. } f \text{ const on } X.$$

Ramsey Theorem: Let $q, d \in \mathbb{N}$ (prove by induction on d; $d=1$ case)

$$f: \binom{\mathbb{N}}{d} \to q \quad \Rightarrow \quad \exists X \in \binom{\mathbb{N}}{\infty} \text{ s.t. } f \text{ const on } \binom{X}{d}$$

($d=2$ case, f = coloring of edges of $K_{\mathbb{N}}$ — complete graph)

What if $d=\infty$? Then as stated, not true.

(necessarily nonconstructive...)

Example: our G_S.

$f: \binom{\mathbb{N}}{\infty} \to 2$ proper 2-coloring of G_S (which exists by AoC)

then $\forall$ ∞ set $X \in \binom{\mathbb{N}}{\infty}$

then $f(X)$, $f(X \setminus \min X)$

must have diff color by def of proper coloring.

The same thing works for $f: \binom{\mathbb{N}}{\infty} \to q$.

So we will prove $\chi_B(G_S) = \aleph_0$ by proving

Thm (Galvin-Prikry): above for $d=\infty$, f Borel. I.e. fix $q \in \mathbb{N}$

Borel $f: \binom{\mathbb{N}}{\infty} \to q \quad \Rightarrow \quad \exists$ ∞ set $X \in \binom{\mathbb{N}}{\infty}$ s.t. f const on $\binom{X}{\infty}$

Exc!

Note: if $X \in \binom{\mathbb{N}}{\infty}$, then $\binom{X}{\infty} \cong \binom{\mathbb{N}}{\infty}$ (homeo.)

So it suffices to prove thm for $q=2$ (then induct on q)

By grouping colors by "blue, not blue"

W/ $q=2$, can now think of f as characteristic fn of a collection $\mathcal{U} \subseteq \binom{\mathbb{N}}{\infty}$

GP thm equivalently formulated as: every Borel set $\mathcal{U} \subseteq \binom{\mathbb{N}}{\infty}$ is Ramsey

("level 3 objects: set of ∞ sets of nat #s.")

Def: A set $\mathcal{U} \subseteq \binom{\mathbb{N}}{\infty}$ is Ramsey if $\forall A \in \binom{\mathbb{N}}{\infty}$

$\exists X \in \binom{A}{\infty}$ s.t.

either $\binom{X}{\infty} \subseteq \mathcal{U}$ or $\binom{X}{\infty} \cap \mathcal{U} = \emptyset$

We will prove by induction, in fact will prove something stronger (easier to induct)

Convention: lower case letter $a, b \ldots x, y \ldots$ for finite set

upper . . $A, B \ldots X, Y \ldots$ ∞ set

Given $a \in \binom{\mathbb{N}}{<\infty}$, $A \in \binom{\mathbb{N}}{\infty}$,

Let $[a, A] := \{X \in \binom{\mathbb{N}}{\infty} : a \subseteq X \subseteq A \cup a\}$ ("must-take", "may-take"; "interval blt a, A")

may assume $a \subseteq A$.

Ex: If $a = \emptyset$, then $[\emptyset, A] = \binom{A}{\infty}$

recover by $a = \emptyset$.

Def: a family $\mathcal{U} \subseteq \binom{\mathbb{N}}{\infty}$ is completely Ramsey if $\forall a \; A$
$\exists X \in \binom{A}{\infty}$
s.t. $[a, X] \subseteq \mathcal{U}$ or $[a, X] \cap \mathcal{U} = \emptyset$

So stronger than
Thm: Every Borel set $\mathcal{U} \subseteq \binom{\mathbb{N}}{\infty} \subseteq G$
is completely Ramsey.

We will actually prove even stronger than! (characterizing completely Ramsey!)

First Note: $[a, A] \cap [b, B] = [a \cup b, A \cap B]$
or $\emptyset$

closed under $\cap$, so is basis of topology on $\binom{\mathbb{N}}{\infty}$,

called Ellentuck topology

Obs: ① Not Polish topology. — "when I said all our topo will be Polish, I LIED!"

② But Baire cat thm still applies.

③ finer than subspace topology $\binom{\mathbb{N}}{\infty} \subseteq G$
(i.e. every fin. determined set in $\binom{\mathbb{N}}{\infty}$ is in fact open in Ellentuck topo)

⇓

every Borel set $\subseteq \binom{\mathbb{N}}{\infty}$ is Ellentuck-Borel.

Thm (Ellentuck): family $\mathcal{U} \subseteq \binom{\mathbb{N}}{\infty}$ is
completely Ramsey $\iff$ Baire msble in the Ellentuck topo.

4/25

$[\mathbb{N}]^{<\infty}$, $\binom{\mathbb{N}}{<\infty}$ (or $\binom{\omega}{<\infty}$) finite subsets denoted by lowercase letters $a, b, x, y \ldots$

$[\mathbb{N}]^{\infty}$, $\binom{\mathbb{N}}{\infty}$ ($\binom{\omega}{\infty}$) infinite subsets, $A, B, X, Y \ldots$

Recall notation

$[a, A] := \{$ must contain a, may contain elts of A (∞ sets) $\} = [a, A \cup a]$

but do not require $a \subseteq A$ for convenience of using this notation

These sets are basic for ~~topology~~ Ellentuck topology on $\binom{\omega}{\infty}$.

Defn: cplttly Ramsey family $\mathcal{U} \subseteq \binom{\omega}{\infty}$ if

$\forall a, A,$

$\exists X \subseteq A$ s.t.

$[a, X] \subseteq \mathcal{U}$ or $[a, X] \subseteq \mathcal{U}^c$

"need some amt of choice to get cplttly Ramsey"?

Thm (Ellentuck $\Rightarrow$ GP): family $\mathcal{U} \subseteq \binom{\omega}{\infty}$ cplttly Ramsey $\iff$ Baire nbhd in Ellentuck top

pf:

($\Rightarrow$) easy direction, good exc. in familiar w/ terminology

boring (we want $\Leftarrow$ direction; we want examples of cplttly Ramsey)

Suppose $\mathcal{U} \subseteq \binom{\omega}{\infty}$ is completely Ramsey

Recall Baire nbhd $\iff$ (100% lemma, hold in any topo space (easy)) symm. diff of some open is meager.

100% lemma remark

Let $\mathcal{V} := \bigcup \{[a, A] : [a, A] \subseteq \mathcal{U}\}$

- clearly $\subseteq \mathcal{U}$
- clearly open (in Ellentuck topo) Ez.

Claim: $\mathcal{U} \setminus \mathcal{V}$ is nowhere dense (hence meager) in $\binom{\omega}{\infty}$, Ez.

pf: take (fix) any basic open $[a, A]$.

Goal, find basic open $\subseteq [a, A]$ disjoint from $\mathcal{U} \setminus \mathcal{V}$

Use cplttly Ramsey $\Rightarrow$ $\exists X \subseteq A$ s.t.

Case 1: $[a, X] \subseteq \mathcal{U} \Rightarrow [a, X] \subseteq \mathcal{V}$ (since $\subseteq [a, A]$) so $[a, X] \cap (\mathcal{U} \setminus \mathcal{V}) = \emptyset$ ☺

Case 2: $[a, X] \cap \mathcal{U} = \emptyset \Rightarrow [a, X] \cap (\mathcal{U} \setminus \mathcal{V}) = \emptyset$. even easier! ☺

"cplty Ramsey directly gives witnesses for Baire nrbility,
I guess this is how Ellentuck concept w/ this characterization"

(⇐). First,

Lemma: cplty Ramsey sets form Boolean alg (closed under finite ∪, complement)

(often the fans; once u know Thm, know σ-alg, but harder to see. Boolean alg is immediate)

Pf: complements obvious

finite unions: say $\mathcal{U}_1, \mathcal{U}_2$ c.R. (completely Ramsey.)

Fix a, A.

Want $X \subseteq A$ s.t. either $[a,X] \subseteq \mathcal{U}_1 \cup \mathcal{U}_2$

or $[a,X] \cap (\mathcal{U}_1 \cup \mathcal{U}_2) = \emptyset$.

well, $\mathcal{U}_1$ is cR $\Rightarrow \exists X_1 \subseteq A$ s.t.

$[a,X_1] \subseteq \mathcal{U}_1 \subseteq \mathcal{U}_1 \cup \mathcal{U}_2$ ☺

or $[a,X_1] \cap \mathcal{U}_1 = \emptyset$.

$\mathcal{U}_2$ is cR $\Rightarrow \exists X_2 \subseteq A$ s.t.

$[a,X_2] \subseteq \mathcal{U}_2 \subseteq \mathcal{U}_1 \cup \mathcal{U}_2$ ☺

or $[a,X_2] \cap \mathcal{U}_2 = \emptyset$.

→ $[a,X] \cap (\mathcal{U}_1 \cup \mathcal{U}_2) = \emptyset$ ☺

Suffices by lemma (and previous 100% lemma remark) to show

① opensets are c.R.

② meager sets are c.R.

First reflect, why our lemma pf fail for ∞ U?

Notate: $A =^* B$ ($\subseteq \mathbb{N}$) "almost-equal" if equal modulo finite

$\Leftrightarrow A \triangle B$ finite

$A \subseteq^* B$ if $A \setminus B$ is finite.

"almost subset"

motivates

Lemma (diagonal intersections)

If $A_0 \supseteq^* A_1 \supseteq^* A_2 \supseteq^* \dots$

then $\exists A \in \binom{\omega}{\infty}$ s.t. $A \subseteq^* A_i \ \forall i \in \mathbb{N}$.

Pf: recursively define $n_0, n_1, \dots \in \mathbb{N}$

by choosing $n_i \in \underbrace{(A_0 \cap \dots \cap A_i)}_{\text{infinite}}$ different from $n_0 \dots n_{i-1}$.

set $A := \{n_i : i \in \mathbb{N}\}$.

Fix family $\mathcal{U} \subseteq \binom{\omega}{\infty}$.

Given $a \in \binom{\omega}{\infty}$, say that $A \in \binom{\omega}{\infty}$

- accepts a if $\exists A' =^* A$ s.t. $[a, A'] \subseteq \mathcal{U}$
  (may assume $A' \subseteq A$)
  (i.e. throw just fin. many elts of A away to shrink $[a,A]$ enough to land in $\mathcal{U}$)

not opposite, possible for A to neither accept nor reject a.

- rejects a if $\forall B \subseteq^* A$, $[a, B] \notin \mathcal{U}$

$\Leftrightarrow$ $\forall B \subseteq A$, B does not accept a.

---

5 observations, necessary in complexity

Obs 1: A accept / reject a, $B \subseteq A \Rightarrow B$ also accept / reject a. (Adj, clear from Acc, remove same fin. many elts from $B \subseteq A$)

Obs 2: $\forall a \in \binom{\omega}{<\infty}$
$\forall A \in \binom{\omega}{\infty}$,
$\exists$ A has a subset $B \subseteq A$ s.t.
{either B accepts / or B rejects} a (clear by in this branch can take "B" $= A$.)

"switch order of quantifiers!!!"

bootstrap 2 to Obs 3: $\forall A \in \binom{\omega}{\infty}$
$\exists D \subseteq A$ s.t.
"decisive" D {either accept / or reject} every $a \in \binom{\omega}{<\infty}$ (choice of (accept / reject may depend on a))

Pf: enum all finite subsets of $\mathbb{N}$

$a_0, a_1, \ldots$

Apply Obs 2 iteratively yields seq

$A \supseteq A_0 \supseteq A_1 \supseteq \ldots$

s.t. each A_i {either accepts / or rejects} a_i

Now, by Diag $\cap$ lemma, $\exists$ $D \subseteq^* A_i$.

May arrange so $D \subseteq A$.

By Obs1 (acc/rej preserve by subset),

$\forall i$, D { either accept or reject } a_i.

Obs4: If A reject a,

then the set

$N := \{n \in A : A \text{ accept } a \cup \{n\}\}$ must be finite

Pf: suppose not, N ∞.

Recursively define $n_i \in N$

and finite sets $f_i \in \binom{\omega}{<\infty}$ as follows:

Pick any $n_i \in N \setminus (f_0 \cup \dots \cup f_{i-1})$

different from $n_0 \dots n_{i-1}$.

As def of N, A accept $a \cup \{n_i\}$.

$\Rightarrow$ (def acc) $\exists f_i$ $[a \cup \{n_i\}, A \setminus f_i] \subseteq \mathcal{U}$

Define

$B := \{n_0 \dots\} \subseteq N \subseteq A$

Since A reject a, Obs1 says B reject a

$\Rightarrow B$ not accept a.

However, we now show $[a, B] \subseteq \mathcal{U}$ ⚡

Pf: take any $a \subseteq X \subseteq a \cup B$

Denote by $i \in \mathbb{N}$ minimum such that $n_i \in X$.

Then

$a \cup \{n_i\}$

4/28/25 **Recall**

$A =^* B$ almost equal (equal mod finite)

$A \subseteq^* B$

so that diagonal intersections nonempty.

Fix family $\mathcal{U} \subseteq \binom{\omega}{\infty}$ (we have not yet assumed it is open, last time, was just combinatorics)

Given a, A (∞ set, ∞ subset of $\mathbb{N}$)

"$\exists$" after removing / can remove finitely many elts of A.

"what we want to happen"

A accepts a $:\Leftrightarrow$ $[a, A'] \subseteq \mathcal{U}$

$\exists A' \subseteq^* A$ subset of A, fin. many removed

that's why accept is not opposite to reject

depend on $\mathcal{U}$, which we fixed

A rejects a $:\Leftrightarrow$ $\forall B \subseteq^* A$ (B would be subset of A, possibly ∞-many removed) $[a, B] \notin \mathcal{U}$

$\Leftrightarrow$ every ∞ subset $\subseteq A$ doesn't accept a

very dramatically "what we want" fails to happen "fails really badly"

Last time,

① acc/rej preserve by $\subseteq^*$

② $\forall A$ $\exists D \subseteq A$ "decisive" ($\forall a$, D acc/or rej a.)

③ If A rejects a then the set $\{n \in A : A \text{ accepts } a \cup \{n\}\}$ is finite.

Today, final pfs

④ If D decisive, then (even stronger than def decisive and ③,) in fact

$\forall a$, either D accepts a or $\exists X \subseteq D$ s.t. D rejects $a \cup x$ for all $x \in X$ (hence X reject a by ①)

Pf: If D acc a, ✓

else, by decisiveness, D must reject a.

We will construct a seq $X_0 \subset X_1 \subset X_2 \cdots$ of finite subsets of D (X by constructing)

s.t. D rejects $a \cup x$ $\forall x \in X_i$

- Start w/ $X_0 = \emptyset$
- Given X_{i-1}, (by inductive hyp, D reject $a \cup x$ $\forall x \in X_{i-1}$) By ③, $\exists$ only fin. $n \in D$ s.t. D accept $a \cup X \cup \{n\}$ (dep on X, $\forall x \in X_{i-1}$)

finitely many $x \in X_{i-1}$,

finite union of finite sets is finite

↳ can pick $n_i \in D \setminus X_{i-1}$ s.t.

D rejects $a \cup X \cup \{n_i\}$ $\forall X \subseteq X_{i-1}$

$\Rightarrow$ set $X_i := X_{i-1} \cup \{n_i\}$.

Let $X := \bigcup_{i \in \mathbb{N}} X_i$

Finally ready to prove

Lemma: Ellentuck-open $\mathcal{U} \subseteq \binom{\omega}{\infty}$ sets are completely Ramsey

Pf: for "Ramsey" (cplttly Ramsey as exe.)
(trivial, replace "$\emptyset$" by "a", but good to write it out yourself)

We fixed $\mathcal{U} \subseteq \binom{\omega}{\infty}$ open.
Let $A \in \binom{\omega}{\infty}$.

Want: $\exists X \subseteq A$ s.t. $\binom{X}{\infty} \subseteq \mathcal{U}$ or $\cap \mathcal{U} = \emptyset$

Let $D \subseteq A$ be decisive ($\exists b_2$ ②)

Then either D accepts $\emptyset$ $\Rightarrow \exists X \subseteq D$, $X =^* D$ s.t. $\binom{X}{\infty} \subseteq \mathcal{U}$ ☺

first encountered disbelieving that our statements built up.

or ④ $\exists X \subseteq D$ that rejects all $x \subset X$.

"rejects all (finite) parts of itself... a sad state of affairs"

We now claim $\binom{X}{\infty} \cap \mathcal{U} = \emptyset$.

Pf: Suppose not, i.e. $\exists Y \subseteq X$ s.t. $Y \in \mathcal{U}$. (∞ set)

$\mathcal{U}$ open, so it contains a basic open nbhd of Y.

only use of $\mathcal{U}$ open

$\exists$ finite $x \subseteq Y \subseteq X$ s.t. $[x, Y] \subseteq \mathcal{U}$.

But then by def, X doesn't reject x. ↯

QED lemma

Prop: fix $\mathcal{F} \subseteq \binom{\omega}{\infty}$ nowhere dense in E_T

Then it is cplttly Ramsey;

in fact $\forall a, A$

$\exists X \subseteq A$ s.t. $[a, X] \cap \mathcal{F} = \emptyset$.

"completely Ramsey in the bad way"

Moreover, given fixed $b \subseteq A$, may pick X so that $b \subset X$.

Pf: may replace $\mathcal{F}$ w/ $\overline{\mathcal{F}}$ (nowhere dense) (bc $\cap \overline{\mathcal{F}} = \emptyset \Rightarrow \cap \mathcal{F} = \emptyset$)

so can assume $\mathcal{F}$ closed

We proved open $\Rightarrow$ c.R., c.R. form Boolean alg., $\Rightarrow$ closed is c.R.

So $\mathcal{F}$ is c.R.

nowhere dense $\Rightarrow$ must be c.R. in the bad way

Regarding the Moreover:

apply Prop but replacing "a" w/ "a∪y" | $\forall y \subseteq b$.

which produces a set $Y_y \subseteq A$

s.t. $[a \cup y, Y_y] \cap \mathcal{F} = \emptyset$

Finitely many $y \subseteq b$

$\Rightarrow Y := \bigcap_{y \subseteq b} Y_y$ still ∞ set

and $[a \cup y, Y] \cap \mathcal{F} = \emptyset \quad \forall y \subseteq$

Now take desired "X" to be $X := Y \cup b$.

Final piece:

Prop: all meager $\mathcal{F}$ in E_2 are nowhere dense!

Pf: may write meager $\mathcal{F} = \bigcup_{i \in \mathbb{N}} \mathcal{F}_i$ (each $\mathcal{F}_i$ nowhere dense)

Fix basic open $[a, A]$.

Want to find nonØ open subset $\subseteq [a, A] \cap \mathcal{F} = \emptyset$.

Recursive pick

$A \supseteq A_0 \supseteq \cdots$

s.t. $[a, A_i] \cap \mathcal{F}_i = \emptyset$.

(should pick additionally $n_0, n_1, \ldots$ distinct integers s.t. $n_0 \ldots n_i \in A_i$ in order to guarantee ∞ set.)

Consider $X := \bigcap A_i$

$\Downarrow$

$[a, X] \cap \mathcal{F} = \emptyset$. QED

4/30/25 Friday, online

Return to coloring. We built this by theory, "surprisingly delicate" to show $\chi_B(G_T) = \aleph_0$.

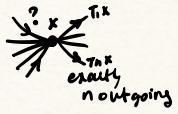
Generalizing to graphs gen'd by multiple fn'l transformations.

A lot of work in Descr. Combo focus on finitely many invtble transformations

$n \in \mathbb{N}$, $T_1 .. T_n : X \to X$ fixed pt free

again, can massage away by saying don't add edges if fixed pt.

$G_{T_1 .. T_n} := G_{T_1} \cup .. \cup G_{T_n}$



combinatorially (analogous to our $n=1$ special case)

$\chi(G_{T_1 .. T_n}) \leq 2n+1$ (exc)

Sharp for K_{2n+1}.



For X st. Borel, T_i Borel $X \to X$,

Lemma: $\chi_B(G_{T_1 .. T_n}) \leq \aleph_0$.

Pf: color each G_{T_i} $c_i : X \to \mathbb{N}$ Borel proper coloring of G_{T_i}

Then $x \mapsto (c_1(x) \dots c_n(x)) \in \underbrace{\mathbb{N}^n}_{\text{ctble}}$

is a Borel proper coloring of $G_{T_1 .. T_n}$

Also, if $\chi_B(G_{T_1 .. T_n}) < \aleph_0$, then each G_{T_i} Borel colorable by ≤ 3 (by lemma from week ago characterizing $\aleph_0$ colorings by recursively set & its complement)

and same arg above gives $\chi_B(G_{T_1 .. T_n}) \leq 3^n$. (uniform bound)

We can do better (uniformly):

Let $f : X \to q \in \mathbb{N}$ be a Borel proper coloring of $G_{T_1 .. T_n}$.

Think $n=2$:



Recall pf thought about it f goes up or down.

Greedily assign colors only for vertices focusing only on edges where f increases.

(which will be proper on subgraph — finite b/c c bdd!)

Recursively define $c(x) := \min\{k : c(T_i x) \neq k \ \ \forall i \in 1..n \text{ s.t. } f(T_i x) > f(x)\}$

($T_i x$: x's children; so that recursion is well-fdd, i.e. has base case.)

This is well defined Borel fn (can depend on vals of f at $T_{i_1}x \dots T_{i_r}x$ w/ $r \leq q$)

By construction, if $f(x) < f(T_i x)$ $\Rightarrow c(x) \neq c(T_i x)$

Now define $c' : X \to \{0, \dots n\}$ — to fix what was colored wrong by c

$c'(x) := \min\{k : c'(T_i x) \neq k \ \ \forall i \text{ s.t. } c(x) = c(T_i x)\}$

$f(x) > f(T_i x)$ ⇒ recursion well-fdd

Again, this is well-defined and Borel.

Now, the map $x \mapsto (c(x), c'(x)) \in \{0, \dots n\}^2$

is Borel proper coloring of $G_{T_1 \dots T_n}$ $\Rightarrow \chi_B(G_{T_1 \dots T_n}) \leq (n+1)^2$.

Can do even more slightly better.

Not all pairs can show up.

Note if $c(x) = k$, then are $\geq k$ indices i s.t. $f(T_i x) > f(x)$

and if $c'(x) = k$ . . . . . $f(x) > f(T_i x)$

$\Longrightarrow$ Hence $c(x) + c'(x) \leq n$.

and $\binom{n+2}{2}$ pairs $\in \{0 \dots n\}^2$ w/ sum $\leq n$.

$\Longrightarrow$ $\chi_B(G_{T_1 \dots T_n})$, if $< \infty$ is in fact $\leq \binom{n+2}{2}$.

(for large n, can -2.)

For $n=2$, lower bd $2n+1 = 5$

upper bd $\binom{2+2}{2} = 6$

Thm (Palamourdas) in UCLA thesis. 5 is correct for $n=2$.

Open for $n \geq 3$! (χ_μ, χ_{BM} also open)

Now, general theory of Borel graphs.

Def. Borel (analy, coanly, etc) (undirected) graph G — graphs generated from T (above) have natural direction, but not play role in our analysis, so treat as undirected

s.t. $V(G)$ st Borel space and set $\{(x,y) \in V(G)^2 : \{x,y\} \in E(G)\}$ is Borel subset $\subseteq V(G)^2$

To make this def abstr nicer, and for other applications,

observe that st. Borel X, $k \in \mathbb{N}$, $\binom{X}{k}$ is also naturally a std Borel space.

3 ways to see:

(I) Quot

let $(X)^k := \{\text{tuples } (x_1 \dots x_k) \in X^k : x_i \text{ all distinct}\} \underset{\text{open}}{\subseteq} X^k$

$\pi \downarrow$ $k!$ covering, sending tuple to set.

$\binom{X}{k}$

Declare Borel subsets of $\binom{X}{k}$ to be quotient,

i.e. $A \subseteq \binom{X}{k}$ Borel $\iff \pi^{-1}(A) \underset{\text{Borel}}{\subseteq} (X)^k$

Fact: this makes $\binom{X}{k}$ st Borel. (finite to 1 or ctble to 1 will be st. Borel)

How to see? (Though in general, st Borel → [1] will be st. Borel)

↑ easiest way to see it is to Prove that it's equiv to next characterization (finding metric)

(II) Metric

Fix a cpat metric on X (cpat w/ std Borel str on X)

Define

$$d^{(k)}(\{x_1 \dots x_k\}, \{y_1 \dots y_k\}) := \min_{\sigma \text{ permutations of } \{1..k\}} \max_{1 \le i \le k} d(x_i, y_{\sigma i})$$



Not complete bc 2 elts set get closer, collapse to 1 elt set.

But, exc, this extends to a complete metric to $\binom{X}{\le k}$ and $\binom{X}{k}$ forms open subset of (hence Polish)

This Polish top generates same Borel σ-alg as quotient one in (I)

($d^{(k)}$ depends on d on X, but Polish top only depends on topo of X)

(III) Borel subset

Fix a Borel lin order $<$ on X.

which exist by Borel isom, $\cong$ ctble ($\Rightarrow$ any lin order is Borel) or $\mathbb{R}$, and $\mathbb{R}$ has Borel lin order.

Then $\binom{X}{k}$ identified w/ $\{\text{tuples } (x_1 \dots x_k) \in X^k : x_1 < \dots < x_k\} \underset{\text{Borel}}{\subseteq} X^k$

(the σ-alg same as (I), indep of lin order we chose)

Can now reduce: Borel (l.f. analytic) graph G st $V(G)$ st Borel, $E(G) \underset{\text{Borel}}{\subseteq} \binom{V(G)}{2}$

Next time: structural thm for when Borel graph has ctble coloring. (recall Hyperarith graphs, not $\aleph_0$ ctble)

5/2/25 Zoom lecture introducing G_0-dichotomy.

5/5/25 Recall $S \subseteq 2^*$
set of finite bit strings

$G_S :=$ vertices C
edges of the form $\{s^\frown 0^\frown t,\ s^\frown 1^\frown t\}$ $\forall s \in S$, $t \in C$
differ in exactly 1 bit, where prefix before deviation $\in S$

} always $\subseteq H$ hypercube.

HW: S contains exactly 1 string of each length (last time discussed iff for ≤ 1, ≥ 1.) then G_S is spanning forest (maximal acyclic subgraph)

Recall
Def: S dense if $\forall t \in 2^*$, $\exists s \exists t$.

Last time, proved (very analogous to previous pf.) for hypercube
$\forall$ Baire-nsbhd G_S-indep set is meager

$\Rightarrow \chi_B(G_S) = \chi_{BM}(G_S) = 2^{\aleph_0}$ uncountable

recall we sneakily defined χ_B to be cardinality of st Borel space, which is either finite countable or $2^{\aleph_0}$.

Rmk: take any prob mstr μ on C
(Miller) $\chi_\mu(G_S) \leq 3$.

"Interesting part of these graphs is comeager but null"

Want to prove

Thm (G_0-dichotomy): if S exactly 1 string of each length $\in \mathbb{N}$ (do not need to assume dense!)

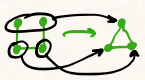
then any analytic graph G on Polish space of uncountable χ_B admits

$$G_S \xrightarrow[\text{cts hom}]{c} G$$

Bic S dense $\Rightarrow \chi_B(G_S) = 2^{\aleph_0}$,

thm says $G_S \to_c G \Leftrightarrow \chi_B(G) = 2^{\aleph_0}$

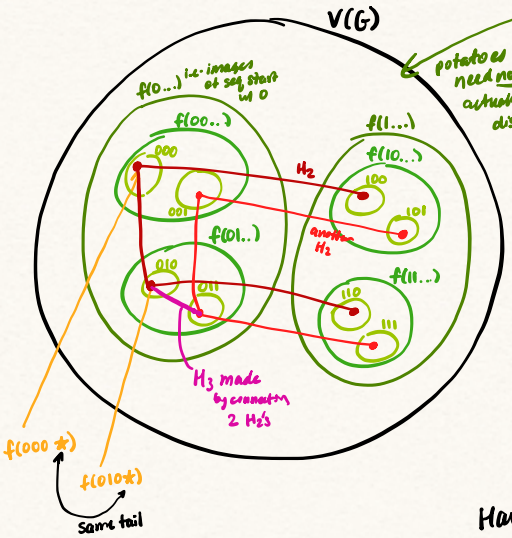
Example of graph hom



though beware, pictures for pf will be drawn as if injective, even though not necessarily the case

Cartoon of pf:

Want to construct $f: C \to X := V(G)$



"Common technique in DST, as long as potatoes nested, diam $\searrow 0$ (wrt. some cpl metric) then this constructs function"

Take eg $S = \{\emptyset, 0, 01, 101 \ldots\}$

Have finite graphs Hn
- vertices 2^n
- edges $\{s^\frown 0^\frown t,\ s^\frown 1^\frown t\}$, $s \in S$ (arbitrary s.t. total len $= n$.)

Need to find "many" copies of H_n in G

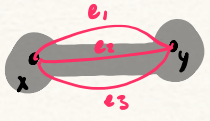
Philosophy: If found "many" H_n, then also have "many" H_{n+1}

Formalization: For analytic graph G on Polish X.

Let E be Polish space

$\pi : E \to \binom{X}{2}$, $\pi(E) = E(G)$.

Intuition: G is multigraph, pts in E are names of edges.



$e_i \in E$,

$\pi(e_i) = \{x, y\}$

"G is underlying simple graph, E dictates multigraph str on G"

formalize "copy of H in G"

Let H be finite graph.

Def: (homomorphic) copy of H in G

is map φ { $\varphi_V : V(H) \to V(G) =: X$, $u \mapsto \varphi(u)$; φ_E edge: $uv \mapsto \varphi(uv) \in E$ } s.t.

① $uv \in E(H)$

② $uv \in E(H) \Rightarrow \pi(\underbrace{\varphi(uv)}_{\in E}) = \{\varphi(u), \varphi(v)\} \in E(G) \subseteq \binom{X}{2}$



(need not be injective)

Next, need to formalize "many" copies of H.

Conceptual leap:

$\mathrm{Hom}(H, G)$ itself is Polish space:

closed subset of $X^n \times E^m$ where $n = \#V(G)$, $m = \#E(G)$

For $\mathcal{H} \subseteq \mathrm{Hom}(H, G)$ ("a family of copies"), $u \in V(H)$, $uv \in E(H)$

$\mathcal{H}(u) := \{\varphi_V(u) : \varphi \in \mathcal{H}\} \subseteq X$

$\mathcal{H}(uv) := \{\varphi_E(uv) : \varphi \in \mathcal{H}\} \subseteq E$

Note: If $\mathcal{H} \subseteq \mathrm{Hom}(H, G)$ Borel, then $\mathcal{H}(u)$ "image of Borel" is analytic:

$x \in \mathcal{H}(u) \iff \exists \varphi \in \mathcal{H} : (\underbrace{\varphi(u) = x}_{\text{a Borel computation}})$

Def: Borel $\mathcal{H} \subseteq \mathrm{Hom}(H, G)$

is **tiny** if $\exists u \in V(H)$, $\mathcal{H}(u)$ is G-indep — "very bad for us b/c our goal is to put edge b/t 2 copies of H_n to get 1 copy of H_{n+1}"

small if covered by ctbly many tiny Borel sets.

large if **not** small

Obs: $\mathcal{H} := \mathrm{Hom}(K_1^{\bullet u}, G)$ is large $\iff \chi_B(G) = 2^{\aleph_0}$

Pf: $\mathcal{H} \subseteq \mathrm{Hom}(\bullet_u, G) \cong X$

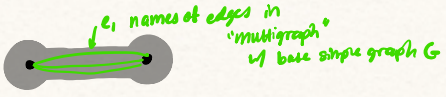
$\mathcal{H}(u)$ G-indep $\iff \mathcal{H} \subseteq X$ is G-indep.

5/7/19 recall setup

G analytic graph on Polish X

E a Polish space "the edge space"

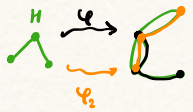
$\pi : E \to \binom{X}{2}$ a cts map s.t. $E(G) = \pi(E)$



Def: H finite graph.

a **homomorphic copy**/homomorphism of H in G is

$\varphi : V(H) \to X$, $E(H) \to E$ s.t. $\forall uv \in E(H)$ $\pi(\varphi(uv)) = \{\varphi(u), \varphi(v)\}$



Def: $\mathrm{Hom}(H, G)$ = space of all hom. copies of H in G

Polish b/c H finite, so really finite tuples.

(H finite, tell me where vtx go $\to X^n$, edges go $\to E^m$)

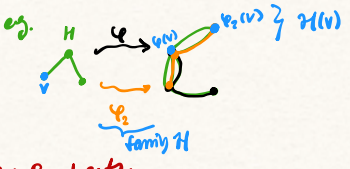
eg $\mathrm{Hom}(\bullet\!\!-\!\!\bullet, G) \subseteq X^2 \times E$.

keydet last time: want many copies

family of copies

Def: **Borel set** $\mathcal{H} \subseteq \mathrm{Hom}(H, G)$

- **tiny** if $\exists v \in V(H)$, the set of G-vertices $V(G) = X$ $\mathcal{H}(v) := \{\varphi(v) : \varphi \in \mathcal{H}\} \subseteq X$ is G-independent.

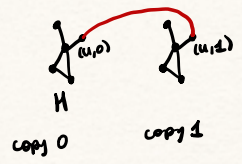


- **small** can be covered by ctbly many tiny Borel sets.

- **large** if **not small**

Key operation: given $u \in V(H)$, let $H +_u H$ be the graph w/

vtx. set $V(H) \times \{0,1\}$, edges $\{(v,i),(w,i)\}$ $\forall vw \in E(H)$, $i=0,1$



and one extra $\{(u,0),(u,1)\}$

example: $S = \{\emptyset, 0, 01 \ldots\}$

$H_\emptyset$ = •named $\emptyset$, $V(H_\emptyset) = \{\emptyset\}$.

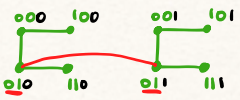
$H_1 = H_\emptyset +_\emptyset H_\emptyset$ =



$H_2 = H_1 +_0 H_1$ =



$H_3 = H_2 +_{01} H_2$ =



Note: each $\varphi \in \mathrm{Hom}(H +_u H, G)$ gives rise to a pair $\varphi^0, \varphi^1 \in \mathrm{Hom}(H, G)$

namely $\varphi^i(v) := \varphi(v,i)$ $\quad \widehat{v} \in V(H)$

$\varphi^i(vw) := \varphi(\{(v,i),(w,i)\})$ $\quad \widehat{vw} \in E(H)$

Keyfunny: H finite graph, $u \in V(H)$. fixed $\rightsquigarrow$ induced map $\mathcal{H} \to X$, $\varphi \mapsto \varphi(u)$, $\mathcal{H}(u)$ is image of projection

Suppose a Borel fam $\mathcal{H} \subseteq \mathrm{Hom}(H,G)$ is **large**!

then, $\mathcal{H} +_u \mathcal{H} := \{\varphi \in \mathrm{Hom}(H +_u H, G) : \varphi^0, \varphi^1 \in \mathcal{H}\}$ is also **large**!

Pf: Note that $\mathcal{H} +_u \mathcal{H}$ is Borel set (if $\mathcal{H}$ Borel, and checking $\varphi^0, \varphi^1 \in \mathcal{H}$ is Borel check) so question of Lebesgue measure makes sense. details are free.

First, show $\mathcal{H} +_u \mathcal{H} \neq \emptyset$.

well, $\mathcal{H}$ is **LARGE** $\Leftrightarrow$ $\mathcal{H}$ not small $\Rightarrow$ $\mathcal{H}$ not tiny

meaning in picture $\mathcal{H}(u) := \{\varphi(u) : \varphi \in \mathcal{H}\} \subseteq X$ is not G-independent

$\Downarrow$

$\exists \varphi_0, \varphi_1 \in \mathcal{H}$ s.t.

$\varphi_0(u), \varphi_1(u)$ adjacent in G

Hence get hom. $\varphi : H +_u H$ in G (copy)

(it's intuitively clear, but definition is $\varphi^0 = \varphi_0$, $\varphi^1 = \varphi_1$, and add one extra)

We want to show LARGE, not just $\neq \emptyset$.

Suppose $\mathcal{H} +_u \mathcal{H}$ is small $\Leftrightarrow \subseteq \bigcup_n \mathcal{F}_n^{\leq \text{Hom}(H+_uH, G)}$ ctbly Borel

$\Leftrightarrow \exists v_n \in V(H),\ i_n \in \{0,1\}$ (data of vtx in $H+_uH$)

s.t. $\mathcal{F}_n(v_n, i_n) \subseteq X$ is G-indep.

$\Leftrightarrow$ $\{\varphi(v_n, i_n) : \varphi \in \mathcal{F}_n\}$

$\Leftrightarrow$ $\{\varphi^{i_n}(v_n) : \varphi \in \mathcal{F}_n\}$

Goal: find ctbly Borel covering of $\mathcal{H}$ ($\lightning$)

Try $\mathcal{H}_n := \{\varphi \in \mathcal{H} : \varphi(v_n) \in G\text{indep}\}$

issue: it is in general analytic, not Borel $\lightning$.

e.g.

$H = \bullet$

$H +_u H$ — "English summary"

many copies of H.

few copies of $H +_u H$ ← remove one of the H's in each copy

↓

get still tons of copies of H, but no copies of $H +_u H$, $\lightning$ (1st half $\neq \emptyset$)

Fix: we can find Borel G-indep $I_n \supseteq \mathcal{F}_n(v_n, i_n)$ (prove later)

Then, let $\mathcal{H}_n := \{\varphi \in \mathcal{H} : \varphi(v_n) \in I_n\}$,

which is Borel and ctbly bc $\mathcal{H}_n(v_n) \subseteq I_n$

G-indep $\subseteq G\text{indep}$

Let $\mathcal{H}' := \mathcal{H} \setminus \bigcup_{n \in \mathbb{N}} \mathcal{H}_n$.

Is LARGE bc (LARGE, small)!

Hence by 1st half of thm, $\mathcal{H}' +_u \mathcal{H}' \neq \emptyset$.

But this is impossible bc

If $\varphi \in \mathcal{H}' +_u \mathcal{H}' \Rightarrow \varphi \in \mathcal{F}_n$ ($\exists n$)

$\Downarrow$

$\varphi^{i_n}(v_n) = \varphi(v_n, i_n) \in \mathcal{F}_n(v_n, i_n) \subseteq I_n$

$\Rightarrow \varphi^{i_n} \in \mathcal{H}_n$, so it's not in $\mathcal{H}'$ $\lightning$

Still need to prove:

Lemma. $A \subseteq X$ analytic, G-indep,

then $\exists$ a Borel G-indep $I \supseteq A$

Pf:



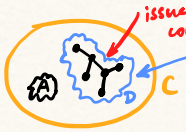
define $B := \{x \in X : x \text{ has nbr in } A\}$

analytic,

disj from A by indep

By analytic sep thm,

$\exists$ a Borel set C disj from B $\Rightarrow$ C has no nbr in A



Idea: $D := \{x \in C : x \text{ has nbr in } C\}$

D disj from A (by), so analytic sep thm again (!!!), $\exists I \supseteq A$, has no nbr in $C \supseteq I$

$\Rightarrow$ indep!

5/9/25 final descent to G0 dichotomy.

recall setup
X, E Polish $\pi: E \to \binom{X}{2}$ cts, $G = (X, \pi(E))$ analytic graph

studying (homomorphic) copies of finite graphs H to G

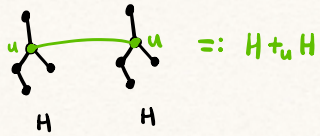
$\varphi: V(H) \to X$, $E(H) \to E$

gives rise to family of all such copies $\mathrm{Hom}(H,G) \subseteq X^{|V(H)|} \times E^{|E(H)|}$ closed b/c π cts fn.

Borel subfamilies of $\mathcal{H} \subseteq$ classified by size: tiny, small (σ-ideal), large = not small

tiny: $\exists v \in V(H)$, $\mathcal{H}(v) = \{$all images of v in X amongst copies of H from $\mathcal{H}\}$ is covered by ctbly many G-indep (no G-edges b/t any such images of v)

"combination" / "stitch"
Key operation:



$=: H +_u H$

Key Lemma: $\mathcal{H} \subseteq \mathrm{Hom}(H,G)$ Borel and LARGE
$\Rightarrow \mathcal{H} +_u \mathcal{H} = \{$copies of $H +_u H$ in G that come from stitching 2 copies from $\mathcal{H}\} = \{\varphi \in \mathrm{Hom}(H +_u H, G) : \varphi', \varphi'' \in \mathcal{H}\}$

is also LARGE

"many copies of H $\Rightarrow$ many copies of $H +_u H$"

One more new lemma, allowing us to take limit at end.

Lemma: fix cpat metrics on X, E (Polish)
If $\mathcal{H} \subseteq \mathrm{Hom}(H,G)$ is LARGE Borel fam,
then $\forall \varepsilon > 0$
$\exists$ LARGE Borel subfam $\mathcal{H}' \subseteq \mathcal{H}$ s.t. $\forall x \in V(H)$, $\forall y \in E(H)$, $\mathrm{diam}(\mathcal{H}'(x)) < \varepsilon$ (or $< \varepsilon$; diam in X / diam in E)

Pf idea: cover $\mathcal{H}$ by ctbly many $\mathcal{H}'$ w/ diam $< \varepsilon$,
LARGE $\Rightarrow$ at least one must still be LARGE.

i.e. want $\mathcal{H} \subseteq \bigcup_{n \in \mathbb{N}} \mathcal{H}_n$ Borel s.t. $\forall x \in V(H) \cup E(H)$, $\mathrm{diam}(\mathcal{H}_n(x)) < \varepsilon$

"instance of fact that 2nd ctble metric space, can cover by ctbly many $< \varepsilon$ diam sets"

(need nothing about dat of $\mathcal{H}$, pure basic topology)

Pf of G_0 dichotomy:

Fix $S \subseteq 2^*$ contain 1 str of each finite len.
ii $\{s_0, s_1, \dots\}$ $\text{len}(s_n) = n$

Assume $\chi_B(G) = 2^{\aleph_0}$.

Want cts hom $G_S \to G$

Define $H_n :=$ graph on $\{0,1\}^n$ w/ edges

$\{\underbrace{s_i \hat{} 0 \hat{} t}_{\text{len}=n}, s_i \hat{} 1 \hat{} t\}$ w/ $i<n$, $\text{len}(t) + i + 1 = n$

As we discussed, $H_{n+1} = H_n +_{s_n} H_n$

And start at our induction, $H_0 = K_1 =$ • named $\emptyset$.

Since $\chi_B(G) = 2^{\aleph_0}$, the set $\underbrace{\text{Hom}(H_0, G)}_{\| \text{ set of vertices in } G.}$ is LARGE

entire vtx set small $\Leftrightarrow$ cover by ctbly many G-indep Borels.

Recursively define
Define $\mathcal{H}_n \subseteq \text{Hom}(H_n, G)$ so that

- $\mathcal{H}_0 = \text{Hom}(H_0, G)$
- $\mathcal{H}_{n+1} \subseteq \mathcal{H}_n +_{s_n} \mathcal{H}_n$ (necessarily so)

  if $v \in V(H_n)$ (n bit str), $i = 0,1$
  $\mathcal{H}_{n+1}(v \hat{} i) \subseteq \mathcal{H}_n(v)$
  $v0, v1$ are 2 copies of v in H_{n+1}, images under hom $\subseteq$ image of v under hom.

- $\forall x \in V(H_{n+1}) \cup E(H_{n+1})$, $\text{diam}(\mathcal{H}_{n+1}(x)) < \frac{1}{n}$

and are LARGE.

Lemmas are precisely what's needed.

$$X \supseteq \mathcal{H}_0(\emptyset) \supseteq \mathcal{H}_1(x_0) \supseteq \mathcal{H}_2(x_0 x_1) \supseteq \dots$$
$$X \supseteq \overline{\mathcal{H}_0(\emptyset)} \supseteq \overline{\mathcal{H}_1(x_0)} \supseteq \overline{\mathcal{H}_2(x_0 x_1)} \supseteq \dots$$

closed, diams $\to 0$ in X
$\Rightarrow \exists!$ pt, called $f(x) \in \bigcap_{n \in \mathbb{N}} \overline{\mathcal{H}_n(x_0 \dots x_{n-1})}$

Claim f is cts $C \to X$
(any use nested closed sets very general fact)

and is also a homomorphism $G_S \to G$ — this is why we had to worry about both vtx & edges above

i.e. $\forall x_{ij} \in C$, $\{x_{ij}\} \in E(G_S)$. WTS $\{f(x), f(y)\}$ edge in G.

say $\begin{cases} x = s_n \hat{} 0 \hat{} t \\ y = s_n \hat{} 0 \hat{} t \end{cases}$.

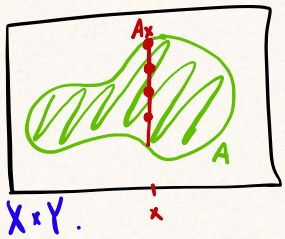
For $k \geq 0$, $f(x) \in \overline{\mathcal{H}_{n+k+1}(s_n \hat{} 0 \hat{} t_0 \dots t_k)}$ — is edge in H_{n+k+1}
$f(y) \in \overline{\dots \cdot (s_n \hat{} 1 \hat{} \cdot \cdot)}$

By diam $\to 0$ in E, $\exists! e \in \bigcap_{k \geq 0} \mathcal{H}_{n+k+1}(\ ) \in E$

Now, $\pi(e)$ must $= \{f(x), f(y)\}$ b/c π is cts. QED

Application!!!

Thm (Luzin–Novikov)



X, Y std. Borel spaces. $A \underset{\text{Borel}}{\subseteq} X \times Y$.

$\forall x \in X$, define slice $A_x := \{y \in Y : (x,y) \in A\} \subseteq Y$

Suppose A_x cble $\forall x \in X$.

Then $\exists$ Borel partial maps $f_n : X \dashrightarrow Y$ $n \in \mathbb{N}$ s.t. $\forall x \in X$,

$A_x = \{f_n(x) : n \in \mathbb{N} \text{ s.t. } x \in \text{dom}(f_n)\}$

"can ~~list~~ enumerate elements of A_x uniformly in x"

Cor: If G loc. cbl Borel graph,

then set of all isolated vertices of G is Borel

Pf: apply Luzin Novikov to $A := \{(u,v) \in V(G) \times V(G) : \{u,v\} \in E(G)\}$

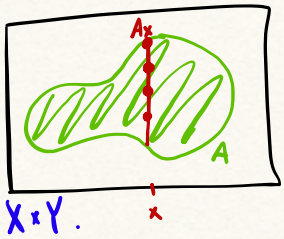
G loc cble $\iff \forall u \in V(G)$ $A_u \overset{\text{nbhood } N_u(G)}{=}$ is cbl

$\Rightarrow f_n : V(G) \dashrightarrow V(G)$ s.t. $N_G(u) = \{f_n(u) : u \in \text{dom}(f_n)\}$

Then $\{\text{iso vertices}\} = V(G) \setminus \bigcup_{n \in \mathbb{N}} N_G(u)$ is Borel.

"Luzin Novikov $\rightsquigarrow$ every cbl graph is automatically Borel"

5/12/15



Thm (Luzin–Novikov)

X, Y std. Borel spaces. $A \subseteq X \times Y$. (Borel)

$\forall x \in X$, define slice $A_x := \{y \in Y : (x,y) \in A\} \subseteq Y$

Suppose A_x ctble $\forall x \in X$.

Then $\exists$ Borel partial maps $f_n : X \to Y$ (domain Borel $\subseteq X$, Borel fn on that domain) $n \in \mathbb{N}$ s.t. $\forall x \in X$,

$$A_x = \{f_n(x) : n \in \mathbb{N} \text{ s.t. } x \in \text{dom}(f_n)\}$$

"can countably ~~list~~ enumerate elements of A_x & unif $\forall x$

ONLY reason is in case $A_x = \emptyset$ for some x.
i.e. if $\forall x$, $A_x \neq \emptyset$, may arrange so not partial, i.e. $\text{dom}(f_n) = X$

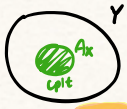
And if A_x infinite $\forall x$, then may arrange so $f_n : X \to Y$ do not repeat, i.e. $\forall x$, $f_1(x), f_2(x) \ldots$ are all distinct.

And if allow $f_n : X \to Y$ partial, can arrange for no repeats, as

Pf (due to Miller): using G_0 dichotomy.
"what even is graph?"

may fix (any) Polish topologies on X, Y.

$\forall x \in X$, let G_x be graph on Y s.t. $\{y,y'\} \in E(G_x) \Leftrightarrow \begin{cases} y \neq y' \\ y, y' \in A_x \end{cases}$

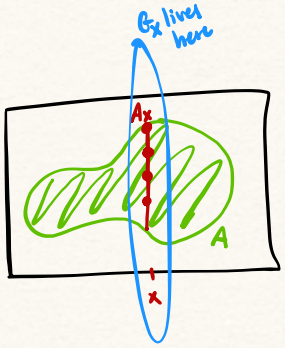
i.e. $G_x =$ 

ctble clique + tons of extra isolated vertices

Obviously, $\chi_B(G_x) \leq \aleph_0$

"disjoint union of G_x's", "place copy of G_x on slice $\{x\} \times Y$"

Now let $G :=$ vtxs $X \times Y$
edge $\{(x,y),(x',y')\} \in E(G) \Leftrightarrow \begin{cases} x = x' \\ \{y,y'\} \in E(G_x) \end{cases}$



"Lightbulb moment": $\chi_B(G) \leq \aleph_0$ also!

This is where G_0-dichotomy useful!

Suppose not f.s.o. Then, uncountable Borel chr #, so G_0-dich says

fixing (any) dense $S \subseteq 2^*$
1 str of each len $\in \mathbb{N}$,

$\exists\ G_S \xrightarrow[\text{(ctbl)}]{} G$, i.e. $\exists\ h : G \to X \times Y$
$\|$ cts hom
(h_1, h_2)

Since h is graph hom, h_1 ($\leq$ x coordinate) must be constant on all conn. comps of G_S. (only edges in G are ctble 2 same x-coords)

Std. Borel parameterized fam of ctble graphs, is

still ctble χ_B.

That's the more natural application of G_0-dich

For example, h_1 takes same value on all pts $x \in C$ w/ fin. many 1's $\subseteq$ same conn comp in H (hypercube graph)

But $\cdot^{C \to X}$ is dense, and h cts, so h_1 constant on C

constant output $x \in X$.

So h_2 homomorphs $G_S \to G_X$

G_S: uncble χ_B; G_X: ctble χ_B ↯

"nonconstructive pf of a constructive fact!"

To conclude, we now observe ctble Borel coloring $c: X \times Y \to \mathbb{N}$ is exactly what we want.

Define $f_n(x) :=$ the unique $y \in A_x$ s.t. $c(x,y) = n$

↑ (if ∃) by proper coloring

"enumerate by color/labeling c" ($n \in \mathbb{N}$)

and $\mathrm{dom}(f_n) = \mathrm{proj}_X c^{-1}(n)$

may look analytic, but recall from discussed/used earlier: 1-1 image under Borel is Borel!

can deduce from analytic separation? or topology changing techniques.

Cor: X, Y std. Borel, $f: X \to Y$ ctble-to-one

Then $f(X)$ still Borel.

Pf: Apply Luzin-Novikov to $A := \{(y,x) : Y \times X : f(x) = y\}$ graph

$A_y := f^{-1}(y)$ is ctbl $\forall y \in Y$

Borel $f_n : Y \to X$, $n \in \mathbb{N}$ s.t.

$f^{-1}(y) = A_y = \{f_n(y) : y \in \mathrm{dom}(f_n)\}$

$\cap$ X

Then $f(X) = \bigcup_{n \in \mathbb{N}} \mathrm{dom}(f_n)$ Borel!

Def: G graph. Let $\sim_G$ be equivrel. same conn. comp.

Note: G Borel $\Rightarrow$ $\sim_G$ analytic ("∃ uv-path")

$\exists k \in \mathbb{N}\ \exists u_0, u_1, \dots, u_k \in V(G)$ s.t. $u_0 = u$, $\{u_0, u_1\}, \dots, \{u_{k-1}, u_k\} \in E(G)$ (issue)

Prop: G Borel, loc. ctble $\Rightarrow$ $\sim_G$ Borel

Pf. LN-thm allows us to replace quantifier over C by quantifier over $\mathbb{N}$!

Apply LN to $A := \{(u,v) \in V(G) \times V(G) : \{u,v\} \in E(G)\}$

↓

Borel $f_n : V(G) \to V(G)$, $n \in \mathbb{N}$ s.t.

$\{u,v\} \in E(G) \iff v = f_n(u)$ for some n

Then, $u \sim_G v \iff \exists k \in \mathbb{N}\ \exists n_1 \dots n_k \in \mathbb{N}$ s.t.

$u \in \mathrm{dom}(f_{n_1})$ (i.e. $u_1 = f_{n_1}(u)$) $\quad f_{n_1}(u) \in \mathrm{dom}(f_{n_2})$ ($u_2 = f_{n_2}(f_{n_1}(u))$) $\quad \dots \quad v = f_{n_k}(f_{n_{k-1}} \dots (f_{n_1}(u)) \dots)$

clearly this pf works for tons of things

Thm: If G Loc finite Borel graph, then $\chi_B(G) \le \aleph_0$

Pf: like our G_T $\chi_B \le 3$ pf. But instead of T,

LN gives us ctbly many f_n.

~~Same as before~~

Apply LN to $A := \{(u,v) \in V(G) \times V(G) : \{u,v\} \in E(G)\}$

↓

Borel $f_n : V(G) \to V(G)$, $n \in \mathbb{N}$ s.t.

$\{u,v\} \in E(G) \iff v = f_n(u)$ for some n

Fix basis $(U_i)_{i \in \mathbb{N}}$ for cpat Polish top. on $V(G)$

Define $c(u) := \min\{i \in \mathbb{N} : u \in U_i \setminus N_G(U_i)\}$



well-defined, proper coloring easy.

LN helps us show c is Borel:

suffices to argue that $\forall i \in \mathbb{N}$, $N_G(U_i)$ is Borel.

(generally, it's analytic)

$N_G(U_i) = \{u \in V(G) : \exists v \in U_i\ (\{u,v\} \in E(G))\}$

$= \{u \in V(G) : \exists n \in \mathbb{N}\ (f_n(u) \in U_i)\} = \bigcup_{n \in \mathbb{N}} f_n^{-1}(U_i)$

Lusin disjoint

strong inj ∀ any used to take limit at end.

wed Zoom.

5/16/25 Feldman Moore:

→ Injective G_0 dichotomy

5/16/25

Lec 21

Next Section of course:

Borel Graphs of Bdd Degree

(we covered loc. ctbl, briefly loc. finite)

Very useful fact ("surprising if you haven't seen it")

Prop: G loc. ctbl Borel graph s.t. $\chi_B(G) \leq \aleph_0$ (e.g. if G is loc. finite)

then G has Borel maximal (wrt inclusion) independent set. (no edges)

i.e. can not add any vtx, i.e. every vtx $\notin$ set has a nbr in I.

("Borel Zorn's lemma")

Pf: $\leq \aleph_0$ coloring

$\Rightarrow \exists$ partition of vtx set into ctbly many Borel indep sets.

$$V(G) = I_0 \sqcup I_1 \sqcup I_2 \sqcup I_3 \dots$$

Try adding in greedy fashion:

define recursively on n: $I_n' := \{x \in I_n : x \text{ has no nbrs in } I_0' \dots I_{n-1}'\}$ (Borel, coanalytic)

$\Rightarrow I_0' := I_0$

i.e. $I_n' = I_n \setminus \underbrace{N_G}_{\text{nbrs}}(I_0' \cup \dots \cup I_{n-1}')$

note we haven't used locally ctble yet... but recall Luzin Novikov reduces to quantify over $\mathbb{N}$

Note I_n' is indeed Borel b/c "in general, loc. ctbl Borel, $N_G(A)$ is Borel $\forall$ Borel $A \subseteq V(G)$"

Take $I := \bigcup_{n \in \mathbb{N}} I_n'$.

indep b/c edge 2 vtxs can't come from same "slice" ($\subseteq I_n$ indep)

. . . . diff. slices b/c

same paper/talk

Thm (KST '99): if G is Borel graph of max deg $d \in \mathbb{N}$

then $\chi_B(G) \leq d+1$. (obvious if χ instd of χ_B) by greedy.

Pf (framed as application of prev. prop's)

induct on d.

$d=0$, ✓ no edges!

Let $I \subseteq V(G)$ be a Borel max^l indep set in G

Then induced subgraph $G - I$ has max degree $\leq d-1$ (b/c every vtx has nbr in I)

By induction, $\chi_B(G-I) \leq d$ Borel, can color by d colors.

Use 1 extra color for vtxs in I

Is this sharp?

Yes, if G has component $\cong \underbrace{K_{d+1}}_{\text{finite graph}}$, then $\chi_B(G) = \chi(G) = d+1$.

or $d=2$, G has component $\cong$ odd cycle, then $\chi_B(G) = \chi(G) = 3 = d+1$.

Boring... not about Borel.

Graph Theory (Brooks '47): If G graph of max deg $d \in \mathbb{N}$, then $\chi(G) \le d$ unless G has comp $\cong K_{d+1}$ ("are only extremal cases") or $d=2$ and G has comp $\cong$ odd cycles.

Example (from 1st lec!): For real # $\alpha \in \mathbb{R}$, let $T_\alpha : S^1 \to S^1$ be rotation by α. (S or T? "1dim sphere or 1dim torus")

Claim: $\frac{\alpha}{\pi} \notin \mathbb{Q}$, G_{T_α} has max deg 2 (in fact 2 regular), has no cycles. but $\chi_B, \chi_\mu, \chi_{BM}(G_{T_\alpha}) = 3$ (i.e. "χ" Brooks)

Leb. / Msrbl. usually top

pf: let's show $\chi_\mu(G_{T_\alpha}) > 2$. (recall $\chi_B \ge \chi_\mu$) (BM exercise)

Suppose not; then partition $S^1 = A_0 \sqcup A_1$ Leb msrable indep sets.

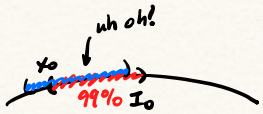
$T_\alpha(A_0) = A_1 \Rightarrow \mu(A_0) = \mu(A_1) = \frac{1}{2}$.

By "99% lemma" (slightly tweaked): $\forall \varepsilon > 0$ (must could just fix any $\varepsilon < \frac{1}{2}$.) can find open interval $I_0^\varepsilon \subseteq S^1$ s.t. A_0 fills $(1-\varepsilon)$% (proportion) of I_0^ε: $\frac{\mu(A_0 \cap I_0^\varepsilon)}{\mu(I_0^\varepsilon)} > 1-\varepsilon$.

Then, for $I_1^\varepsilon := T_\alpha(I_0^\varepsilon)$, have $\frac{\mu(A_1 \cap I_1)}{\mu(I_1)} > 1-\varepsilon$ ($= \mu(I_0)$)

Fact (Dirichlet): If $\frac{\beta}{\pi} \in \mathbb{Q}$, then $\forall x \in S^1$, $\{T_\beta^n(x) : n \in \mathbb{N}\}$ is dense in S^1.

So let x_i be "left" endpt of I_i. Given any $\delta > 0$ (much smaller than len of I_0), $\exists n \in \mathbb{N} : T_{2\alpha}^n(x_1) = T_\alpha^{2n}(x_1)$ is at distance $< \delta$ from x_0 ("mult. is commutative")





"Coloring graph w/ 2 colors is hard, color one, immediately color entire connected comp"

"large scale interactions"

Brief Survey

Q: what's maximal $\chi_B, \chi_\mu, \ldots$ for Borel forest of max deg d? ("choose as aggressively as possible" / "worst possible msr for worst-possible graph" ($\Leftrightarrow$ fix e.g. coinflip msr on G, choose worst graph on G") (definitely nonzero Brooks extrema)

Boils down to atomless. all atomless are iso to Leb.

G_{T_α}							
d	0	1	2	3	4	5	d
χ	1	2	2	2	2	2	2
χ_B	1	2	3	4	5	6	$d+1$ ← Thm of Marks!
χ_μ	1	2	3	3	3 or 4 (?)	3/4/5 (?)	$\frac{d}{\log_2(1+o(1))}$ (??)
χ_{BM}	1	2	3	3	3	3	3 !!

ooo! Measurable Brooks

5/19/15 Next lec on zoom (Wed)
+1 (Fri) asynchronous

Small discussion on hw.

Ely: KW3p1 $f: X \to Y$ μ-mble

(X, μ), Y (Std Borel)

Def 1: $\exists$ Borel g s.t. $f \underset{\mu}{=} g$

Def 2: preimages of Borel are μ-mble sets $\subseteq X$.

$2 \Rightarrow 1$ approx by simple functions. (from Grad Anly days)

Msr Iso Thm: any 2 atomless standard msr spaces are iso: Borel bijection $f: X \to Y$ $\mu(A) = \nu(f[A])$ $\forall$ Borel $A \subseteq X$

(X, μ) ← prob msr, ↑ Std Borel

More carefully:

R: $f_n: X \to Y$ Borel

$f_n(x) \to f(x)$ $\forall x \in X$

is f Borel?

in fact, set on which $\lim$ $\exists$ is Borel

Recall subtlest to check graph is Borel!

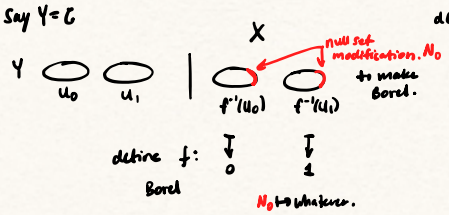
i.e. given x, y, check $f(x) = y \Leftrightarrow \lim_n f_n(x) = y$

which is a ctble computation.

$\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ $\forall n \geq N$

$d(f_n(x), y) < \varepsilon$

Say $Y = C$



define f: 0 on $f^{-1}(U_0)$, 1 on $f^{-1}(U_1)$, Borel

N_0 to whatever.

U_{00} U_{01} U_{10} U_{11}

ponder preimages of ctble basis of Y

(don't need to normalize $Y = C$)

"even simpler to write down"

Returning to Borel graphs of bdd degree

Overview of general facts. Let G be Borel of max degree $d \in \mathbb{N}$

We've proved Thm (KST): $\chi_B(G) \leq d+1$.

Thm (Marks, 2016): $\forall d \in \mathbb{N}$, $\exists$ Borel forest G of max deg d w/ $\chi_B(G) = d+1$

"requires cleverness b/c not true for χ_μ, χ_{BM}"

Thm (Conley-Marks-Tucker-Drob '16): (Measurable Brooks) for $d \geq 3$, Brooks thm holds mably / Baire-mably

i.e. if G has no K_{d+1}, then $\chi_\mu(G), \chi_{BM}(G) \leq d$

(μ: any Borel prob msr on $V(G)$; BM: w/ any cpat Polish top)

+ structural analysis of the case when $\chi_B(G) = d+1$

Thm (Anton Bernshteyn '20): if $d \geq 10^{100}$, if G has no clique of size d (K_d), then $\chi_\mu(G), \chi_{BM}(G) \leq d-1$

(this specializes to thm about χ, $K_d \not\subseteq G \Rightarrow$ proven by Reed '91: $\chi(G) \leq d-1$ "pretty nontrivial graph theory")

Thm: if $a \leq \sqrt{d + \frac{1}{4}} - \frac{5}{2}$,

and $d \geq 10^{100}$, then $\chi_\mu(G) \geq d-a \Leftrightarrow \chi_{BM}(G) \geq d-a \Leftrightarrow \chi(G) \geq d-a$

This fails for $a \geq \sqrt{d - \frac{3}{4}} - \frac{1}{2}$. gap ≤ 2.

Molloy-Reed '14 (ordinary graph theory)

Bernshteyn-Esperet '19 (computer science) algo.

Thm (Conley–Miller 2016): G Loc. finite Borel graph where $\chi(G) < \infty$
then $\chi_{BM}(G) \le 2\chi(G) - 1$
(which is best possible by Wellacker '22)

↓ started whole active area
"large scale geometry of Borel graphs"
"viewing graph as metric space and understanding the geometry"
need
prob. paper in Annals of Prob.

Thm (Kolman–Virág '17): $\forall d \in \mathbb{N}$, $\exists$ Borel forest G of max deg d (in fact, Marks' Borel forests)
bipartite
and prob msr μ on $V(G)$ s.t.
every μ-msble G-indep set I has $\mu(I) \le (1+o(1))\frac{\log d}{d}$ $\Rightarrow$ $\chi_\mu(G) \ge (1-o(1))\frac{d}{\log d}$

Huge prob in prob graph theory
Buildings on:
finite graphs:
- Bollobás, McKay 80's
- Frieze–Luczak '92
- Lyons–Nazarov '11 realized connection to χ_μ → 2
- Gamarnik–Sudan '14 better const → $1+\frac{1}{12}$

Thm (AB '23): if G has no triangles & no 4-cycles, then $\chi_\mu(G) \le (1+o(1))\frac{d}{\log d}$

Builds on Kim '95 prove $\chi(G) \le \frac{d}{\log d}$ graph theory
Chung Pettie Su '17 CS paper algos

Thm: G triangle free, $\chi_\mu(G) \le (4+o(1))\frac{d}{\log d}$
Open: reduce 4 to 1
(and 1 has matching lower bd)

(Molloy '18 $\chi \le (1+o(1))\frac{d}{\log d}$
now, 3 qualitatively different pfs, none work for χ_μ)

I asked about seems like often $\chi_{BM} \le \chi_\mu$.
AB gives some thms in this direction
acyclic vs. grids.

5/11/23
Lec 23

structure of $\chi_B = d+1$

$F: X \to X$
F-descending seq = reverse/backward orbit
(borbit? Loc)

F one ended := no infinite F-desc. seq
"can only go forward indefinitely"
⇓
G_F is a forest. (no cycles!)

Def: G graph, $A \subseteq V(G)$
A-one-ended subforest is one-ended $F: A^c \to V(G)$ s.t. all $\{x, F(x)\} \in E(G)$ (i.e. $G_F \subseteq G$)
!!!
i.e. all trees in G_F go ∞ upward or hit A and stop.

Prev prop said A-1-ended subfor ☺
Now we show when can get

Prop: G Loc. fin. Borel graph.
$A \subseteq V(G)$ Borel
Suppose A meets every conn. comp. of G.
Then G has a Borel A-one-ended subforest.

Cor: G Borel, max deg $d \in \mathbb{N}$ at least
if ∀ component of G has $\exists$ vtx of deg $< d$
then $\chi_B(G) \le d$. Wow!

Pf: def $A := \{x \in V(G) : \deg x < d\}$. Borel (ctbl computation)
Take Borel max indep set $I \subseteq A$.
Then I meets every conn comp of G (also)
prop ⇒ $\exists$ Borel 1-one-ended subforest.
Prev Prop ⇒ $\chi_B(G - I) \le d$. $I \subseteq A$ has $< d$ nbrs. ⇒ can color $\forall x \in I$

All of stuff to right is to set up:
$\forall x \in A^c \cap W_n$,
$c(x) := \min\{k : c(y) \neq k \ \forall \text{ nbrs } y \text{ of } x \text{ in } W_0 \cup \dots \cup W_{n-1}\}$
(recursively defined on n)

Prop: G Borel graph, max deg $d \in \mathbb{N}$
$A \subseteq V(G)$ Borel
Suppose $\exists$ Borel A-one-ended subforest F of G.
Then $\chi_B(G - A) \le d$
($A = \emptyset$ → ☺)

Pf: have $F: A^c \to V(G)$ that is A-one-ended
Idea: A-one-ended tells us order in which to do greedy coloring
Define $\text{rank}_F(x) :=$ max length of F-desc seq starting at x.
(argument: König's lemma) (" borbit)
BIG Idea #1 (König's lemma) is legit, rank_F always finite.
Why? Suppose not. "must run out" ∞
$\infty = \text{rank}_F(x) = 1 + \sup\{\text{rank}_F(y) : F(y) = x\}$
max ble G is Loc. Finite!!!
$\Rightarrow \exists y$ s.t. $\text{rank}_F(y) = \infty$.
Iterate to get $x \leftarrow y \leftarrow y_2 \leftarrow \dots$ ∞ seq! ※ 1-endedness = finite borbits

Note: $\text{rank}_F(F(x)) > \text{rank}_F(x)$ $\forall x$.

Define $V_n := \{x : \text{rank}_F(x) = n\}$, is Borel (by Lusin–Novikov)

(key property) $\forall \text{vtx } x \in V_n \cap A^c$ has $\le d-1$ nbrs in $V_0 \cup \dots \cup V_n$ ($F(x)$ is nbr in a higher rank)

Refine to: $\exists$ partition $V(G) = \bigsqcup_n W_n$ Borel G-indep sets s.t. $\forall \text{vtx in } W_n \cap A^c$ has $\le d-1$ nbrs in $W_0 \cup \dots \cup W_{n-1}$ ($\cup W_n$)

Why? $\chi_B(G) \le d+1 \Rightarrow V(G) = I_0 \sqcup \dots \sqcup I_d$ Borel G-indeps.
Partition each V_n into $d+1$ indep sets
($W_{(d+1)n+r} := V_n \cap I_r$ $\forall n \in \mathbb{N}$, $0 \le r \le d$.

Rot Prop: A in red.



"Run away from A if you can"

$\forall x \in V(G)$,

define $dist(x,A) :=$ min len path in G from x to vtx in A.

by def "A meet all conn comp of G", $dist(x,A) < \infty \ \forall x \in V(G)$.

Say path $x_0 \cdots \in G$ is **A-repulsive** if $dist(x_{n+1},A) > dist(x_n,A) \ \forall n$

$R := \{x \in A^c : \exists$ an ∞ A-repulsive path starting from $x\}$

= x from which can run away from A.

which is Borel (seems like "$\exists \infty$ path" is not usable)

by König trick ↕ $\exists$ arb long finite paths!

(G loc. finite)

Indeed: $\forall L \in \mathbb{N}$,

$R_L := \{x \in A^c : \exists$ len L A-repulsive path starting at $x\}$

Borel by Luzin Novikov.

Clearly, $R \subseteq \bigcap_L R_L$

But in fact $\supseteq$ b/c G loc. finite (Idea: if $x \in \bigcap_L R_L$, then x has nbr y s.t. $dist(y,A) > dist(x,A)$ and $y \in \bigcap_L R_L$.)

Anton's final scheme

- 48 hrs in advance, learn which 5 problems selected for you
- Anton picks 3/5 to ask you to present

≤7 min per problem

present the main heart of the problem

identify &

5/28/25 previous lectures, started 3 BIG Ideas so far:

BIG Idea 1: König lemma argument

BIG Idea 2:

BIG Idea 3: AoC for Borel component-finite graphs ← (in general, Borel combinatorics "trivial" for or reduce to std combo.)

Stage: G Borel graph

d max degree

Want: Borel d-coloring of graph G.

Plan: ① find (suitable...) Borel set $A \subseteq V(G)$ s.t. A meets every comp of G. and the induced subgraph $G[A]$ is comp-finite

(next lecture deal w/ when plan fails)

finite sized islands hitting all components.

A

② find Borel d-coloring of $G-A$.

③ extend the coloring to A (seems hard... but CS ppl thought about this a lot.)

b/c these colors not available!

but, # colors available $\geq \deg_{G[A]}$!

All G are loc. finite.

proved by 2 camps separated by Iron Curtain Soviet

Def (Vizing / Erdős-Rubin-Taylor, 70's) (to formalize)

List assignment for G is

fn $L: V(G) \to [\mathbb{N}]^{<\infty}$

$v \mapsto$ "available colors" for v

L-coloring is $c: V(G) \to \mathbb{N}$, $v \mapsto c(v) \in L(v)$ $\forall v$.

proper L-coloring if it's proper (adj vertices get different colors) $c(u) \neq c(v)$ $\forall uv \in E(G)$

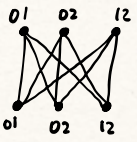
L-colorable if $\exists$ a

3-colorable means $L: V(G) \to \binom{\mathbb{N}}{<\infty}$, $v \mapsto \{0,1,2\}$

Intuitively, changing list for different v makes problem easier (can make each list disjoint!)

but surprisingly, can also make it harder:

bipartite but for this example of L, not L-colorable!



Def: G is **degree-list-colorable** if G is L-colorable $\forall$ list assignments L s.t. $|L(v)| \geq \deg_G(v)$ $\forall v \in V(G)$

"off the shelf graph theory results"

Thm (Borodin / Erdős-Rubin-Taylor) A finite conn. graph G is deg-list-colorable unless G is a **Gallai tree**: every block is a clique or odd cycle

actually the thm is true

example



K_2 included

Note: – could be ∞.

– all trees are Gallai trees bc

no proof provided

also cpctness principle:

∞ conn. loc finite graph G is Gallai tree $\iff$ all its finite induced subgraphs are Gallai tree

Thm: G Borel graph, d max deg

If no comp of G is a Gallai tree, then $\chi_B(G) \leq d$.

pf: every comp has finite subset S s.t. $G[S]$ is conn & not Gallai tree. e.g. (could intersect)



non Gallai-ness witnessed

BIG Idea 4: take max'l disjoint subfamilies!

Prop: Let $\mathcal{H} \subseteq \binom{X}{<\infty}$ Borel be family of finite sets s.t. every set $A \in \mathcal{H}$ intersects $\leq$ ctbly many other (finite) sets.

Then, $\mathcal{H}$ has Borel maximal disjoint subfamily

pf: $G_{\mathcal{H}}$ = intersection graph of $\mathcal{H}$

by assumption, locally ctbl

"Generalized Feldman-Moore (GJKL)" $\Rightarrow \chi_B(G_{\mathcal{H}}) \leq \aleph_0$

$\Rightarrow$ learned from prev lecture $G_{\mathcal{H}}$ has Borel max'l indep set $\mathcal{H}' \subseteq \mathcal{H}$

containing pf of Thm: apply prop to $\mathcal{H} := \{ S \in [V(G)]^{<\infty} : G[S]$ conn not Gallai tree $\}$ (vertices are finite sets; S finite, so ctbl computation, just compare to finite list of Gallai trees on that # of vertices)

$\Rightarrow$ Borel max'l disjoint $\mathcal{H}' \subseteq \mathcal{H}$.

ISSUE: still could be edge.

Fix define Γ w/ $V(\Gamma) = \mathcal{H}'$

$E(\Gamma) := \{ (S,S') \in [\mathcal{H}']^2 :$ $\exists$ edge in G from S to $S' \}$

Γ loc finite blc:



potatoes disjoin (by def of $\mathcal{H}'$)

G max deg $\leq d$

$\Longrightarrow \chi_B(\Gamma) \leq \aleph_0 \Rightarrow$ have Borel maxl indep set $\mathcal{H}'' \subseteq \mathcal{H}'$

So now, picture looks like



$$A := \bigcup \mathcal{H}'' = \{v \in V(G) : \exists S \in \mathcal{H}'' \text{ s.t. } S \ni v\}$$

Borel blc if such S $\exists$, it is unique, so Lusin thm.

(or, use Lusin Novikov)

Finally, $G[A]$ is comp. finite,
meets every comp of G
no comp is Gallai tree

5/30/25 recall Thm:

G Borel, max deg $d \in \mathbb{N}$

No component of G is Gallai tree

$\Rightarrow \chi_B(G) \leq d$

Pf: we found

$\mathcal{H}'' \subseteq \binom{V(G)}{<\infty}$ s.t.

① $\forall S \in \mathcal{H}''$ $G[S]$ conn, not a Gallai tree

easy

② every comp. of G contains some $S \in \mathcal{H}''$

that's why we got all the way to $\mathcal{H}''$ "down here"

→ ③ the (finite) sets in $\mathcal{H}''$ pairwise disjoint and non adj in G

Defined $A := \{v \in V(G) : \exists S \in \mathcal{H}'' : v \in S\}$

$G' := G[A]$

By constr, comps of G' are exactly the sets $S \in \mathcal{H}''$

$\Rightarrow$ G' component-finite (comps are finite)

and none of its comps are Gallai trees

A meets every comp of G

⬇ (Borel A-one-ended subforest)

$\chi_B(G-A) \leq d$, so $\exists\ c : A^c \to A$ be Borel proper c-coloring.

deg-list colorable:

$L(v) := \{0 \leq i < d : c(w) \neq i \ \ \forall w \in A^c \text{ adjacent to } v\}$

Note: $|L(v)| \geq \deg_{G'}(v)$

Since G' components are not Gallai trees,

$\exists$ proper L-coloring ("set theoretic")

Since G' is component finite, it has a Borel proper L-coloring

recall BIG Idea, "Borelness disappears"

Now $c \cup c'$ is Borel proper d-coloring of G

QED

What to do w/ Gallai trees?

(d-regular) since $\mathbb{Z}^d$

the most difficult part

χ_μ gives us freedom to look at Borel A-one-ended subforest

A $\top$ null

Avoid working w/ deg 2 ∞ paths

(since know from irrational rotation graph bad)

also want to avoid these (bc only one road out)

Def: forest G **ample** if

① No vertices of deg ≤ 1 (i.e. mindeg ≥ 2)

② No ∞ path of deg 2 vertices

Thm: Let G be an ample (crucial for delicate induction) Borel forest of finite max deg
μ prob measure $V(G)$

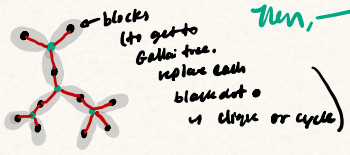
Then $\exists$ μ-null Borel $A \subseteq V(G)$ and Borel "A one ended" A-subforest

(intuitively, ample forest = subdivision of forest of mindeg ≥ 3)

Cor: G Borel Gallai forest of fin. max deg

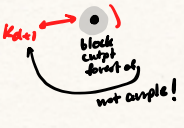
Suppose block-cutpt forest of G is ample.

Then,



Pf of Cor: Exc.

Rmk: If G is a d-regular Gallai forest, $d \geq 3$, then its block-cutpt forest is ample



→ **Hence**

Thm: G Borel graph
max deg $d \geq 3$

Then $\forall$ prob meas μ on $V(G)$,

$\exists$ μ-null Borel set $A \subseteq V(G)$ s.t. $\chi_B(G-A) \leq d$

Pf:

↪ may assume components are d-regular Gallai trees

split G into 3 parts ...

G loc.ctble, $\chi \leq k$
TFAE
① $\forall \mu$, $\chi_\mu \leq k$
② $\forall \mu$, χ_B of μ-full set is $\leq k$

Pf: **Lemma:** G loc fin. Borel

Suppose $V(G) = A_0 \supseteq A_1 \supseteq \cdots$ seq of Borel sets s.t.

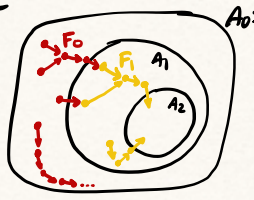
$\forall n$, A_{n+1} intersects any comp. of $G[A_n]$.

Then, G has a Borel A_∞-subforest

$\bigcap_{n \in \mathbb{N}} A_n$ (could be empty! Determ might not meet all comps of G!)

Pf: $F_n : A_n \setminus A_{n+1} \to A_n$ be Borel A_{n+1}-subst. of $G[A_n]$

$A_0 := V(G)$



Then, $F := \bigcup_{n \in \mathbb{N}} F_n$ is A_∞-subforest.

Pf of Thm: due to Lemma:

G ample, Borel, first meaning $\leq d$ — STS $\exists$ Borel $A \subseteq V(G)$ s.t.

① A meets every comp of G — "set large"

② $\mu(A) \leq C(d) < 1$ — "set small"

} actually easy, trivial parts

③ $G[A]$ ample → enables induction

(Will subtract blc inductively iterate $\rightsquigarrow$ $V(G) = A_0 \supseteq A_1 \supseteq \cdots$

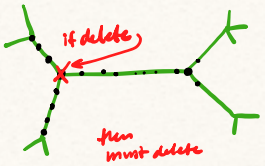
① A_{n+1} meets $\forall$ comp. of $G[A_n]$

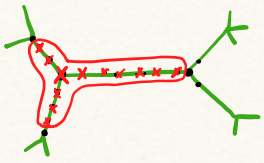
② $\mu(A_n) \leq C^n \xrightarrow{n\to\infty} 0$

③ $G[A_n]$ ample!

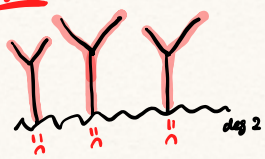
Lemma → A_∞ ← subfirst!

an ample forest = subdiv of forest min deg ≥ 3.





Danger:



Final exam review 6/9/25 6221 3-5pm

6/2/25 in process of constructing A ← subforest, null

In order for induction, we made definition: ample

① min degree ≥ 2

② no ∞ paths of deg 2 vertices $\iff$ G is subdivision of a forest of min deg 3

($\Rightarrow$ no finite components)

last case in our journey towards settling subtle Brooks thm

Thm: G Borel ample forest, max degree $d \in \mathbb{N}$, μ prob meas on $V(G)$

Then $\exists$ Borel μ-null $A \subseteq V(G)$ and $\exists$ Borel $A \subseteq$ s.t. of G.

(recall we reduced non Baire to ...?)

Pf: Last time saw (blc done by iterating) it STS $\exists$ a Borel set $A \subseteq V(G)$ s.t.

① A meets every comp of G (there was way we know how to get $A \subseteq$ s.t.)

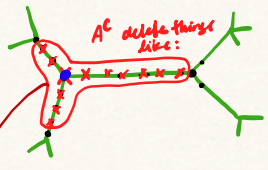
② $\mu(A) \leq C = C(d) < 1$

③ $G[A]$ still ample

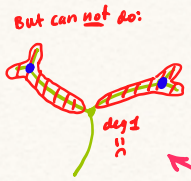
Pictures: instead of thinking what A should look like, think about what A^c should look like.

→ want ① A^c not contain all of 1 comp of G

② meas > 0



But can not do:



and should also not do



in order to not do

Idea: don't delete guys "too close to each other" like in ↔ formalize as independent set in auxiliary graph

Formalize their Y shape around • y

$V_Y \subseteq X := \{v \in V(G) : \deg_G(v) \geq 3\}$

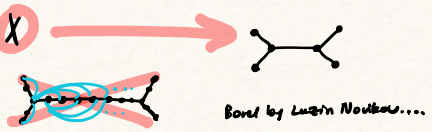
Define: $\hat{Y} := Y \cup \{u \in X^c : \exists$ path in G going from u to Y avoiding $X \setminus Y\}$

(X^c: deg 2 vertices; $X \setminus Y$: deg 3 vertices not already in Y)

Our vision is to make $A^c := \hat{Y}$ for some $Y \subseteq X$.

"add more edges to sort of undo subdivision of many ≥ 3 forest"

Let H be graph w/ vtx set X, edge set



Borel by Lusin Novikov....

H^3, has edge $\{u,v\}$ $\Leftrightarrow$ $0 < \text{dist}_H(u,v) \leq 3$ $\Rightarrow$ new connect



make sure Y is H^3-indep

(this) ensures that $G[A]$ simple and A meets every comp of G

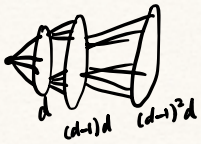
↳ if component



contained entirely in $A^c = Y^c$ then Y is not H^3-indep.

Remains to show $\mu(\hat{Y}) > 0$ depend on d quantitatively

Note: $\deg H \leq d$



$d + (d-1)d + (d-1)^2 d < d^3$

$\Rightarrow$ max deg of $H^3 < d^3$

$\Rightarrow \chi_B(H^3) \leq d^3$

$\Rightarrow$ can cover X by d^3 Borel H^3-indep sets $Y_1 \dots Y_{d^3}$

Then, $\hat{Y}_1 \dots \hat{Y}_{d^3}$ cover $V(G)$.

But then pigeonhole $\Rightarrow \exists i$ s.t. $\mu(\hat{Y}_i) \geq \frac{1}{d^3}$



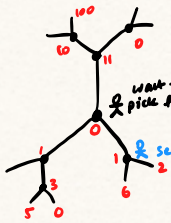
↺ BIG ideas

Thm (Marks): $\forall d \in \mathbb{N}$ ($d=1,2$ done) so $d \geq 3$

$\exists$ Borel d-regular forest G s.t. $\chi_B(G) \geq d+1$.

(i.e. no way to color w/ $\leq d$ colors)

Construction:



algo: label you see $\mapsto$ color 1..d s.t. proper

2 labels same after 1 shift $\mapsto$ different colors.

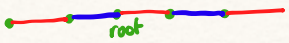
we show this also must fail.

Makes construct G s.t. every coloring of it corresponds to

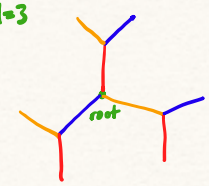
$\chi_B(G)$ can not be $\leq d$.

Let T_d be
rooted d-regular tree
whose edges are properly colored w/ colors $1..d$

e.g. $d=2$



$d=3$



Observe: $\forall$ vtx $v \in T_d$,
$\exists!$ color preserving automorphism of T_d
sending root $\mapsto v$

We work in the space
$Col(T_d, \mathbb{N})$ of all proper $\mathbb{N}$-colorings of T_d

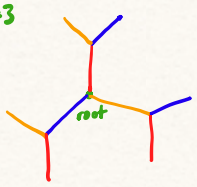
---

6/4/25 Thm (Marks) $\forall d \in \mathbb{N}$
$\exists$ acyclic Borel graph G, max deg $\leq d$
w/ $\chi_B(G) = d+1$.

e.g. $d=2$



$d=3$



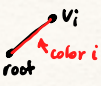
Pf: Def T_d rooted d-regular tree w/ edges properly colored using $\{1 \dots d\}$ colors

Rmk: up to isomorphism, only 1 coloring possible?

(also T_d = Cayley graph of $(\mathbb{Z}/2\mathbb{Z})^{*d}$ ← free product
colors = the generators)

Observe: $\forall$ vtx $v \in T_d$,
$\exists!$ color preserving automorphism $F_v : T_d \to T_d$
root $\mapsto v$

Let v_i be nbr of root connected via edge of color i



$F_i := F_{v_i}$ "shift" root to v_i
i.e. move along edge of color i.

Note: $F_i^2 = id$ involution

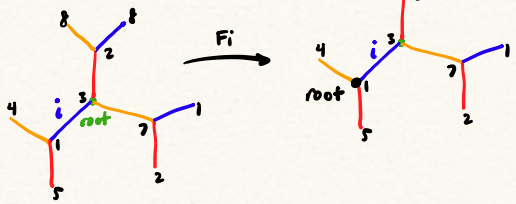
Def $Col(T_d, N)$ ($\{1..d\}$? also works, but $\mathbb{N}$ easier to write)
be space (Polish) of all proper N-colorings $x : T_d \to N$
(closed subspace of N^N)

Of course, can "lift" automorphisms to act on colorings, by attaching colors to vertices
setting permuted

$F_i[x] \in Col(T_d, N) \quad \forall x \in Col(T_d, N)$

$F_i[x] := x \circ F_i$

e.g.



$G := G_{F_1 \dots F_d}$, i.e.
$V(G) = Col(T_d, N)$
$E(G) = \{\{x, F_i[x]\}$: $x \in Col(T_d, N)$, $i \in \{1..d\}\}$
diff bc color of root diff by preserves

the adjacent vertices in graph

Fact: G has max deg $\leq d$.

But... G not forest bc could $\exists$ pattern in labelling
(s.t. walking around, see same pattern again.)
Intuition: for most/generic X, don't have this issue

Fix: $G' :=$ subgraph of G induced by set of vtxs x s.t. the comp. of x in G is tree
Fact: $V(G')$ is meagre in $Col(T_d, N)$ so $G' \neq \emptyset$.
Borel by Lusin–Novikov

$\Rightarrow$ G' Borel forest max deg $\leq d$.

---

we show $\chi_B(G') > d$.

Actually, we show $\forall$ Borel $f: V(G') \to \{1..d\}$,
$\exists\ x \in V(G')$; $i \in \{1..d\}$ s.t.
$f(x) = f(F_i(x)) = i$
"f not proper" — the twist "edge bt them prevents color of endpts"

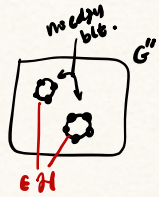
Actually can forget about G' bc enough to prove for G in place of G'

Pf sketch "uses previous lec's tricks, good review"

Def: function f failing to have above prop
a ~~coo~~ coloring.
"no ii" "no iii" "no iii"
WTS: G has no coot Borel colony $\Rightarrow$ G' has no coal Borel colony

Change perspective: Write $G = G' \sqcup G''$ where every comp of G'' has cycle.

Claim G'' does have a Borel "no iii" coloring.
Why? "Similar to Gallai tree arguments"



① find Borel maximal fam $\mathcal{H}$ of disjoint nonadj cycles in G''

② $A := \{v \in V(G'') : \exists C \in \mathcal{H} : v \in C\}$
i.e. vertices in cycles in $\mathcal{H}$.

$\Rightarrow$ A is Borel set
meeting $\forall$ comp of G'',
and comps of $G''[A]$ are
exactly cycles $\in \mathcal{H}$.

③ find Borel proper coloring of complement $G'' - A$ using colorings $\{1..d\}$

④ $\forall$ vtx $x \in A$,
has ≥ 2 available color $\Rightarrow$ $\exists$ extension of to yield a no iii coloring of G''
combinatorics
(perhaps nonproper tho e.g. if C odd length and only 2 colors available)

⑤ Bc $G''[A]$ is component-finite, can be Borel extension.

Return to our G: $V(G) = Col(T_d, N)$
$E(G) := \{\{x, F_i(x)\}\}$

Given Borel $f: V(G) \to \{1..d\}$,
want to find $x \in Col(T_d, N)$, $i \in \{1..d\}$ s.t. $f(x) = f(F_i(x)) = i$

we know mble. BM approach fail.

Our last new idea: **Infinite Games** (ω many rounds)

Framework:

"payoff set" $A \subseteq \mathbb{N}^{\mathbb{N}}$ ← in fact any other set works

| e.g. Go
- finite set
- moves you aren't allowed
- game ends

} can still fit in our framework by adding ∞ dummy moves and restricting A.

can define game $\Gamma(A)$ defined as

Alice (1)	n_0	n_2	...
Bob (2)	n_1	n_3	

Alice wins $\Leftrightarrow (n_0 \dots) \in A$

Def: strategy for Alice is alg (/oracle/function)

σ: (position Alice may find herself facing) $\to \mathbb{N}$

i.e. $\mathbb{N}^* \to \mathbb{N}$

A play of game $(n_0 \dots)$ is consistent w/ σ means $n_0 = \sigma(\phi)$, $n_2 = \sigma(n_0, n_1)$, ⋮ ("Alice played according to strategy σ")

and σ is winning strat (for Alice) means ∀ play $(n_0 \dots)$ consistent w σ ends up $\in A$.

analogous def for Bob

Observe: obviously impossible that a game $\Gamma(A)$ $\exists\, \sigma_A, \sigma_B, \dots$ both winning strat for Alice

Punchline: **Thm (Martin)** (, UCLA!): If $A \subseteq \mathbb{N}^{\mathbb{N}}$ is Borel (aka "nice" subset), then either Alice or Bob (exactly one of the other) has a winning strat. (i.e. $\Gamma(A)$ is determined)

(Martin first proved analytic determinacy assuming Large Cardinals)

reduce to closed set

but closed set ... ∎

6/6/25 Finish Mark's proof using ∞ games

Recall graph we've been working w/.

T_d: v_d —d— root —1, 2, 3— v_1, v_2, ... (1 = color of edge)

$F_i : T_d \to T_d$ unique (edge) color preserving automorphism sending root $\mapsto v_i$

which lifts to fn on labelings of VRTs:

$x : T_d \to \mathbb{N} \rightsquigarrow F_i[x] := x \circ F_i$

i.e. $F_i[x]$ is move the "root" label to v_i

G $V(G) = Col(T_d, \mathbb{N}) := \{$proper $\mathbb{N}$-colorings x of $T_d\} \subseteq \mathbb{N}^{T_d} \cong \mathbb{N}^{\mathbb{N}}$

$E(G) = \{x, F_i[x]\}$ $\forall x$, $\forall i$ (only reason do this is to guarantee no loops) (simple reason)

"proper colorings vs arbitrary/general $\mathbb{N}$-labelings

Now, fix Borel fn $f: V(G) \to \{1..d\}$
$\quad\quad\quad

↓

"Bob's job is hard (has to = i)
but he can win!"

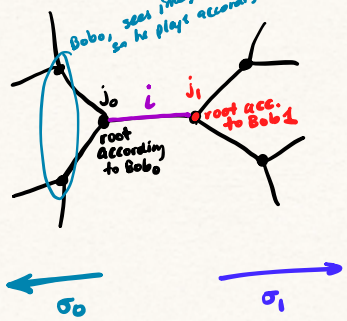
Next idea: pigeonhole (that's why can replace N by $\{1..d+1\}$)

$\exists j_0 \neq j_1$ and same $i \in \{1..d\}$ s.t.

Bob wins both Γ_{i,j_0} and Γ_{i,j_1}

Final idea: 2 Bobs play against each other.

Bob$_0$ sees imaginary Alice play j_1 "across the board", so he plays according to his winning strat σ_0.



Then Bob$_1$ will see imag. Alice 1 "across the board" play j_0, ...

So, both Bobs win!

But then σ_0 winning for Bob$_0$ $\Rightarrow f(x) = i$

σ_1 . . . . . $\Rightarrow f(F_i(x)) = i$

QED

Try to gen to not trees.

What else is out there??

* matching theory (Banach Tarski)

Thm (Hall): G loc. fin. bipartite graph
if $\forall$ finite $S \subseteq V(G)$, it $|N_G(S)| \geq |S|$
$\Rightarrow$ G has perfect matching.

As stated, fail w/ μ, BM. But

Thm (Marks & Unger): G Borel loc fin. graph on Polish space

s.t. $\exists \varepsilon > 0$:
$\forall S \subseteq V(G)$ finite,
$|N_G(S)| \geq (1+\varepsilon)|S|$

then G has BM perfect matching (μ fails)

$\Rightarrow$ BM version of Banach Tarski

* Application in ergodic theory

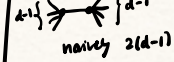
* Applications / connections w/ CS.
(distributed computing)

* connections to metric geometry / geom. group theory
often allow improving BM/μ to Borel

Leave w/ open questions ("I had many to choose from, but I'll give a particularly interesting one")

normal graph theory

Thm (Vizing 64): G has max deg $\leq d \in \mathbb{N}$,
then $\chi'(G) \leq d+1$
index = coloring of edges



naively $2(d-1)+1 = 2d-1$

Borel? $\chi'_B(G) \leq 2d-1$. ✓ Optimal (Marks again, by Gamer)

UCLA 2023
Thm (Grebik / Grebik-Pikhurko): $\chi'_\mu(G) \leq d+1$

BM? **Thm (Bean-Weilacher):** $\chi'_{BM}(G) \leq d+2$. Anton: 😠!

"one of few natural problems that BM harder than μ"

6/9/25 OH

G_{fs}: If $A \subseteq G$ Borel, μ-null (fair coin flip)

$\Rightarrow [A]_{G_{fs}} := \{x : x \text{ has path to } A \text{ in } G_{fs}\}$ (saturation)

is also null.

— useful for thinking our Brooks ? noble

μ "quasi inv" for G_{fs}.

Generally, G Borel, loc ctbl, μ prob'mr on $V(G)$

Claim: $\exists \tilde{\mu}$ G-quasi-inv s.t.

A is $\tilde{\mu}$ null $\Rightarrow A$ is μ-null.