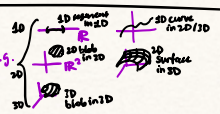
5/19/25

General situation: ω (eats oriented 1D/2D/3D segment) = a # (omega)

ρ (eats unoriented 1D/2D/3D seg.) = a # (rho)

for S is 1D/2D/3D space, e.g.: 1D, 2D, 3D



$\int_S \omega :=$ { cut S into small pieces; apply ω to those small pieces; add those #s all up; take limit as size of small pieces → 0. }

(or $\int_S \rho$)

lowercase f, scalar-valued function = for every input point, give a #.

uppercase F, vector-valued function = for every input point, give a vector.
↳ can visualize as vector field: at every input pt, draw small copy of output vector at that pt.

Definition of "1D/2D/3D segment enters"

Conceptual/Notation

Geom/Physical interpretation

Calculational

$\int_{\text{1D seg. in 1D}} \overset{1}{f} = \int_{1D} f \overset{1}{\rho}$ = { cut up 1D space into many small 1D pieces; apply $\overset{1}{f}$, apply $\overset{1}{\rho}$; multiply & sum; take limit }

For $f: \frac{1}{3} \to 1$, have Geometric interpretation of "area under function f"

in 1D $f: \mathbb{R} \to \mathbb{R}$,

$\int_{[a,b]} f \overset{1}{\rho} = \int_a^b f\,dx$ ✔ standard/familiar 1dim integral

$\overset{1}{\rho}(\overrightarrow{\ }) := \text{len}(\overrightarrow{\ })$ eats unoriented line seg.

Scalar valued $f: \frac{1}{3} \to 1$, i.e. $\#$ maps 1D, 2D, or 3D pts to a #.

can define $\overset{1}{f}_{\rho} = f \cdot \overset{1}{\rho}$, i.e. $\overset{1}{f}_{\rho}(\overrightarrow{\ }) = f(\circ) \cdot \text{len}(\overrightarrow{\ })$

$\int_{\text{1D curve in 2D/3D}} \overset{1}{f}_{\rho} = \int_C f\,ds$ arclength integral.

Physically: For $f: \frac{1}{3} \to 1$, can think of f as "density", then integral = total mass. If $f=1$, = total arclength.

in 2D,3D, $f: \mathbb{R} \to \mathbb{R}^{2,3}$ (weighted) arclength integral.

parameterize



$\int_C \overset{1}{f}_{\rho} = \int_C f\,ds = \int_{t_{init}}^{t_{final}} f(r(t))\,\|r'(t)\|\,dt$

$\overset{2}{\rho}(\square) := \text{area}(\square)$ eats unoriented 2D segment

Scalar valued $f: \frac{2}{3} \to 1$; $\overset{2}{f}_{\rho} = f \cdot \overset{2}{\rho}$

$\int_{\text{2D region in 2D}} \overset{2}{f}_{\rho} = \iint_{2D} f\,dA$ = { cut up 2D space into many small 2D pieces; apply $\overset{2}{f}_{\rho}$, apply $\overset{2}{\rho}$ by area; multiply & sum; take limit } "area integral"

Geometric

$f: 2 \to 1$, volume under function (over 2D region R in 2D)

$f: \frac{2}{3} \to 1$ "density", integral = total mass.

If $f=1$, then surface area of S

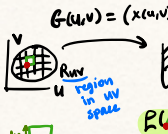
don't worry, the "dS" is just pure notation to tell you it's a surface integral. No deeper meaning.

$\int_{\text{2D surface in 3D}} \overset{2}{f}_{\rho} = \iint_S f\,dS$ surface integral

in 2D, $f: \mathbb{R}^2 \to \mathbb{R}$ $\int_R f \overset{2}{\rho} =$ ~~def~~ integral of f on R. $= \iint_R f\,dx\,dy = \iint_R f\,dA$ (week 3)

in 3D, must parameterize (any param works)

$G(u,v) = (x(u,v), y, z)$



$\int_S \overset{2}{f}_{\rho} = \iint_{R_{uv}} f(G(u,v)) \cdot \|N\|\,du\,dv$

$F(\square) = \text{area}($ parallelogram made by $\partial_u G(\bullet)\,du$, $\partial_v G(\bullet)\,dv$ $)$

$= \text{area}($ parallelogram made by $\partial_u G(\bullet)$, $\partial_v G(\bullet)$ $)$

$\partial_u G(\bullet) \in \mathbb{R}^3$, $\partial_v G(\bullet)$

$= \|T_u \times T_v\| = \|N\|$

the x,y,z wiggled as I wiggle u in positive u dir.

not a focus of this class. SZB

Though, you are actually secretly quite familiar w/ it.

for $f: 1 \to 1$,

$\int_{[a,b]} \overset{1}{\omega}_f = \int_a^b f'(x)\,dx$ ← 1D seg. in 1D

Scalar valued $f: \frac{1}{3} \to 1$ i.e. # maps 1D, 2D, or 3D pts to a #.

$\overset{1}{\omega}_f(\overrightarrow{\ }) := f(\text{head}) - f(\text{tail})$ eats oriented 1D segment (could be denoted $\overset{1,0}{\omega}_f$)

$\int_{\text{1D}} \overset{1}{\omega}_f = \lim \sum_{\text{pieces of}} \overset{1}{\omega}_f(\overrightarrow{\ }) = f(b) - f(a)$ only boundary contributions remain b/c all interior things telescoped/cancelled away!

$= \lim \sum_{\text{pieces of}} \frac{\overset{1}{\omega}_f(\overrightarrow{\ })}{\text{len}(\overrightarrow{\ })} \cdot \text{len}(\overrightarrow{\ })$ → $\frac{d}{dx} f(a)$ "dx"

→ FTC: $f(b) - f(a) = \int_{1D} f'\,dx$ (for $f: 1 \to 1$) ← see my week 6 recordings

calculationally always param!

pure notation



$\int_C \vec{F} \cdot d\vec{r} = \int_{t_{start}}^{t_{end}} \vec{F}(r(t)) \cdot r'(t)\,dt$

Vector valued $F: \frac{2}{3} \to \frac{3}{3}$

$\overset{1}{\omega}_F(\overrightarrow{\ }) := F(\circ) \cdot \overrightarrow{\ }$ (also $dx_{\rho}(\overrightarrow{\ }) := f(\circ)\,dx(\overrightarrow{\ })$ scalar x vol of proj.)

$\int_{\text{1D curve}} \overset{1}{\omega}_F = \int_C \vec{F} \cdot d\vec{r}$ line integral

minor geom interp when $\int_C \vec{F} \cdot d\vec{r} = \int_C f_1\,dx + \int_C f_2\,dy$, $\vec{F} = (f_1, f_2)$ = area of projection to xy plane.

Physics: work done by a force field on particle moving along curve C.

exterior derivative

$\overset{2,1}{\omega}_F(\square) :=$ line integral of $\vec{F}$ around square = $\sum$ of $\overset{1}{\omega}_F(\overrightarrow{\ })$ for 4 1D seg. on $\partial\square$

(running from a 1-form)

eats oriented 2D segment

$\int_{\text{2D}} \overset{2,1}{\omega}_F = \lim \sum_{\text{pieces of}} \overset{2,1}{\omega}_F(\square) = \int_{\partial \square} \overset{1}{\omega}_F$ only boundary contributions remain b/c all interior things telescoped/cancelled away!

$= \lim \sum_{\text{pieces of}} \frac{\overset{2,1}{\omega}_F(\square)}{\text{area}(\square)} \cdot \text{area}(\square)$ → curl(F)(•) "dA"

FTC (Green): $\int_C \vec{F} \cdot d\vec{r} = \iint_R \text{curl}(F)\,dA$ (line int)

$\overset{2}{\omega}_F(\square) := \vec{F}(\bullet) \cdot$ "cross product" $\perp$ to oriented plane seg, right hand rule, magnitude = area.

Normal Vector ($\perp$ to $\square$)

eats oriented plane seg (bivector)

"Flux"

$= \vec{n} \cdot \text{area}(\square)$ ($\vec{n}$: unit direction)

(b/c $d\vec{S} = \vec{n}\,dS$) =

$\iint_S \vec{F} \cdot \vec{n}\,dS$ scalar valued!!!

Surface integral!

$\int_{\text{2D surface}} \overset{2}{\omega}_F = \iint_S \vec{F} \cdot d\vec{S}$ flux integral

no geom interp.

Physical: how much fluid flowing through the membrane S

$\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$ describes fluid motion.

Param

$\iint_S \vec{F} \cdot d\vec{S} = \iint_S F(G(u,v)) \cdot \left(\partial_u G(u,v) \times \partial_v G(u,v)\right) du\,dv$

(3D vector dot product; pt in 3D; vector in 3D; vector in 3D; Scalar!)

Tip: Type checking!

exterior derivative

$\overset{3,2}{\omega}_F(\square) = \sum_{\text{6 faces of } \square} \overset{2}{\omega}_F(\square)$ (with $\partial\square$ oriented w/ outward orientation)

eats oriented 3D segment

$\int_{\text{3D}} \overset{3,2}{\omega}_F = \lim \sum_{\text{pieces of}} \overset{3,2}{\omega}_F(\square) = \int_{\partial S} \overset{2}{\omega}_F$ only boundary contributions remain b/c all interior things telescoped/cancelled away!

$= \lim \sum_{\text{pieces of}} \frac{\overset{3,2}{\omega}_F(\square)}{\text{vol}(\square)} \cdot \text{vol}(\square)$ → div(F)(•) "dV"

FTC (Stokes): $\iiint_W \text{div}(F)\,dV = \iint_S \vec{F} \cdot d\vec{S}$

volume integral ($\frac{3}{\rho}(\square) = \text{vol}(\square)$) ; flux integral