335 Final Problems

Daniel Rui - 3/18/20

Problem 1

We are given that $\int_a^\infty |f(x)|\,dx < \infty$, and we want to prove that

$$\lim_{s\to\pm\infty} \int_a^\infty f(x)e^{isx}\,dx = 0$$

First, I will point out that this integral is absolutely convergent because $|f(x)e^{isx}| = |f(x)|$, so it is conditionally convergent; now we just have to prove that is converges to 0. Now it is tempting to think that this is just an application of the Riemann-Lebesgue lemma; however, the R-L lemma requires us to be on a compact set $[a, b]$. Thus, we break the integral into two pieces and consider each piece separately:

$$\begin{aligned}\left|\int_a^\infty f(x)e^{isx}\,dx\right| &\le \left|\int_a^b f(x)e^{isx}\,dx\right| + \left|\int_b^\infty f(x)e^{isx}\,dx\right| \\ &\le \left|\int_a^b f(x)e^{isx}\,dx\right| + \int_b^\infty |f(x)|\,dx\end{aligned}$$

Let's look at the second term first. Fix any $\epsilon > 0$. Because $\int_a^\infty |f(x)|\,dx < \infty$, we know there must be some large enough b s.t. $\int_b^\infty |f(x)|\,dx < \frac{\epsilon}{2}$. We have now fixed a b. Looking at the first term, we can use the R-L lemma, which tells us that

$$\lim_{s\to\pm\infty} \int_a^b f(x)e^{isx}\,dx = 0$$

Thus, for s s.t. $|s| > S$ for some large enough S, $\left|\int_a^b f(x)e^{isx}\,dx\right| < \frac{\epsilon}{2}$. Thus putting our two bounds together, we have that for all s satisfying $|s| > S$,

$$\left|\int_a^\infty f(x)e^{isx}\,dx\right| \le \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

which is exactly what we wanted to prove.

Problem 2

Fix $r \in [0, 1)$ and consider the series

$$1 + 2re^{ix} + 3r^2e^{i2x} + \ldots = \sum_{n=0}^{\infty}(n+1)r^n e^{inx}$$

(a) This series is absolutely and uniformly convergent by the Weierstrass M-test; $|(n+1)r^n e^{inx}| = (n+1)r^n =: M_n$ for all $x \in \mathbb{R}$, and $\sum_0^\infty (n+1)r^n$ converges by the ratio test for any r we are

considering:

$$\frac{(n+2)r^{n+1}}{(n+1)r^n} = \frac{(n+2)r}{(n+1)} \to r < 1$$

Thus, this series pointwise and uniformly converges to some function $f(x)$ for all $x \in \mathbb{R}$. Furthermore, because $(n+1)r^n e^{inx}$ is a continuous function of x on $[-\pi, \pi]$ and $\sum_{n=0}^{\infty}(n+1)r^n e^{inx}$ converges pointwise and uniformly on $[-\pi, \pi]$, we can integrate term by term, and so the coefficients of our series must be exactly the complex Fourier coefficients of f (see middle of page 358 for Folland's explanation). In other words, this series is the Fourier series of f.

(b) We have already proved that the series converges uniformly on $\mathbb{R}$ (as a function of x with a fixed $r \in [0, 1)$) above; now we just need to find the limit.

Consider the series $\sum_{n=0}^{\infty} r^{n+1}e^{inx} = \sum_{n=1}^{\infty} r^n e^{i(n-1)x}$ (we are thinking of it as a power series of r, not x). This power series in r as coefficients $e^{i(n-1)x}$, and its radius of convergence is $R = 1$ because for $|r| < 1$ the series converges absolutely by the ratio test (for all x):

$$\frac{|r^{n+1}e^{inx}|}{|r^n e^{i(n-1)x}|} = |r| < 1$$

It diverges for $|r| > 1$ because the terms do not go to 0. Furthermore, the series is exactly $\sum_{n=0}^{\infty} r^{n+1}e^{inx} = r\sum_{n=0}^{\infty}(re^{ix})^n = \frac{r}{1-re^{ix}}$ by geometric series because $|re^{ix}| = |r| < 1$. Thus by Theorem 7.20, the derivative of $\frac{r}{1-re^{ix}}$ is equal to the term by term derivative of the series, i.e. $\sum_{n=0}^{\infty}(n+1)r^n e^{inx}$ for all $r \in (-1, 1)$. Again recall that we have no restrictions on x anywhere, so this is also for all $x \in \mathbb{R}$. Thus, we have that

$$\frac{1 - re^{ix} - r(-e^{ix})}{(1-re^{ix})^2} = \frac{1}{(1-re^{ix})^2} = \sum_{n=0}^{\infty}(n+1)r^n e^{inx}$$

for $r \in [0, 1)$ and $x \in \mathbb{R}$.

Problem 3

We are given that f is a 2π-periodic, piecewise smooth function. The definition of piecewise smooth (middle of page 336) is that on any bounded interval, f is C^1 except at finitely many points, at which the one sided limits $f(x+), f(x-), f'(x+)$, and $f'(x-)$ all exist and are finite. Let $\hat{f}(n)$ be the complex Fourier coefficients of f. We want to show that there is a constant M (dependent on f) s.t. $|\hat{f}(n)| < \frac{M}{|n|}$ for all $n \neq 0$.

Consider the bounded interval $[-\pi, \pi]$. By piecewise smooth, there are $N+1$ points $-\pi = x_0 < x_1 < \ldots < x_N = \pi$ where f is discontinuous. However, we know that f is C^1 on every (x_k, x_{k+1}). Thus,

we can integrate by parts on this interval to get that

$$\begin{aligned}
\left|\int_{x_{k-1}}^{x_k} f(x)e^{-inx}\,dx\right| &= \left|\left(\frac{1}{-in}f(x)e^{-inx}\right)\Big|_{x_{i-k}}^{x_k} - \left(-\frac{1}{in}\right)\int_{x_{k-1}}^{x_k} f'(x)e^{-inx}\,dx\right| \\
&= \frac{1}{|-in|}\left|\left(f(x)e^{-inx}\right)\Big|_{x_{i-k}}^{x_k} - \int_{x_{k-1}}^{x_k} f'(x)e^{-inx}\,dx\right| \\
&\leq \frac{1}{|n|}\left(\left|\left(f(x)e^{-inx}\right)\Big|_{x_{i-k}}^{x_k}\right| + \left|\int_{x_{k-1}}^{x_k} f'(x)e^{-inx}\,dx\right|\right) \\
&= \frac{1}{|n|}\left(\left|f(x_k)e^{-inx_k} - f(x_{k-1})e^{-inx_{k-1}}\right| + \left|\int_{x_{k-1}}^{x_k} f'(x)e^{-inx}\,dx\right|\right) \\
&\leq \frac{1}{|n|}\left(\left|f(x_k)e^{-inx_k}\right| + \left|f(x_{k-1})e^{-inx_{k-1}}\right| + \int_{x_{k-1}}^{x_k} |f'(x)e^{-inx}|\,dx\right) \\
&\leq \frac{1}{|n|}\left(\left|f(x_k)\right| + \left|f(x_{k-1})\right| + \int_{x_{k-1}}^{x_k} |f'(x)|\,dx\right)
\end{aligned}$$

But we know that both f and f' are piecewise continuous and hence are bounded (take the maximum bound over the finitely many continuous pieces, recalling at even the discontinuous places, the one sided limits $f(x+), f(x-), f'(x+)$, and $f'(x-)$ all exist and are **finite**). Let $|f| < M_1$ on $[-\pi, \pi]$ and let $|f'| < M_2$ on $[-\pi, \pi]$. Then, the above bound is

$$\left|\int_{x_{k-1}}^{x_k} f(x)e^{-inx}\,dx\right| \leq \frac{2M_1 + M_2(x_k - x_{k-1})}{|n|}$$

Thus, we have

$$\begin{aligned}
|\hat{f}(n)| &= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx\right| = \frac{1}{2\pi}\left|\sum_{k=0}^{N}\int_{x_{k-1}}^{x_k} f(x)e^{-inx}\,dx\right| \\
&\leq \frac{1}{2\pi}\sum_{k=0}^{N}\left|\int_{x_{k-1}}^{x_k} f(x)e^{-inx}\,dx\right| \leq \frac{1}{2\pi}\sum_{k=0}^{N}\left(\frac{2M_1 + M_2(x_k - x_{k-1})}{|n|}\right) \\
&= \frac{1}{2\pi|n|}\left(2M_1(N+1) + M_2\sum_{k=0}^{N}(x_k - x_{k-1}))\right) = \frac{1}{2\pi|n|}\left(2M_1(N+1) + M_2(x_N - x_0)\right) \\
&= \frac{1}{2\pi|n|}\left(2M_1(N+1) + M_2(\pi - (-\pi))\right) = \frac{2M_1(N+1) + 2\pi M_2}{2\pi|n|}
\end{aligned}$$

and so $|\hat{f}(n)| \leq \frac{M}{|n|}$ for

$$M = \frac{2M_1(N+1) + 2\pi M_2}{2\pi} = \frac{2\sup\limits_{x\in[-\pi,\pi]}\{f(x)\}(N+1) + 2\pi\sup\limits_{x\in[-\pi,\pi]}\{f'(x)\}}{2\pi}$$

Problem 4

We will use a theorem proved on the website. I will copy it down here because I think it is important and I would like to have a written record of it:

Theorem: let f be 2π-periodic and suppose

$$\frac{f(x_0+t)-f(x_0)}{t} = g(t) \in R[-\pi,\pi]$$

As always, $S_N f(x_0) = \sum_{-N}^{N} c_n e^{inx_0}$, and $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx$. Then $\lim_{N\to\infty} S_N f(x_0) = f(x_0)$.

Proof: $f(x_0+t)-f(x_0) = tg(t) \in R[-\pi,\pi] \implies f(x_0+t) \in R[-\pi,\pi]$ because $f(x_0)$ is a constant, and this implies that $f(t) \in R[-\pi,\pi]$ because f is 2π-periodic so shifting left or right doesn't change it's Riemann integrability. c_n is defined because $f \in R[-\pi,\pi]$. Let $D_n(t) = \frac{1}{2\pi}\sum_{-N}^{N} e^{int} = \frac{1}{2\pi}(\frac{e^{-iNt}-e^{i(N+1)t}}{1-e^{it}})$. Then, because we can interchange integrals and finite sums, we have

$$S_N f(x_0) = \int_{-\pi}^{\pi} f(x)\left(\frac{1}{2\pi}\sum_{-N}^{N} e^{in(x_0-t)}\right)dt = \int_{-\pi}^{\pi} f(x)D_N(x_0-t)\,dt = \int_{-\pi}^{\pi} f(x_0+t)D_N(t)\,dx$$

where again the bounds don't need to change because f is 2π-periodic so shifting left or right doesn't change the integral. Furthermore, because from Lemma 8.15 the integral $\int_{-\pi}^{\pi} D_N(t)\,dt = 1$, we have that

$$\begin{aligned}(S_N f(x_0)) - f(x_0) &= \int_{-\pi}^{\pi} f(x_0+t)D_N(t)\,dt - f(x_0)\int_{-\pi}^{\pi} D_N(t)\,dt \\ &= \int_{-\pi}^{\pi} (f(x_0+t)-f(x_0))D_N(t)\,dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f(x_0+t)-f(x_0)}{t}\cdot\frac{t}{(1-e^{it})}\cdot[e^{-iNt}-e^{i(N+1)t}]\,dt\end{aligned}$$

But we know that

$$\frac{f(x_0+t)-f(x_0)}{t}\cdot\frac{t}{1-e^{it}} = g(t)\cdot\frac{t}{1-e^{it}} \in R[-\pi,\pi],$$

so

$$(S_N f(x_0)) - f(x_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f(x_0+t)-f(x_0)}{t}\cdot\frac{t}{(1-e^{it})}\cdot[e^{-iNt}-e^{i(N+1)t}] \to 0$$

by the Riemann-Lebesgue lemma.

NOW, we apply it to the problem. We have that $h = f - g$ is zero on a neighborhood around some point x_0 (then the neighborhood contains $(x_0-\delta, x_0+\delta)$ for some $\delta > 0$). Thus, for t with $0 < |t| < \delta$, $H(t) = \frac{h(x_0+t)-h(x_0)}{t} = 0$, so we can set the value at $t = 0$ to be 0, and so H is 0 on $t \in (-\delta, \delta)$. Outside this interval (but still on $t \in [-\pi,\pi]$), $H(t)$ is the difference and quotient (where the denominator is not 0) of Riemann integrable functions, so it is in $R[-\pi,\pi]$. Furthermore note that the Fourier coefficients of h are $\hat{h}(n) = \hat{f}(n) - \hat{g}(n)$ by the linearity of multiplication and integration. The above

theorem then gives that

$$\sum_{-\infty}^{\infty} \hat{h}(n)e^{inx_0} = \sum_{-\infty}^{\infty} (\hat{f}(n) - \hat{g}(n))e^{inx_0} = h(x_0) = 0$$

If we assume WLOG that $\sum_{-\infty}^{\infty} \hat{g}(n)e^{inx_0}$ is convergent to say some constant α, then because both $\sum_{-\infty}^{\infty} \hat{g}(n)e^{inx_0}$ and $\sum_{-\infty}^{\infty} \hat{h}(n)e^{inx_0}$ converge, we can add them term by term to get a new series that converges to the sum of what those two series converged to. Thus,

$$0 + \alpha = \sum_{-\infty}^{\infty} \hat{h}(n)e^{inx_0} + \sum_{-\infty}^{\infty} \hat{g}(n)e^{inx_0} = \sum_{-\infty}^{\infty} \hat{(}(\hat{f}(n) - \hat{g}(n) + \hat{g}(n))e^{inx_0} = \sum_{-\infty}^{\infty} \hat{f}(n)e^{inx_0}$$

And so

$$\sum_{-\infty}^{\infty} \hat{f}(n)e^{inx_0} = \sum_{-\infty}^{\infty} \hat{g}(n)e^{inx_0}$$

Problem 5

We want to prove that

$$\lim_{n\to\infty} \int_0^{\pi} \frac{\sin(nx)}{x}\, dx = \frac{\pi}{2}$$

Making the substitution $u = nx \implies du = n\, dx$,

$$\int_0^{\pi} \frac{\sin(nx)}{x}\, dx = \int_0^{n\pi} \frac{\sin u}{u}\, du$$

But we know that $\int_0^{\infty} \frac{\sin u}{u}\, du = \frac{\pi}{2}$, and so for any $\epsilon > 0$ we fix, there is large enough N s.t. $n > N \implies |\int_{n\pi}^{\infty} \frac{\sin u}{u}\, du| < \epsilon$. Thus, for any $n > N$,

$$\left| \int_0^{\pi} \frac{\sin(nx)}{x}\, dx - \frac{\pi}{2} \right| = \left| \int_0^{n\pi} \frac{\sin u}{u}\, du - \int_0^{\infty} \frac{\sin u}{u}\, du \right| = \left| \int_{n\pi}^{\infty} \frac{\sin u}{u}\, du \right| < \epsilon$$

which by the definition of limit is exactly what we wanted to show.

Problem 6

We want to prove that $\sum_1^{\infty} \frac{\sin nx}{\sqrt{n}}$ converges for all x, and uniformly on any interval of the form $[\delta, 2\pi - \delta]$ for $\delta > 0$ small. We proceed by Dirichlet's test: we have that $\frac{1}{\sqrt{n}}$ is monotonically decreasing to 0, and that for $x \neq 2k\pi$,

$$\begin{aligned} |B_N| &= \left| \sum_{n=1}^{N} \sin(nx) \right| = \left| \mathrm{Im} \left(\sum_{n=1}^{N} e^{inx} \right) \right| = \left| \mathrm{Im} \left(e^{ix} \frac{1 - e^{iNx}}{1 - e^{ix}} \right) \right| \\ &\leq \left| e^{ix} \frac{1 - e^{iNx}}{1 - e^{ix}} \right| = \frac{|1 - e^{iNx}|}{|1 - e^{ix}|} \leq \frac{|1| + |e^{iNx}|}{|1 - e^{ix}|} = \frac{2}{|1 - e^{ix}|} \end{aligned}$$

which is a bound that holds for all N. For $x = 2k\pi$, the sum is trivially 0 (alternatively, we could use Corollary 6.27 which gives the result trivially). Thus, the sum is convergent for all $x \in \mathbb{R}$. Furthermore, if we restrict ourselves to $[\delta, 2\pi - \delta]$, this bound can be made independent of x because $\frac{2}{|1-e^{ix}|}$ is a continuous function, which has a maximum on a compact interval $[\delta, 2\pi - \delta]$ (continuous because on the interval the denominator is never 0). Let's say this maximum bound is C (keep in mind this bound is for ALL N, and for all $x \in [\delta, 2\pi - \delta]$). Then, for $a_k = \frac{1}{\sqrt{k}}, b_k = \sin kx$, and $B_n = \sum_1^n b_k$, we have via summation by parts for $m > n > N$:

$$\begin{aligned}
&\left|\sum_{k=1}^{n} a_k b_k - \sum_{k=1}^{m} a_k b_k\right| = \left|\frac{B_n}{\sqrt{n}} - \sum_{k=1}^{n-1}\left(\frac{1}{\sqrt{k+1}} - \frac{1}{\sqrt{k}}\right) B_k - \frac{B_m}{\sqrt{m}} + \sum_{k=1}^{m-1}\left(\frac{1}{\sqrt{k+1}} - \frac{1}{\sqrt{k}}\right) B_k\right| \\
&\leq \left|\frac{B_n}{\sqrt{n}}\right| + \left|\frac{B_m}{\sqrt{m}}\right| + \sum_{n}^{m-1}\left|\left(\frac{1}{\sqrt{k+1}} - \frac{1}{\sqrt{k}}\right) B_k\right| \leq \frac{C}{\sqrt{n}} + \frac{C}{\sqrt{m}} + C\sum_{n}^{m-1}\left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}\right) \\
&= \frac{C}{\sqrt{n}} + \frac{C}{\sqrt{m}} + C\left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{m}}\right) = \frac{2C}{\sqrt{n}} \leq \frac{2C}{\sqrt{N}}
\end{aligned}$$

Fixing any $\epsilon > 0$, there is always large enough N s.t. the Cauchy sum for all $x \in [\delta, 2\pi - \delta]$ is $< \epsilon$ for any $m > n > N$. Therefore, the series is uniformly convergent on $[\delta, 2\pi - \delta]$ for any small $\delta > 0$.

Lastly, this series is not a Fourier series of a Riemann integrable function, because Riemann integrable $\implies$ bounded and measurable $\implies$ in $L^2(-\pi, \pi)$. But if this were the case, by Parseval's identity, we would have $\sum_1^\infty a_n^2 = \sum_1^\infty \frac{1}{n} = \frac{1}{2}\int_{-\pi}^{\pi} |f|^2\, dx < \infty$, which is not true.

Problem 7

We want to compute

$$\lim_{n\to\infty} \int_a^b \sin^2(nx)\, dx$$

Well, we know that

$$\int_a^b \sin^2(nx)\, dx = \int_a^b \frac{1 - \cos(2nx)}{2}\, dx = \frac{b-a}{2} + \frac{\sin(2na) - \sin(2nb)}{4n}$$

and so

$$\begin{aligned}
\left|\int_a^b \sin^2(nx)\, dx - \frac{b-a}{2}\right| &= \left|\frac{\sin(2na) - \sin(2nb)}{4n}\right| \leq \frac{|\sin(2na) - \sin(2nb)|}{4n} \\
&\leq \frac{|\sin(2na)| + |\sin(2nb)|}{4n} \leq \frac{2}{4n} = \frac{1}{2n}
\end{aligned}$$

Fixing any $\epsilon > 0$, there will always be N s.t. $n > N \implies \frac{1}{2n} < \epsilon$, which is exactly what it means to have

$$\lim_{n\to\infty} \int_a^b \sin^2(nx)\, dx = \frac{b-a}{2}$$

Problem 8

(a) By the alternating series test, the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{\sqrt{n}}$ converges conditionally because $\frac{1}{\sqrt{n}}$ is monotonically decreasing to 0.

(b) Consider a rearrangement of the series,

$$1+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{5}}+\frac{1}{\sqrt{7}}-\frac{1}{\sqrt{4}}+\ldots=\sum_{n=1}^{\infty}b_n$$

It is useful to group these in three, and so observe the following:

$$S_{3N}=\sum_{n=1}^{3N}b_n=\sum_{n=1}^{N}\frac{1}{\sqrt{4n-3}}+\frac{1}{\sqrt{4n-1}}-\frac{1}{\sqrt{2n}}$$

But note that

$$\frac{1}{\sqrt{4n-3}}+\frac{1}{\sqrt{4n-1}}-\frac{1}{\sqrt{2n}}\geq\frac{1}{\sqrt{4n}}+\frac{1}{\sqrt{4n}}-\frac{1}{\sqrt{2n}}=\frac{1}{n}-\frac{1}{\sqrt{2n}}=\left(1-\frac{1}{\sqrt{2}}\right)\frac{1}{\sqrt{n}}$$

and so

$$S_{3N}\geq\left(1-\frac{1}{\sqrt{2}}\right)\sum_{n=1}^{N}\frac{1}{\sqrt{n}}$$

But the sum on the right diverges; i.e. for large enough N the sum gets larger than any fixed bound we set. Thus, S_{3N} gets larger than any fixed bound we set (i.e. it diverges), so S_n can't possibly converge (because if it did, then S_{3N} would too).

335 Homework 10

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§8.4 - Problem 2ab

Note that for these problems we are only asked to find the series, not worry about convergence.

(a) We want to compute the sine series of $f(x) = 1$ on $[0, 1]$. By the definition, this is $\sum_0^\infty b_n \sin\left(\frac{n\pi x}{1}\right)$ where $b_n = \frac{2}{1}\int_0^1 f(x)\sin\left(\frac{n\pi x}{1}\right)\,dx = 2\int_0^1 \sin(n\pi x)\,dx = 2(-\frac{1}{n\pi}\cos(n\pi x))|_0^1 = \frac{2}{n\pi}(-\cos(n\pi)+1)$ which is $\frac{4}{n\pi}$ when n is odd and 0 when n is even. Thus we get our old friend

$$\frac{4}{\pi}\sum_1^\infty \frac{\sin(\pi(2n-1)x)}{2n-1}$$

(b) We want to compute the cosine series of $f(x) = 1\cdot 1_{(0,2)} + (-1)\cdot 1_{(2,4)}$ on $[0, 4]$. By the definition, this is $\frac{1}{2}a_0 + \sum_1^\infty a_n \cos\left(\frac{n\pi x}{4}\right)$ where $a_n = \frac{2}{4}\int_0^4 f(x)\cos\left(\frac{n\pi x}{4}\right)\,dx$. First, notice that for even n, $\cos\left(\frac{n\pi x}{4}\right)$ is even centered at 2. Thus because $f(x)$ is odd centered at 2, the resulting integral will be 0. However, for odd n, $\cos\left(\frac{n\pi x}{4}\right)$ is odd centered at 2, and so the resulting integral is $\frac{2}{4}\cdot 2\int_0^2 1\cdot\cos\left(\frac{n\pi x}{4}\right)\,dx = \int_0^2 \cos\left(\frac{n\pi x}{4}\right)\,dx = \frac{4}{n\pi}(\sin\left(\frac{n\pi x}{4}\right))|_0^2 = \frac{4}{n\pi}\sin\left(\frac{n\pi}{2}\right)$ which is positive for $n \equiv 1 \pmod 4$ and negative for $n \equiv 3 \pmod 4$, and so we get

$$\frac{4}{\pi}\sum_1^\infty (-1)^{n-1}\frac{\cos\left(\frac{(2n-1)\pi x}{4}\right)}{2n-1}$$

§8.5 - Problem 2

We want to find the temperature function $u(\theta, t)$ $t > 0$ for a rod bent into the shape of a circle, given the inital temperature $u(\theta, 0) = f(\theta)$. The boundary conditions are that u needs to be a 2π periodic function of θ.

This problem and the one studied by Folland on page 382 are essentially the same, other than boundary conditions. Thus, we may adopt Folland's solution to the heat equation, i.e. $u(\theta, t) = \varphi(\theta)\psi(t)$, where

$$\psi(t) = C_0 e^{-k\alpha t} \quad \text{and} \quad \varphi(\theta) = C_1\cos(\sqrt{\alpha}\theta) + C_2\sin(\sqrt{\alpha}\theta)$$

Our boundary condition is that $\varphi(\theta + 2\pi) = \varphi(\theta)$, and so $\sqrt{\alpha}$ must be an integer, say n. And so $u_n(\theta, t) = e^{-kn^2 t}(C_1\cos(n\theta) + C_2\sin(n\theta))$ is a solution to this problem, without the condtion that $u(\theta, 0) = f(\theta)$. Now I claim the the following infinite linear combination of the u_n is a solution to the full problem:

$$u(\theta, t) = \sum_1^\infty u_n(\theta, t) = \sum_1^\infty e^{-kn^2 t}(a_n\cos(n\theta) + b_n\sin(n\theta)$$

where a_n and b_n are exactly the Fourier coefficients of f. This finally makes it so that $u(\theta, 0) = f(\theta)$.

Folland does not specify conditions on f, so let's just borrow his assumptions from page 383; i.e that f is continuous and piecewise smooth (and of course 2π periodic). Similarly, by Theorem 8.29, the Fourier series of f is uniformly and absolutely convergent to f (see also Theorem 8.16). Finally, we justify taking infinite linear combinations exactly as Folland does at the end of page 383 (i.e. the exponential terms overpower any n^k that arise from differentiation, and so $u \in C^\infty$ and termwise differentiation is permissible).

Problem 7

The Dirichlet problem on the unit disc $\partial_x^2 u + \partial_y^2 u = 0$ for $x^2 + y^2 < 1$, $u(\cos\theta, \sin\theta) = f(\theta)$. If we go to polar coordinates instead, the equation becomes $r^2\partial_r^2 u + r\partial_r u + \partial_\theta^2 u = 0$ for $r < 1$, $u(1, \theta) = f(\theta)$.

(a) Using separation of variables we can find solutions of the DE of the form $u(r, \theta) = \varphi(r)\psi(\theta)$. Plugging this form into the DE, we see that we must have

$$r^2\varphi''(r)\psi(\theta) + r\varphi'(r)\psi(\theta) + \varphi(r)\psi''(\theta) = 0 \implies \frac{r^2\varphi''(r) + r\varphi'(r)}{\varphi(r)} = -\frac{\psi''(\theta)}{\psi(\theta)}$$

for all $r < 1$ and $\theta \in [0, 2\pi)$. The left side only depends on r and the right side only depends on θ so they both must be some constant, say α (let's look only at $\alpha \geq 0$; that will suffice to solve this problem). This means that $\psi(\theta) = C_1\cos(\sqrt{\alpha}\theta) + C_2\sin(\sqrt{\alpha}\theta)$ (which I copy directly from Folland page 385). But because $\psi(\theta)$ must equal $\psi(\theta + 2\pi)$ (one of our boundary conditions), $\sqrt{a}$ must be an integer (non-negative, because we only look at $\alpha \geq 0$), say n for notation's sake. Then, we have that

$$r^2\varphi''(r) + r\varphi'(r) - n^2\varphi(r) = 0 \implies \varphi(r) = \begin{cases} ar^n + br^{-n} & \text{if } n \neq 0 \\ a + b\log r & \text{if } n = 0 \end{cases}$$

by the hint given in the book. We don't want $\varphi(r)$ to blow up at the origin so the coefficients of the r^{-n} term and $\log r$ term must be 0 (because both r^{-c} and $\log r$ are bad as $r \to 0$). This simplifies matters because now $\varphi(r) = ar^n$ for all n we are concerned with, i.e. $n \in \{0, 1, \ldots, \}$. Thus, we have that $u_n(r, \theta) = ar^n(C_1\cos(n\theta) + C_2\sin(n\theta))$ satisfies the DE. Now I claim the the following infinite linear combination of the u_n is a solution to the full problem:

$$u(r, \theta) = \sum_1^\infty u_n(r, \theta) = \sum_1^\infty r^n(a_n\cos(n\theta) + b_n\sin(n\theta))$$

where a_n and b_n are exactly the Fourier coefficients of f. This finally makes it so that $u(1, \theta) = f(\theta)$. Folland does not specify conditions on f, so let's just borrow his assumptions from page 383; i.e that f is continuous and piecewise smooth (and of course 2π periodic). Similarly, by Theorem 8.29, the Fourier series of f is uniformly and absolutely convergent to f (see also Theorem 8.16). Finally, we justify taking infinite linear combinations exactly as Folland does at the end of page 383 (i.e. the exponential terms overpower any n^k that arise from differentiation, and so $u \in C^\infty$ and termwise differentiation is permissible).

(b) Rewriting our $u(r,\theta)$ in the exponential form, we do in fact see that $u(r,\theta) = A_r f(\theta) = \sum_{-\infty}^{\infty} r^{|n|} c_n e^{in\theta}$ for c_n being the Fourier coefficients in this exponential form. 8.20 tells us that

$$A_r f(\theta) = \int_{-\pi}^{\pi} f(\theta+\varphi) P_r(\varphi)\, d\varphi = \int_{\pi}^{-\pi} f(\theta-\varphi) P_r(-\varphi)(-1)\, d\varphi = \int_{-\pi}^{\pi} f(\theta-\varphi) P_r(-\varphi)\, d\varphi$$

where 8.22 tells us that

$$P_r(\varphi) = \frac{1-r^2}{2\pi(1+r^2-2r\cos\varphi)} = P_r(-\varphi)$$

Hence,

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1+r^2-2r\cos\varphi} f(\theta-\varphi)\, d\varphi$$

§8.6 - Problem 2

We want to prove that $f \in L^2(0,\pi) \implies$ the Fourier cosine and sine series of f both converge to f in norm.

Let f_e and f_o be the even and odd extensions resp. of f to $[-\pi,\pi]$. f_e and f_o are both Lebesgue measurable on $[-\pi,\pi]$ because f was Lebesgue measurable on $[0,\pi]$ and we are just reflecting/flipping f to make f_e, f_o. They are both also square integrable because $\int_{-\pi}^{\pi} |f_e|^2\, dx = \int_{-\pi}^{\pi} |f_o|^2\, dx = 2\int_0^{\pi} |f|^2\, dx < \infty$. Thus $f_e, f_o \in L^2(-\pi,\pi)$. Theorem 8.43 tells us that

$$\lim_{N\to\infty} \int_{-\pi}^{\pi} \left| f_e(\theta) - \frac{1}{2}a_o - \sum_1^N a_n \cos(n\theta) \right|^2 d\theta = 0 \quad \text{and} \quad \lim_{N\to\infty} \int_{-\pi}^{\pi} \left| f_o(\theta) - \sum_1^N b_n \sin(n\theta) \right|^2 d\theta = 0$$

where the series in question are exactly the cosine and sine Fourier series of f (see page 378). The integrand of the 1st integral is even, and $f_e = f$ on $(0,\pi)$, so

$$\int_{-\pi}^{\pi} \left| f_e(\theta) - \frac{1}{2}a_o - \sum_1^N a_n \cos(n\theta) \right|^2 d\theta = 2\int_0^{\pi} \left| f(\theta) - \frac{1}{2}a_o - \sum_1^N a_n \cos(n\theta) \right|^2 d\theta$$

means that

$$\lim_{N\to\infty} \int_0^{\pi} \left| f(\theta) - \frac{1}{2}a_o - \sum_1^N a_n \cos(n\theta) \right|^2 d\theta = 0$$

Similarly, the stuff inside the absolute values of the second integral is odd, so when squared it's even, and the same argument applies, and so we get that

$$\lim_{N\to\infty} \int_0^{\pi} \left| f(\theta) - \sum_1^N a_n \cos(n\theta) \right|^2 d\theta = 0$$

and so f converges to its cosine and sine Fourier series in the L^2 norm on $(0,\pi)$.

Problem 3

$f_0(x) = 1,\ f_1(x) = x + a,\ f_2(x) = x^2 + bx + c$ form an orthogonal set on $[0, 1]$. We want that $\langle f_0, f_1\rangle, \langle f_0, f_2\rangle$, and $\langle f_1, f_2\rangle$ to all be 0:

$$\int_0^1 1 \cdot x + a \, dx = \frac{1}{2} + a = 0 \implies a = -\frac{1}{2}$$

$$\int_0^1 (x - \tfrac{1}{2}) \cdot (x^2 + bx + c) \, dx = \frac{1}{4} + \frac{1}{3}b + \frac{1}{2}c - \frac{1}{2}\left(\frac{1}{3} + \frac{1}{2}b + c\right) = \frac{1}{2} + \frac{1}{2}b = 0 \implies b = -1$$

$$\int_0^1 1 \cdot (x^2 + bx + c) \, dx = \frac{1}{3} + \frac{1}{2}b + c = 0 \implies c = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

Problem 4

$\{\varphi_n\}_{n=1}^{\infty}$ is an orthonormal set of of functions on $[0, l]$, and let φ_n^+ and φ_n^- be the even and odd extensions of φ_n to $[-l, l]$. Show that $\{\varphi_n^+/\sqrt{2}\} \cup \{\varphi_n^+/\sqrt{2}\}$ form an orthonormal set on $[-l, l]$. Well, an even function times another even function is even, and an odd function times an odd function is even, so

$$\int_{-l}^{l} \varphi_n^+ \overline{\varphi_m^+} \, dx = 2\int_0^l \varphi_n^+ \overline{\varphi_m^+} \, dx = 2\int_0^l \varphi_n \overline{\varphi_m} \, dx = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{else} \end{cases}$$

by orthonormality of $\{\varphi_n\}_{n=1}^{\infty}$. Similarly,

$$\int_{-l}^{l} \varphi_n^- \overline{\varphi_m^-} \, dx = 2\int_0^l \varphi_n^- \overline{\varphi_m^-} \, dx = 2\int_0^l \varphi_n \overline{\varphi_m} \, dx = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{else} \end{cases}$$

Finally, $\varphi_n^+ \cdot \varphi_m^-$ is an even times odd so odd function, and so the integral is 0.

Problem 5

$\{\varphi_n\}_{n=1}^{\infty}$ is an orthonormal set of of functions on $[a, b]$. Given $c > 0$ and $d \in \mathbb{R}$, let $\psi_n(x) = \sqrt{c}\varphi_n(cx + d)$. Then, $\{\psi_n\}$ is an orthonormal set on $[\frac{a-d}{c}, \frac{b-d}{c}]$. Using the substitution $u = cx + d \implies du = c\,dx$, we have

$$\int_{\frac{a-d}{c}}^{\frac{b-d}{c}} \sqrt{c}\varphi_n(cx + d)\sqrt{c}\overline{\varphi_m(cx + d)} \, dx = \int_a^b \varphi_n \overline{\varphi_m} \, du = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{else} \end{cases}$$

Problem 6

$\{\varphi_n\}_{n=1}^{\infty}$ is an orthonormal set of of functions on $[0, 1]$. Let $\psi_n(x) = \sqrt{2x}\varphi_n(x^2)$. Then, $\{\psi_n\}$ is an orthonormal set on $[0, 1]$. Using the substitution $u = x^2 \implies du = 2x\,dx$, we have

$$\int_0^1 \sqrt{2x}\varphi_n(x^2)\sqrt{2x\varphi_m(x^2)} \, dx = \int_0^1 \varphi_n \overline{\varphi_m} \, du = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{else} \end{cases}$$

Problem 9acd

Evaluate the following series by applying Parseval's identity:

$$\int_{-\pi}^{\pi} |f(\theta)|^2 \, d\theta = \frac{\pi}{2}|a_0|^2 + \pi \sum_1^\infty (|a_n|^2 + |b_n|^2)$$

to some of the Fourier series from previous exercises in §8.1, §8.3.

(a) $\sum_1^\infty \frac{1}{n^4}$: using Folland's §8.1.4 (back of book),

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^\infty \frac{(-1)^n}{n^2}\cos(nx)$$

for all $x \in [-\pi, \pi]$, and so $a_0 = \frac{2\pi^2}{3}$, $a_n = 4\frac{(-1)^n}{n^2}$ for $n \geq 1$, and $b_n = 0$. Using Parseval's identity (because x^2 sure is in $L^2(-\pi, \pi)$),

$$\int_{-\pi}^{\pi} x^4 \, dx = \frac{2\pi^5}{5} = \frac{\pi}{2}\frac{4\pi^4}{9} + \pi\sum_1^\infty \frac{16}{n^4} \implies \sum_1^\infty \frac{1}{n^4} = \pi^4 \frac{1}{16}\left(\frac{2}{5} - \frac{2}{9}\right) = \frac{\pi^4}{90}$$

(b) $\sum_1^\infty \frac{1}{n^8}$: using Folland's §8.3.3b (the page the 8.3 problems are on),

$$x^4 - 2\pi^2 x^2 = -\frac{7\pi^4}{15} + 48\sum_{n=1}^\infty \frac{(-1)^{n+1}}{n^4}\cos(nx)$$

for all $x \in [-\pi, \pi]$, and so $a_0 = -\frac{14\pi^4}{15}$, $a_n = 48\frac{(-1)^{n+1}}{n^4}$ for $n \geq 1$, and $b_n = 0$. Using Parseval's identity (because $x^4 - 2\pi^2 x^2$ sure is in $L^2(-\pi, \pi)$),

$$\int_{-\pi}^{\pi} (x^4 - 2\pi^2 x^2)^2 \, dx = 2\left(\frac{\pi^9}{9} - \frac{4\pi^9}{7} + \frac{4\pi^9}{5}\right) = \frac{214\pi^9}{315} = \frac{\pi}{2}\frac{196\pi^8}{225} + \pi\sum_1^\infty \frac{48^2}{n^8}$$

and so

$$\sum_1^\infty \frac{1}{n^8} = \pi^8 \frac{1}{48^2}\left(\frac{214}{315} - \frac{98}{225}\right) = \frac{128\pi^8}{48^2 525} = \frac{\pi^8}{9450}$$

(c) $\sum_1^\infty \frac{\sin^2 na}{n^2}$. First for $a \in (0, \pi)$, we have that from Folland's §8.1.7 (back of book), for any $a \in (0, \pi)$,

$$f(x) = \begin{cases} 1/a & \text{for } |x| < a \\ -1/(\pi - a) & \text{for } a < |x| < \pi \end{cases} = \frac{2}{a(\pi - a)} \sum_1^\infty \frac{\sin na}{n} \cos nx$$

for all $x \in [-\pi, \pi] \setminus \{a, -a\}$, and so $a_0 = 0$, $a_n = \frac{2}{a(\pi-a)}\frac{\sin na}{n}$ for $n \geq 1$, and $b_n = 0$. Using Parseval's identity (because $f(x)$ sure is in $L^2(-\pi, \pi)$),

$$\int_{-\pi}^{\pi} f(x) \, dx = 2\left(a \cdot \frac{1}{a^2} + \frac{1}{(\pi - a)^2}(\pi - a)\right) = \frac{2\pi}{a(\pi - a)} = \pi \frac{4}{a^2(\pi - a)^2} \sum_1^\infty \frac{\sin^2(na)}{n^2}$$

and so

$$\sum_{1}^{\infty} \frac{\sin^2(na)}{n^2} = \frac{a(\pi - a)}{2}$$

We know this holds for $a \in (0, \pi)$; furthermore it holds on the endpoints $a = 0, \pi$ because both sides are 0. Finally, for any value outside $[0, \pi]$, we can write it as $a + k\pi$ for $a \in [0, \pi]$ and $k \in \mathbb{Z}$. We know that $\sin^2(na + kn\pi) = \sin^2(na)$ by π-periodicity of $\sin^2$, and so for any value $b = a + k\pi$, we have that

$$\sum_{1}^{\infty} \frac{\sin^2(nb)}{n^2} = \sum_{1}^{\infty} \frac{\sin^2(na)}{n^2} = \frac{a(\pi - a)}{2}$$

335 Homework 9

Daniel Rui - 3/5/20

§8.1 - Problem 1

We want to find the Fourier series of $f(\theta) = -1$ on $(-\pi, 0)$ and 1 on $(0, \pi)$. As per the formula in the book,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta\, d\theta \quad \text{and} \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta\, d\theta$$

Because $f(\theta)$ is odd, the $a_n = 0$. However, the b_n take more work:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta\, d\theta = \frac{1}{\pi}2\int_{0}^{\pi} \sin n\theta\, d\theta = \frac{2}{n\pi}(-\cos n\theta)|_0^{\pi}$$

which equals $\frac{4}{n\pi}$ for odd n but 0 for even n. Thus, the Fourier series is

$$\sum_{n=1}^{\infty} \frac{4}{\pi}\frac{\sin((2n-1)\theta)}{(2n-1)}$$

Problem 2

$f(\theta) = \sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$ is even, and so the b_n will be 0. The a_n are just a tad more annoying:

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta\, d\theta = \frac{1}{\pi}\left(\frac{1}{2}\int_{-\pi}^{\pi}\cos n\theta\, d\theta - \frac{1}{2}\int_{-\pi}^{\pi}\cos 2\theta\cos n\theta\, d\theta\right) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos n\theta\, d\theta - \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{2}[\cos(n-2)\theta + \cos(n+2)\theta]\, d\theta \end{aligned}$$

For $n = 0$, the first term is 1, and for $n \neq 0$, the first term is 0. For $n = 2$, the second term is $-\frac{1}{2}$, and for $n \neq 2$, the second term is 0. Thus, the Fourier series is exactly as we expected; $\frac{1}{2}a_0 + \sum_1^{\infty}(a_n\cos n\theta + b_n\sin n\theta) = \frac{1}{2}(1 - \cos 2\theta)$.

§8.2 - Problem 1

Folland already proves that the Fourier series for the sawtooth wave converges at every point at which the sawtooth wave is continuous. We follow Folland's example in Example 3 and just scale:

(a) We simply flip over the x axis and shift right by π:

$$\sum_1^{\infty}\frac{(-1)^{n+2}}{n}\sin(n(x-\pi)) = \sum_1^{\infty}\frac{(-1)^{n+2}}{n}(-1)^n\sin(nx) = \sum_1^{\infty}\frac{\sin(nx)}{n}$$

(b) This is just part (a) flipped over the x axis, shifted up by $\pi/2$ and scaled by a factor of $2/\pi$; i.e.

$$\frac{2}{\pi}\left(\frac{\pi}{2} - \sum_1^{\infty}\frac{\sin(nx)}{n}\right) = 1 - \frac{2}{\pi}\sum_1^{\infty}\frac{\sin(nx)}{n}$$

Problem 3ab

(a) $f(\theta) = 0$ on $(-\pi, 0)$ and 1 on $(0, \pi)$. We found the Fourier series of the function from Problem 8.1.1 above; by Theorem 8.16 in the text, that function is piecewise continuous (and we are only looking at an interval of length 2π), and so the Fourier series converge pointwise everywhere that original function was continuous (and it converges to $\frac{1}{2}((-1)+1) = 0$ elsewhere). Thus, scaling the Fourier series of Problem 8.1.1 by a half and moving it up by a half, we get that the Fourier series of this question's function is

$$\frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi} \frac{\sin((2n-1)\theta)}{(2n-1)}$$

(b) As per the hint, we want to find the Fourier series of $f(\theta) = \frac{1}{2}(|\sin\theta| + \sin\theta)$ for $|\theta| < \pi$. We need to find the Fourier series of $|\sin\theta|$: it is even so the $b_n = 0$; $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |\sin\theta|\, d\theta = \frac{2}{\pi}\int_0^{\pi} \sin\theta\, d\theta = \frac{4}{\pi}$, and for $n \neq 0$,

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} |\sin\theta| \cos n\theta\, d\theta = \frac{1}{\pi} 2 \int_0^{\pi} \sin\theta \cos n\theta\, d\theta \\ &= \frac{2}{\pi}\int_0^{\pi} \frac{1}{2}[\sin(n+1)\theta - \sin(n-1)\theta]\, d\theta \\ &= \frac{1}{\pi}\left(\frac{\cos(n-1)\theta}{n-1} - \frac{\cos(n+1)\theta}{n+1}\right)\Bigg|_0^{\pi} = \frac{1}{\pi}\left(\frac{1}{n+1} - \frac{1}{n-1}\right)(\cos\pi n + 1) \end{aligned}$$

which is 0 for odd n (the case where $n = \pm 1$ is handled separately, but it is a very trivial integral that also yield 0), and $\frac{2}{\pi}\left(\frac{1}{n+1} - \frac{1}{n-1}\right) = -\frac{4}{\pi(n^2-1)}$ for even n. Thus, the Fourier series of the function given by this problem is

$$f(\theta) = \frac{1}{2}\left(\sin\theta + \frac{1}{2}\frac{4}{\pi} - \frac{4}{\pi}\sum_{1}^{\infty} \frac{\cos(2n)\theta}{(2n)^2 - 1}\right) = \frac{1}{2}\sin\theta + \frac{1}{\pi} - \frac{2}{\pi}\sum_{1}^{\infty} \frac{\cos(2n)\theta}{(2n)^2 - 1}$$

following similar logic as above as to how we can add two things together.

§8.3 - Problem 4

We know that $f(\theta) = |\sin\theta|$ is continuous and piecewise smooth, and its derivative is $\cos\theta$ on $(0, \pi)$ and $-\cos\theta$ on $(-\pi, 0)$ (also piecewise smooth, but not continuous). We can simplify this by using Problem 8.1.1's function, the square wave. Let's call that function $\sigma(x)$, for "sign" of x. We have satisfied the conditions of Corollary 8.27, so

$$f'(\theta) = \frac{4}{\pi}\sum_{1}^{\infty} \frac{2n \sin 2n\theta}{(2n)^2 - 1}$$

for any θ where $f'(\theta)$ exists, i.e. $0 < |\theta| < \pi$. On the flip side, integrating $\sin\theta$ from 0 to θ (for $\theta \in (0, \pi)$) yields

$$\int_0^\theta \sin x\, dx = 1 - \cos\theta = \frac{2}{\pi}\theta - \frac{4}{\pi}\sum_1^\infty \frac{\sin 2n\theta}{2n((2n)^2-1)}$$

Now we seemingly have

$$\frac{8}{\pi}\sum_1^\infty \frac{n\sin 2n\theta}{(2n)^2-1} = \cos\theta = 1 - \frac{2}{\pi}\theta + \frac{2}{\pi}\sum_1^\infty \frac{\sin 2n\theta}{n((2n)^2-1)}$$

These are not contradictory because of $\frac{4n}{(2n)^2-1} = \frac{1}{n((2n)^2-1)} + \frac{1}{n}$ in addendum with the fact that

$$\frac{2}{\pi}\sum_1^\infty \frac{\sin 2n\theta}{n} = 1 - \frac{2}{\pi}\theta \iff \sum_1^\infty \frac{\sin 2n\theta}{n} = \frac{\pi}{2} - \theta$$

(for $\theta \in (0, \pi)$), which is true because substituting $\pi - 2\theta$ for θ in Example 1, §8.2 yields the result; that is:

$$\sum_1^\infty \frac{\sin 2n\theta}{n} = \sum_1^\infty \frac{(-1)^{n+1}\sin n(\pi - 2\theta)}{n} = \frac{\pi - 2\theta}{2} = \frac{\pi}{2} - \theta$$

Problem 5

Theorem 8.26 says that if we have a continuous piecewise smooth 2π periodic function, then the Fourier coefficients of f' are $c'_n = inc_n$. The function we are given, $f(\theta) = e^\theta$ is NOT continuous at $-\pi$ and π (and also at any odd multiple of π), and so we can not use Theorem 8.26.

Problem 6

From Folland's "general principle", the degree of smoothness of a periodic function is closely related to the rate of decay of its Fourier coefficients. More specifically on page 376, Folland says that if $|c_n| \le C|n|^{-k-\epsilon}$ for some $C, \epsilon > 0$, then $f \in C^{k-1}$.

(a) The series $\sum_{n\neq 0} \frac{e^{in\theta}}{n^{6/5}(1+n^6)}$ has coefficients $c_n = n^{-6/5}(1+n^6)^{-1}$, and so for $k = 7$ and e.g. $\epsilon = 1/5$ and $C = 1$:

$$|c_n||n|^{k+\epsilon} = \frac{|n|^{k+\epsilon-6/5}}{1+n^6} \le C$$

This doesn't work for $k \ge 8$ because the degree of the numerator then becomes strictly greater than that of the denominator. Thus, $f \in C^6$.

(b) The series $\sum_0^\infty \frac{\cos n\theta}{2^n}$ has coefficients $c_n = \frac{1}{2}(a_n - ib_n) = \frac{1}{2^{n+1}}$, and $|c_n||n|^{k+\epsilon} \le C$ for any k because the limit of some polynomial over 2^n tends to 0. Thus, $f \in C^\infty$.

(c) The series $\sum_0^\infty \frac{2^n\theta}{2^n}$ is a Fourier series, but the coefficients are not easy to work with (because there are so many 0's everywhere), so we must make do with other methods. By the Weierstrass M-test, all the terms in absolute value are $\le |\frac{1}{2^n}|$ and so the sum converges uniformly. However differentiating the series term by term, we have the terms $\cos 2^n\theta$, which don't even go to 0 on most points, and so none of the theorems about f' equaling some series apply. So $f \in C^0$.

335 Homework 8

Daniel Rui - 2/27/20

§7.5 - Problem 3

We want to verify that for any $x > 0$, $\int_0^\infty e^{-xt}\,dt = \frac{1}{x}$:

$$\int_0^b e^{-xt}\,dt = \left(-\frac{1}{x}e^{-xt}\right)\Big|_0^b = \frac{1}{x} - \frac{e^{-xb}}{x}$$

and as $b \to \infty$ (and keeping in mind $x > 0$), the exponent on e goes to $-\infty$, and so we get a final answer of $\frac{1}{x}$. Keeping in mind that $t^n e^{-xt}$ is a continuous function on $\{(x,t) : x > 0, t \geq 0\}$, we can use Theorem 7.41, which tells us that if $\int_0^\infty f(x,t)\,dt$ converges for $x \in I$ and $|\partial_x f(x,t)| \leq g(t)$ for all $x \in I, t \geq 0$ where $\int_0^\infty g(t)\,dt < \infty$, we can differentiate under the integral sign. If we take $I = [\delta, \infty)$ for any $\delta > 0$, we have already verified the first condition, so all we need to do is find appropriate g:

$$|\partial_x f(x,t)| = \left|-te^{-xt}\right| = te^{-xt} \leq te^{-\delta t} =: g(t)$$

satisfying $\int_0^\infty g(t)\,dt$. In fact, we can do this over and over again, because $t^n e^{-\delta t}$ always satisfies $\int_0^\infty g(t)\,dt < \infty$ (a rigorous inductive proof based on integration by parts will be omitted due to its trivial nature). Thus, for every $x \in [\delta, \infty)$,

$$\int_0^\infty (-1)^n t^n e^{-xt}\,dt = (-1)^n n! x^{-1-n} \implies \int_0^\infty t^n e^{-xt}\,dt = \frac{n!}{x^{n+1}}$$

for every $x > 0$, we can choose δ s.t. $0 < \delta < x$, and so this equality is satisfied by all $x > 0$.

Problem 5

On any compact interval $I = [a,b]$, $a, b > 0$, e^{-xt} is a continuous function on the set $\{(x,t) : x \in I, t \geq 0\}$, and $|f(x,t)| = e^{-xt} \leq e^{-at} =: g(t)$ for $x \in I$, where $\int_0^\infty g(t)\,dt < \infty$. Thus by Theorem 7.41, the following integral interchange is legal:

$$\int_0^\infty \int_a^b e^{-xt}\,dx\,dt = \int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\,dt = \int_a^b \int_0^\infty e^{-xt}\,dt\,dx = \int_a^b \frac{1}{x}\,dx = \ln\left(\frac{b}{a}\right)$$

Problem 7

On $I = \mathbb{R}$, $e^{-t}\sin(xt)$ is a continuous function on the set $\{(x,t) : x \in I, t \geq 0\}$, and $|f(x,t)| = e^{-t}|\sin(xt)| \leq e^{-t} =: g(t)$, for $x \in I$, where $\int_0^\infty g(t)\,dt < \infty$. Thus by Theorem 7.41, the following integral interchange is legal (where $[0,a] \subset \mathbb{R}$ for any a):

$$\int_0^\infty \int_0^a e^{-t}\sin(xt)\,dx\,dt = \int_0^\infty e^{-t}\frac{1-\cos at}{t}\,dt = \int_a^b \int_0^\infty e^{-t}\sin(xt)\,dt\,dx = \int_a^b \frac{x}{x^2+1}\,dx$$

where $\int_0^\infty e^{-t}\sin(xt)\,dt$ converges absolutely and uniformly (for $x \in \mathbb{R}$ because $|f(x,t)| \le e^{-t}$) to the above function (by WolframAlpha). Furthermore, the last integral is $= \frac{1}{2}\ln(1+a^2)$ from basic calculus knowledge.

Problem 8

For $x > 0$, take $u = xt \implies du = x dt$, and so $\int_0^\infty \frac{\sin xt}{t}\,dt = \int_0^\infty \frac{\sin xt}{xt} \cdot x\,dt = \int_0^\infty \frac{\sin u}{u}\,du = \frac{\pi}{2}$. For $x = 0$, the integral is obviously 0, and for $x < 0$, it's just the $x > 0$ case but with $(-x)$ instead of x, and because $\sin(x)$ is odd, this would just result in the negative of the integral, i.e. $-\frac{\pi}{2}$.

Obviously, it would not be uniformly convergent on a compact interval containing 0, because the resulting function is not continuous (which it would be had the convergence been uniform, by Theorem 7.39a); however, we will now show that it is uniformly convergent on any compact interval $[a,b]$ not containing 0:

Unfortunately, we can not use Theorem 7.38, because the conclusions of that are that the integral converges absolutely and uniformly, which we know it is not. Thus, we must go back to the basics (w.l.o.g. take $a, b > 0$; the negative case is just the positive case with the integral negated, which under the absolute value makes no difference):

$$\sup_{x\in[a,b]} \left| \int_d^\infty \frac{\sin xt}{t}\,dt \right| \to 0 \text{ as } d \to \infty$$

We do this via integration by parts:

$$\begin{aligned}\left| \int_d^\infty \frac{\sin xt}{t}\,dt \right| &= \left| \left(\frac{-\cos xt}{xt} \right) \Big|_d^\infty - \int_d^\infty \frac{\cos xt}{x} \cdot \frac{1}{t^2}\,dt \right| \\ &\le \left| \frac{\cos xd}{xd} \right| + \left| \frac{1}{x} \int_d^\infty \frac{\cos xt}{t^2}\,dt \right| \le \frac{1}{ad} + \frac{1}{a}\int_d^\infty \frac{1}{t^2}\,dt = \frac{2}{ad}\end{aligned}$$

which is a uniform bound for all $x \in [a,b]$ which goes to 0 as $d \to \infty$, so we have uniform convergence.

Problem 10

$$I(a,b) = \int_0^\infty \frac{\cos bt - \cos at}{t^2}\,dt$$

(a) We want to find $g(t)$ s.t. $\int_0^\infty g(t)\,dt < \infty$ and $|f(a,b,t)| \le g(t)$ for all fixed b and a in a finite interval (and vice versa), because then we would have that $\int_0^\infty f(a,b,t)\,dt$ converges uniformly and absolutely on the aforementioned intervals (Theorem 7.38). For $t \ge 1$, $|f(a,b,t)| \le \frac{1}{t^2}$. For $t \in [0,1]$, note that $\cos bt - \cos at = -2\sin\left(\frac{at+bt}{2}\right)\sin\left(\frac{bt-at}{2}\right)$ and $|\sin x| \le \min\{|x|, 1\}$, so

$$\begin{aligned}\left| \frac{\cos bt - \cos at}{t^2} \right| &= \left| -2\frac{\sin\left(\frac{at+bt}{2}\right)}{t} \frac{\sin\left(\frac{bt-at}{2}\right)}{t} \right| \le 2\frac{\min\{|\frac{a+b}{2}|t, 1\}}{t} \frac{\min\{|\frac{b-a}{2}|t, 1\}}{t} \\ &= \min\left\{ \frac{|b^2-a^2|}{2}, \frac{|a+b|}{t}, \frac{|b-a|}{t}, \frac{1}{t^2} \right\} \le \min\left\{ \frac{|b^2-a^2|}{2}, \frac{1}{t^2} \right\}\end{aligned}$$

Because a, b are either fixed or in a finite interval, we can definitely bound $\frac{|b^2-a^2|}{2}$ by some constant M. (To be more accurate above, we should not include $t = 0$ in our intervals, but if we define $f(a, b, 0) = \frac{a^2-b^2}{2}$, i.e. the limiting value, then that'll be fine because we still get a continuous function and our uniform bound is not affected).

Combining this with our bound for $t \geq 1$, we get that $|f(a, b, t)| \leq \min\{\frac{|b^2-a^2|}{2}, \frac{1}{t^2}\} =: g(t)$, which in fact satisfies $\int_0^\infty g(t)\,dt < \infty$. Thus, Theorem 7.38 gives that the convergence is uniform on the aforementioned intervals.

(b) On any compact interval $I = [a, b]$, $a, b > 0$, $f(x, t) = -\frac{\sin(xt)}{t}$ (defining it to be $-x$ at $t = 0$) is a continuous function on the set $\{(x, t) : x \in I, t \geq 0\}$. Furthermore, $\int_0^\infty f(x, t)\,dt$ is uniformly convergent for $x \in I$ (by Problem 8 from above). Thus by Theorem 7.39, the following integral interchange is legal:

$$\int_0^\infty \int_a^b -\frac{\sin(xt)}{t}\,dx\,dt = \int_0^\infty \frac{\cos bt - \cos at}{t^2}\,dt = \int_a^b \int_0^\infty -\frac{\sin(xt)}{t}\,dt\,dx = \int_a^b -\frac{\pi}{2}\,dx$$

which is of course $\frac{\pi(a-b)}{2}$.

(c) Because $\cos(x)$ is even, $I(a, b) = I(|a|, |b|) = \frac{\pi}{2}(|a| - |b|)$.

Problem 11

Define $F(x) = \int_0^\infty e^{-t^2} \cos xt\,dt$

(a) On $I = \mathbb{R}$, $f(x, t) = e^{-t^2}\cos xt$ is a continuous function on the set $\{(x, t) : x \in I, t \geq 0\}$, and $\int_0^\infty |e^{-t^2}\cos xt|\,dt \leq \int_0^\infty e^{-t^2}\,dt$ and so it converges for all $x \in \mathbb{R}$, and so just the integral without the absolute values also converges for all $x \in \mathbb{R}$. Moreover, $|\partial_x f(x, t)| = |te^{-t^2}\cos(xt)| \leq te^{-t^2} =: g(t)$ where $\int_0^\infty g(t)\,dt < \infty$. Thus by Theorem 7.41, we can differentiate under the integral sign:

$$\begin{aligned} F'(x) &= \int_0^\infty \partial_x[te^{-t^2}\cos(xt)]\,dt = \int_0^\infty -te^{-t^2}\sin(xt)\,dt \\ &= \left(\frac{1}{2}e^{-t^2}\sin(xt)\right)\Big|_0^\infty - \int_0^\infty \frac{1}{2}e^{-t^2}x\cos(xt)\,dt = -\frac{1}{2}xF(x) \end{aligned}$$

(b) Using very unrigorous methods (but what option do we have...we've not done a single differential equation this entire year), we get that $F'(x)/F(x) = -\frac{1}{2}x \implies \int F'(x)/F(x)\,dx = \int -\frac{1}{2}x\,dx \implies \ln|F(x)| + C_1 = -\frac{1}{4}x^2 \implies |F(x)| = C_2e^{-x^2/4} \implies F(x) = C_2e^{-x^2/4}$. $F(0)$ is the Gaussian integral which we know from previous sections is $= \frac{\sqrt{\pi}}{2}$, and so $C_2 = \frac{\sqrt{\pi}}{2}$. Thus $F(x) = \frac{\sqrt{\pi}}{2}e^{-x^2/4}$

§7.6 - Problem 2

$a, b > 0$, so taking $u = \ln\frac{1}{t} = -\ln t \implies t = e^{-u} \implies dt = -e^{-u}\,du$,

$$\int_0^1 \left(\ln\frac{1}{t}\right)^{a-1}\,dt = \int_\infty^0 u^{a-1}(-e^{-u})\,du = \int_0^\infty u^{a-1}e^{-bu}\,du$$

and using the same substitution:

$$\int_0^1 \left(\ln \frac{1}{t}\right)^{a-1} t^{b-1}\, dt = \int_\infty^0 u^{a-1} e^{-u(b-1)}(-e^{-u})\, du = \int_0^\infty u^{a-1} e^{-bu}\, du = \Gamma(a)$$

and $v = bu \implies dv = b\, du$ and so this is $= \int_0^\infty v^{a-1} \frac{1}{b^{a-1}} e^{-v} \frac{1}{b}\, dv = \frac{1}{b^a} \int_0^\infty v^{a-1} e^{-v}\, dv = b^{-a}\Gamma(a)$.

Problem 3

(a) Using the substitution $u = x^2 \implies du = 2x\, dx$, we have $\int_0^\infty x^4 e^{-x^2}\, dx = \int_0^\infty \frac{1}{2} u^{1.5} e^{-u}\, du = \frac{1}{2}\Gamma(2.5) = \frac{1}{2}\frac{3}{2}\Gamma(\frac{3}{2}) = \frac{3}{4}\frac{\sqrt{\pi}}{2} = \frac{3}{8}\sqrt{\pi}$.

(b) Using the substitution $u = 3x \implies du = 3\, dx$, we have $\int_0^\infty e^{-3x}\sqrt{x}\, dx = \int_0^\infty \frac{1}{3} e^{-u}(u/3)^{1/2}\, du = \frac{1}{3\sqrt{3}} \int_0^\infty e^{-u} u^{1/2}\, du = \frac{1}{3\sqrt{3}}\Gamma(1.5) = \frac{\sqrt{\pi}}{6\sqrt{3}}$.

(c) Using the substitution $u = x^4 \implies du = 4x^3\, dx$, we have $\int_0^\infty x^9 e^{-x^4}\, dx = \int_0^\infty u^{6/4} e^{-u} \frac{1}{4}\, du = \frac{1}{4}\Gamma(2.5) = \frac{1}{4}\frac{3}{2}\Gamma(1.5) = \frac{3}{16}\sqrt{\pi}$

Problem 5

Using the substitution $u = x^b \implies du = bx^{b-1}\, dx \implies dx = \frac{1}{bu^{(b-1)/b}}\, du$, we have (for $a, b, c > 0$)

$$\begin{aligned}
\int_0^1 x^a (1-x^b)^c\, dx &= \int_0^1 u^{a/b}(1-u)^c \frac{1}{bu^{(b-1)/b}}\, du = \frac{1}{b}\int_0^1 u^{(a+1-b)/b}(1-u)^c\, du \\
&= \frac{1}{b}\int_0^1 u^{((a+1)/b-1)}(1-u)^{(c+1)-1}\, du = \frac{1}{b} B\left(\frac{a+1}{b}, c+1\right) \\
&= \frac{1}{b}\frac{\Gamma(\frac{a+1}{b})\Gamma(c+1)}{\Gamma(\frac{a+1}{b}+c+1)}
\end{aligned}$$

335 Homework 7

Daniel Rui - 2/20/20

§7.1 - Problem 1c

$f_k(x) = \sin^k x, x \in [0, \pi]$: on $[0, \pi]$, $0 \leq \sin x \leq 1$, where $\sin x = 1 \iff x = \pi/2$. Thus for $x = \pi/2$, $f_k(\pi/2) = 1$ for all k, and so the limit is obviously 1. For all other x in the given interval, $\sin x < 1$, and so raising to successive powers just makes it smaller and so the limit in all of those cases is 0.

The limit function is not continuous, so $f_k(x)$ can not be uniformly convergent on $[0, 1]$. I claim that for any δ, $f_k(x)$ is uniformly continuous on $[0, \frac{1}{2}\pi - \delta]$, because $\sin x$ and x^k are increasing on $[0, \pi/2]$ and so for any $x \in [0, \frac{1}{2}\pi - \delta]$, $0 \leq \sin^k x \leq \sin^k(\frac{\pi}{2} - \delta)$ which goes to 0 as $k \to \infty$. This means that if we pick an $\epsilon > 0$, we can find K such that $\sin^K(\frac{\pi}{2} - \delta) < \epsilon$. In other words, we've found a K such that for every x, $|\sin x| < \epsilon$, which means that $f_k(x)$ is uniformly continuous on $[0, \frac{1}{2}\pi - \delta]$. Similarly it is uniformly convergent on $[\frac{1}{2}\pi + \delta, \pi]$.

Problem 1g

$f_k(x) = \frac{x^k}{1+x^{2k}}, x \in [0, \infty)$. On $[0, 1)$, x^k and x^{2k} go to zero, so $f_k(x)$ goes to 0. On $(1, \infty)$, $f_k(x) = \frac{x^{-k}}{x^{-2k}+1}$ and x^{-k} and x^{-2k} go to zero, so $f_k(x)$ also goes to 0. For $x = 1$, $f_k(1)$ stays $\frac{1}{2}$. Because the limit function is not continuous, it is not uniformly continuous on $[0, \infty)$. However, it IS uniformly continuous on $[0, 1 - \delta]$ and $[1 + \delta, \infty)$ for any δ:

On $[0, 1 - \delta]$, $0 \leq f_k(x) \leq \frac{(1-\delta)^k}{1+0} = (1 - \delta)^k$ for all x, and this obviously goes to 0. On $[1 + \delta, \infty)$, $0 \leq f_k(x) \leq \frac{(1+\delta)^{-k}}{1+0} = (1 + \delta)^{-k}$ for all x, and this obviously goes to 0 as well.

Problem 2c

$\sum_1^\infty \frac{nx^n}{2^{n+3}}$: By the ratio test,

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)|x|^{n+1}2^{n+3}}{2^{n+4}nx^n} = \frac{(n+1)|x|}{2n}$$

which goes to $|x|/2$ as $n \to \infty$, meaning that the sum is absolutely convergent when $|x| < 2$ and not when $|x| > 2$. Finally on the 2 cases where $|x| = 2$, the sum's terms to not tend to 0 so it is obviously divergent.

We consider $|x| < 2$ only because it diverges otherwise. I claim that the series is uniformly convergent on $[-2 + \delta, 2 - \delta]$ for any $\delta > 0$: for any x in the aforementioned interval,

$$|f_n(x)| = \left|\frac{nx^n}{2^{n+3}}\right| = \frac{n|x|^n}{2^{n+3}} \leq \frac{n(2-\delta)^n}{2^{n+3}}$$

which converges from the above analysis using the ratio test.

Problem 2e

$\sum_1^\infty \frac{1}{x^2+n^2}$: this is a simple application of Weierstrass's M-test: taking $M_n = \frac{1}{n^2}$, $|f_n(x)| \leq M_n$ for all x, and $\sum_1^\infty M_n$ is convergent by the p-test, and so $\sum_1^\infty \frac{1}{x^2+n^2}$ must be absolutely and uniformly convergent on $\mathbb{R}$.

Problem 3

Let $f_k(x) = g(x)x^k$ where g is continuous on $[0,1]$ and $g(1) = 0$. show that $f_k \to 0$ uniformly on $[0,1]$: g is continuous on $[0,1]$ and $g(1) = 0$, so for every $\epsilon > 0$, I can find δ_ϵ s.t. $x \in [\delta_\epsilon, 1] \implies |g(x)| < \epsilon$. Then on $[\delta_\epsilon, 1]$, $|f_k(x)| = |g(x)x^k| < \epsilon \cdot 1$ (for all k, $x^k \leq 1$ on $[\delta_\epsilon, 1] \subseteq [0,1]$). We also know that g is bounded (because $[0,1]$ is compact), so say that $|g| < M$ on $[0,1]$. Furthermore, we know that for every $x \in [0, \delta_\epsilon]$, there is K s.t. for any $k \geq K$, $x^k < \epsilon/M$. Thus, on $[0, \delta_\epsilon]$, $|f_k(x)| = |g(x)x^k| < M \cdot \epsilon/M = \epsilon$. Thus, we've proved that for all $x \in [0,1]$, $|f_k(x)| < \epsilon$. This is true for all $\epsilon > 0$, so $f_k(x)$ is uniformly continuous on [0,1].

Problem 4

Show that the series $\sum_1^\infty f_n(x) := \sum_1^\infty \frac{1}{x^2-n^2}$ converges uniformly on any compact interval that does not contain a nonzero integer, and conclude that the sum of the series is a continuous function on $\mathbb{R} \setminus \{\pm 1, \pm 2, ...\}$:

Consider any compact interval $[a,b]$ that does not contain any non-zero integers. Because the interval is bounded, there must be N s.t. $|x| < N$ for all x in the interval. Then for every $n \geq N$, $0 \leq n^2 - N^2 \leq n^2 - x^2 \implies \frac{1}{n^2-x^2} \leq \frac{1}{n^2-N^2}$. Thus for every $n \geq N$,

$$|f_n(x)| = \left|\frac{1}{x^2-n^2}\right| \leq \frac{1}{n^2-N^2} =: M_n$$

and because $\frac{M_n}{1/n^2}$ tends to 1, we have by the limit comparison test, Theorem 6.12, (recalling that $\sum_1^\infty 1/n^2 < \infty$) that $\sum_N^\infty M_n$ is finite. Adding back the first $N-1$ terms doesn't change this, and so we have by the M-test that the series converges uniformly.

Problem 5

Show that the series

$$\sum_1^\infty \frac{(-1)^{n-1}}{x^2+n}$$

converges uniformly on $\mathbb{R}$, although the converges is conditional at every point. We prove this by showing that the series is uniformly Cauchy which implies uniformly convergent by Theorem 7.7. The partial sums are $\frac{1}{x^2+n}$ is decreasing to 0 as $n \to \infty$ for all x, and so the alternating series test tells us

that it converges conditionally for all x. This also tells us that for any $i, j > N$,

$$\left|\sum_{i}^{j} \frac{(-1)^{n-1}}{x^2+n}\right| \leq \left|\sum_{1}^{\infty} \frac{(-1)^{n-1}}{x^2+n} - \sum_{1}^{N} \frac{(-1)^{n-1}}{x^2+n}\right| < \frac{1}{x^2+N+1} \leq \frac{1}{N+1}$$

So as $N \to \infty$, we can make this difference as small as we want. The bound does not depend on x, and so this is a uniform bound proving uniform Cauchy.

Problem 6

Given a sequence $\{c_n\}$ of real numbers such that $\sum_1^\infty c_n$ converges, consider the series $\sum_1^\infty c_n \frac{x^n}{1-x^n} (x \neq \pm 1)$. Such a series is called a Lambert series.

(a) Show that the series converges absolutely and uniformly on $[-a, a]$ for any $0 \leq a < 1$: because the sum is convergent $c_n \to 0$, so $|c_n| < C$ for some C. Furthermore, all $x \in [-a, a]$ satisfy $\frac{x^n}{1-x^n} \leq \frac{a^n}{1-a}$ (the numerator is maximized and the for the denominator $x^n \leq a^n \leq a$ and so it's minimized). Thus we have a uniform bound $|c_n \frac{x^n}{1-x^n}| \leq C \frac{a^n}{1-a}$, and $\frac{c}{1-a} \sum_1^\infty a^n < \infty$ obviously by geometric series. Thus by the M-test, the series converges uniformly on any $[-a, a]$, $a < 1$.

(b) Show that the series converges uniformly on $(-\infty, -b]$ and on $[b, \infty)$ for any $b > 1$, and that the convergence is absolute if and only if $\sum_1^\infty |c_n| < \infty$. Hint $x^n(1-x^n)^{-1} = (1-x^n)^{-1} - 1$.

$\sum_1^\infty c_n \frac{x^n}{1-x^n} = \sum_1^\infty \frac{c_n}{1-x^n} - c_n$. This converges uniformly by the M-test on the above intervals because first the $-c_n$ doesn't depend on x, so we can ignore that and second $|\frac{c_n}{1-x^n}| \leq |\frac{c_n}{1-x^n}| \leq \frac{C}{b^n-1}$ (because $|x|^n \geq b^n \implies$ for $x > 1$ $x^n - 1 \geq b^n - 1$ and for $x < 1$, and even n, $x^n - 1 \geq b^n - 1$ and for odd n, $1 - x^n = 1 + |x|^n \geq b^n + 1 \geq b^n - 1$). This converges by the limit comparison test to $1/b^n$. In order for it to converge absolutely, we need only that the $\sum_1^\infty c_n$ to converge absolutely.

Problem 8

Let $\{f_k\}$ be a sequence of continuous functions defined on a $[a, b]$, show that if $\{f_k\}$ converges uniformly on (a, b), then it converges uniformly on $[a, b]$

To show this, we show that the f_k are uniformly Cauchy on $[a, b]$ (Theorem 7.7). We already know that it's uniformly convergent (hence uniformly Cauchy) and (a, b), or mathematically: for all ϵ there exists K s.t. for all $x \in (a, b)$ and $i, j > K$, $|f_i(x) - f_j(x)| < \epsilon/3$. But the $f_k(x)$ are continuous on $[a, b]$! So for $\epsilon/3$, there is a δ s.t. $x \in [a, a+\delta] \implies |f_i(a) - f_i(x)| < \epsilon/3$. Doing this for $f_j(a)$ as well, we get that for an $x \in [a, a+\delta]$ and for i, j greater than the K from above,

$$|f_j(a) - f_i(a)| \leq |f_j(a) - f_j(x)| + |f_j(x) - f_i(x)| + |f_i(x) - f_i(a)| \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

The argument for b is similar. In other words, we've shown that for all $x \in [a, b]$, for $i, j > K$, $|f_i(x) - f_j(x)| < \epsilon$, which means that the f_k are uniformly Cauchy on $[a, b]$, as desired.

Problem 9

Let $\{f_k\}$ be a sequence of continuous functions on a compact set $S \subset \mathbb{R}^n$. Suppose the (a) the sequence $\{f_k(\mathbf{x})\}$ is bounded and increasing (and hence has a limit) for each $x \in S$, and (b) the function $f = \lim_{k\to\infty} f_k$ is continuous on S. Show that $f_k \to f$ uniformly on S (Hint: Given $\epsilon > 0$, apply Exercise 5 in §1.6 to the sets $S_k = \{X \in S : f(\mathbf{x}) - f_k(\mathbf{x}) \geq \epsilon\}$.)

All the functions f and f_k are continuous, so $f - f_k$ is continuous, meaning that S_k is closed. Furthermore, they are a subset of S, meaning that they are bounded and therefore compact. Because f_k increases to f, for any $i < j$ we have $f_i(\mathbf{x}) \leq f_j(\mathbf{x}) \leq f(\mathbf{x}) \implies 0 \leq f(\mathbf{x}) - f_j(\mathbf{x}) \leq f(\mathbf{x}) - f_i(\mathbf{x})$ implying that $f(\mathbf{x}) - f_i(\mathbf{x}) \geq \epsilon \implies f(\mathbf{x}) - f_j(\mathbf{x}) \geq \epsilon$. Thus $S_1 \supseteq S_2 \supseteq \ldots$ — Exercise §1.6-5 tells us that if each compact S_k is nonempty $\bigcap_1^\infty S_k$ is nonempty as well. However, this set must be empty because if say x were in it, then $f(x) - f_k(x) \geq \epsilon$ for all k, contradicting that $\lim_{k\to\infty} f_k(x) = f(x)$. Hence there must be some K for which every S_k, $k \geq K$, is empty, or in mathematical terms $|f(x) - f_k(x)| < \epsilon$ for all $x \in S$ for $k \geq K$, i.e. f_k converge uniformly to f on S.

§7.2 - Problem 1

$f(x) = \sum_1^\infty \frac{\sin nx}{n^2}$: $|f_n(x)| = |\frac{\sin nx}{n^2}| \leq \frac{1}{n^2} =: M_n$ and so by the M-test $\sum_1^\infty f_n(x)$ converge uniformly everywhere. Thus by Theorem 7.13 we can integrate term by term to get the integral of f, or more explicitly

$$\int_0^{\pi/2} f(x)\,dx = \sum_1^\infty \frac{1}{n^3}(1 - \cos(\tfrac{\pi}{2}n)) = \sum_{n \text{ odd}} \frac{1}{n^3} + \sum_{n\equiv_4 2} \frac{2}{n^3}$$

Problem 3

$f_k(x) = x \arctan kx$.

1. $\lim_{k\to\infty} \arctan kx$ is 0 if $x = 0$, $\frac{\pi}{2}$ if $x > 0$, and $-\frac{\pi}{2}$ if $x < 0$, and so $\lim_{k\to\infty} x \arctan kx = \frac{\pi}{2}|x|$.
2. $f_k'(x) = \arctan kx + kx\frac{1}{1+(kx)^2}$. If $x > 0$, then the limit is $\frac{\pi}{2}$; if $x = 0$, then the limit is 0, and if $x < -\frac{\pi}{2}$. This is not continuous at 0, and so it is impossible that the $f_k'(x)$ uniformly converge on an interval containing 0 because continuous uniformly converging functions must converge to a continuous function.

Problem 5

For $x \neq \pm 1, \pm 2, \ldots$, $f(x) = 2x\sum_1^\infty \frac{1}{x^2-n^2}$. We want to show that f is C^1 on its domain and that $f'(x) = -\sum_1^\infty \frac{1}{(x-n)^2} + \frac{1}{(x+n)^2}$.

Theorem 7.13 tells us that if the f_n's are C^1 on a compact interval and if the series $\sum f_n'$ converges uniformly on that compact interval, then the $\sum f_n$ is C^1 on the interval and has derivative equal to $\sum f_n'$. The f_n' are obviously C^1 on any compact interval not including a non-zero integer, so all we need to show is that $\sum f_n'$ is uniformly convergent.

$f_n(x) = \frac{2x}{x^2-n^2} = \frac{1}{x+n} + \frac{1}{x-n} \implies f_n'(x) = -\frac{1}{(x+n)^2} - \frac{1}{(x-n)^2}$. On any compact interval not including a non-zero integer, the series is uniformly convergent: because the interval is bounded, there must be N s.t. $|x| < N$ for all x in the interval. Then for every $n \geq N$, $0 \leq n^2 - N^2 \leq n^2 - x^2 \implies \frac{1}{n^2-x^2} \leq \frac{1}{n^2-N^2} \implies \frac{1}{(n^2-x^2)^2} \leq \frac{1}{(n^2-N^2)^2}$. Thus for every $n \geq N$,

$$|f_n'(x)| = \left|\frac{x^2 - 2xn + n^2 + x^2 + 2xn + n^2}{(x^2-n^2)^2}\right| = \left|\frac{2x^2+2n^2}{(x^2-n^2)^2}\right| < \frac{2N^2+2n^2}{(n^2-N^2)^2} =: M_n$$

and because $\frac{M_n}{1/n^2}$ tends to 2, we have by the limit comparison test that $\sum_N^\infty M_n$ is finite. Adding back the first $N-1$ terms doesn't change this, and so we have by the M-test that the series converges uniformly.

Problem 6

Let f be continuous function on $[0,\infty)$ s.t. $0 \leq f(x) \leq Cx^{-1-\epsilon}$ for some $C, \epsilon > 0$ and let $a = \int_0^\infty f(x)\,dx$. Let $f_k(x) = kf(kx)$.

(a) $f_k(x) = kf(kx)$ is ≥ 0 and $\leq k \cdot C(kx)^{-1-\epsilon} = C \cdot 1/k^\epsilon \cdot \frac{1}{x^{1+\epsilon}}$ which goes to 0 as $k \to \infty$ for all $x > 0$. If we only look at $[\delta, \infty)$, then we can bound $C \cdot 1/k^\epsilon \cdot \frac{1}{x^{1+\epsilon}} \leq C \cdot 1/k^\epsilon \cdot \frac{1}{\delta^{1+\epsilon}}$ and then we have a uniform bound that tends to 0 as $k \to \infty$ and so $f_k(x)$ is uniformly convergent on $[\delta, \infty)$ for any $\delta > 0$.

(b) $\int_0^1 f_k(x)\,dx$ undergoing $u = kx \implies du = k\,dx$ is the same as $\int_0^k f(u)\,du$ and so $\lim_{k\to\infty} \int_0^1 f_k(x)\,dx = \lim_{k\to\infty} \int_0^k f(u)\,du = a$.

(c) We want to show that for any integrable function g continuous at 0,

$$\lim_{k\to\infty} \int_0^1 f_k(x)g(x)\,dx = ag(0)$$

For any $\eta > 0$, we can find $\delta > 0$ s.t. $x \in [0,\delta] \implies |g(x) - g(0)| < \eta$. We can split the integral into

$$\int_0^1 f_k(x)g(x)\,dx = \int_0^1 g(0)f_k(x)\,dx + \int_0^\delta f_k(x)\big(g(x)-g(0)\big)\,dx + \int_\delta^1 f_k(x)\big(g(x)-g(0)\big)\,dx$$

and so

$$\left|\int_0^1 f_k(x)g(x)\,dx - g(0)a\right| \leq \left|\int_0^1 g(0)f_k(x)\,dx - g(0)a\right| + \left|\int_0^\delta f_k(x)\big(g(x)-g(0)\big)\,dx\right| + \left|\int_\delta^1 f_k(x)\big(g(x)-g(0)\big)\,dx\right|$$

Regarding the first term, for any ϵ_1 there is K_1 s.t. $g(0)\int_0^1 f_k(x)\,dx$ is $> g(0)(a-\epsilon_1)$ and $< g(0)(a+\epsilon_1)$ for $k \geq K_1$ (we know this from (b)). This means that $-\epsilon_1 < g(0)\int_0^1 f_k(x)\,dx < \epsilon_1$,

and so the first term is $< \epsilon_1$. The second term is (again for $k \geq K_1$)

$$\left| \int_0^\delta f_k(x)\big(g(x) - g(0)\big)\, dx \right| \leq \int_0^\delta f_k(x)|g(x) - g(0)|\, dx < \eta \int_0^\delta f_k(x) \leq \eta \int_0^1 f_k(x)\, dx < \eta(a + \epsilon_1)$$

Finally g is integrable and hence is bounded, meaning that $|g(x)| < M$ for some M. Thus $|g(x) - g(0)| \leq |g(x)| + |g(0)| < 2M$. The third term is therefore

$$\begin{aligned} \left| \int_\delta^1 f_k(x)\big(g(x) - g(0)\big)\, dx \right| &\leq \int_\delta^1 f_k(x)|g(x) - g(0)|\, dx < 2M \int_\delta^1 f_k(x)\, dx \\ &\leq 2M \int_\delta^1 C\frac{1}{k^\epsilon}\frac{1}{x^{1+\epsilon}}\, dx = 2CM \int_\delta^1 \frac{1}{x^{1+\epsilon}}\, dx \cdot \frac{1}{k^\epsilon} \end{aligned}$$

which tends to 0, meaning that for all $k \geq K_2$, this third term is bound by ϵ_2. Thus in total we have that for $k \geq \max\{K_1, K_2\}$

$$\left| \int_0^1 f_k(x)g(x)\, dx - g(0)a \right| \leq \epsilon_1 + \eta(a + \epsilon_1) + \epsilon_2$$

The ϵ's and η's can get arbitrarily small (pushing the K_1 and K_2 larger and larger), and so we have that

$$\lim_{k\to\infty} \left| \int_0^1 f_k(x)g(x)\, dx - g(0)a \right| = 0 \implies \lim_{k\to\infty} \int_0^1 f_k(x)g(x)\, dx = g(0)a$$

§7.3 - Problem 1

We are given $\{a_n\}_0^\infty$ a sequence of real or complex numbers.

(a) Suppose that $|a_{n+1}/a_n|$ converge to L as $n \to \infty$. Observe that

$$\frac{|a_{n+1}x^{n+1}|}{|a_n x^n|} = |x| \cdot |a_{n+1}/a_n| \to L|x|$$

Then, by the ratio test, the series $\sum_0^\infty a_n x^n$ is absolutely convergent for $|x| < \frac{1}{L}$ and divergent for $|x| > \frac{1}{L}$. This is exactly the definition of having a radius of convergence of $\frac{1}{L}$.

(b) Suppose that $|a_n|^{1/n}$ converge to L as $n \to \infty$. Observe that

$$\sqrt[n]{|a_n x^n|} = \sqrt[n]{|a_n|}\sqrt[n]{|x|^n} = \sqrt[n]{|a_n|}|x| \to L$$

By the root test, the series $\sum_0^\infty a_n x^n$ is absolutely convergent for $|x| < \frac{1}{L}$ and divergent for $|x| > \frac{1}{L}$. This is exactly the definition of having a radius of convergence of $\frac{1}{L}$.

Problem 3

Suppose that the radius of convergence of $\sum_0^\infty a_n x^n$ is R. This means that this converges absolutely for $|x| < R$ and diverges for $|x| > R$. So for $k \in \mathbb{N}$, $\sum_0^\infty a_n (x^k)^n$ converges absolutely for $|x^k| <$

$R \iff |x|^k < R \iff |x| < R^{1/k}$ and diverges for $|x^k| > R \iff |x|^k > R \iff |x| > R^{1/k}$. But $\sum_0^\infty a_n x^{kn} = \sum_0^\infty a_n (x^k)^n$! And so $\sum_0^\infty a_n x^{kn}$ has a radius of convergence of $R^{1/k}$.

Problem 7

Suppose we have a power series $f(x) = \sum_0^\infty a_n x^n$ with positive radius of convergence. If $f(-x) = f(x)$. then $\sum_0^\infty a_n (-x)^n = \sum_0^\infty a_n (-1)^n \cdot x^n = \sum_0^\infty a_n x^n$. Because $f(x)$ was absolutely convergent with positive radius of convergence, both these sums are also absolutely convergent, and they must agree on the entire set on which $f(x)$ was absolutely convergent. By Corollary 7.22, if two series agree on for $|x| < R$, $R > 0$, then their coefficients must be exactly equal for all n. And so $a_n(-1)^n = a_n$, meaning that for all odd n, $a_n = 0$.

Similarly, if $f(-x) = -f(x)$, $\sum_0^\infty a_n (-x)^n = \sum_0^\infty a_n (-1)^n \cdot x^n = \sum_0^\infty -a_n x^n$, and as above $a_n(-1)^n = -a_n$ for all n. This means that for all even n, $a_n = 0$.

Problem 9

$$1 + \frac{x^3}{2 \cdot 3} + \frac{x^6}{2 \cdot 3 \cdot 5 \cdot 6} + \ldots + \frac{x^{3n}}{2 \cdot 3 \cdot 5 \cdot 6 \cdots (3n-1)(3n)} + \ldots$$

has

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|x|^3}{(3n+3)(3n+2)}$$

which tends to 0 as $n \to \infty$. Thus the series converges absolutely for all x. Furthermore, differentiating twice, the n-th term becomes

$$\frac{x^{3n-2}}{2 \cdot 3 \cdot 5 \cdot 6 \cdots (3n-4)(3n-3)} = \frac{x^{3(n-1)+1}}{2 \cdot 3 \cdot 5 \cdot 6 \cdots (3n-4)(3n-3)} = x \frac{x^{3(n-1)}}{2 \cdot 3 \cdot 5 \cdot 6 \cdots (3n-4)(3n-3)}$$

which is exactly the $(n-1)$-th term of the original sequence. Because we have infinitely many terms and the first 1 dies after differentiating once, we get that $a''_{n+1} = x a_n \implies f''(x) = x f(x)$.

Problem 11

$f(x) = \int_0^x \arctan t \, dt$

(a) Integrating by parts (recalling that $\arctan'(x) = \frac{1}{1+x^2}$, we get $f(x) = x \arctan x - \frac{1}{2} \ln(1 + x^2)$.

(b) We know from Example 2 that $\arctan x = \sum_0^\infty (-1)^n \frac{x^{2n+1}}{2n+1}$ on $(-1, 1)$. At $x = 1$ and -1 respectively, the sums are $\sum_0^\infty (-1)^n \frac{1}{2n+1}$ and $\sum_0^\infty (-1)^{n+1} \frac{1}{2n+1}$, and we know by the alternating series test that these both converge conditionally. By Abel's Theorem (7.26), the series converges uniformly on $[-1, 0]$ and $[0, 1]$, and so it converges uniformly on $[-1, 1]$ to a continuous function on $[-1, 1]$. Thus, because the series is equal exactly to $\arctan x$ on $(-1, 1)$, the series must equal $\arctan x$ on $[-1, 1]$ due to continuity of $\arctan x$.

Finally, from Theorem 7.13, we know that the integral of the series is the sum of the integral of each of the terms; and so we get that

$$f(x) = \sum_0^\infty (-1)^n \frac{x^{2n+2}}{(2n+2)(2n+1)}$$

(c) Plugging in $x = 1$, we get that

$$\frac{\pi}{4} - \frac{\ln 2}{2} = \sum_0^\infty (-1)^n \frac{1}{(2n+2)(2n+1)} = \sum_0^\infty (-1)^n \frac{1}{2n+1} - \frac{1}{2n+2} = 1 - \frac{1}{2} + \left(-\frac{1}{3} + \frac{1}{4}\right) + \dots$$

335 Homework 6

Daniel Rui - 2/13/20

§6.4 - Problem 5

$$\sum_0^\infty \frac{(-1)^n(x-4)^n}{(2^n-3)\ln(n+3)}$$

This converges absolutely for $|x-4|<2$: $\frac{|a_n|}{|a_{n-1}|} = \frac{|(x-4)(2^{n-1}-3)\ln(n+2)|}{|(2^n-3)\ln(n+3)|} = |x-4|\left|\frac{2^{n-1}-3}{2^n-3}\right|\left|\frac{\ln(n+2)}{\ln(n+3)}\right| = |x-4|\left|\frac{\frac{1}{2}-3/(2^n)}{1-3/(2^n)}\right|\left|\frac{\ln(n+2)}{\ln(n+3)}\right|$. The limit as $n\to\infty$ of $\frac{|a_n|}{|a_{n-1}|}$ is thus $\frac{|x-4|}{2}$ which is <1 for $|x-4|<2$. For $|x-4|>2$, the sum becomes $\sum_0^\infty \frac{(-1)^n(x-4)^n}{(2^n-3)\ln(n+3)} = \sum_0^\infty \frac{(-1)^n(\frac{x-4}{2})^n}{(1-3/2^n)\ln(n+3)}$, the terms of which do not go to zero (so the sum can not possible converge, absolutely or conditionally).

Now for $x=2$, the sum becomes $\sum_0^\infty \frac{(-1)^n(-2)^n}{(2^n-3)\ln(n+3)}$. However, notice that $\frac{2^n}{(2^n-3)\ln(n+3)} \geq \frac{1}{\ln(n+3)}$, and $\sum_0^\infty \frac{1}{\ln(n+3)}$ diverges.

For $x=6$, the sum is $\sum_0^\infty (-1)^n \frac{2^n}{(2^n-3)\ln(n+3)}$, which is conditionally convergent by the alternating series test: $\frac{2^n}{(2^n-3)\ln(n+3)}$ is decreasing because its derivative $-\frac{2^n(2^n+3n\ln(2)\ln(n+3)+9\ln(2)\ln(n+3)-3)}{(2^n-3)^2(n+3)\ln^2(n+3)}$ is clearly negative for all large n, and it goes to 0 because $\frac{2}{\ln(n+3)} \geq \frac{2^n}{(2^n-3)\ln(n+3)} \geq 0$.

Problem 11

Regarding absolute convergence, note that $x>1 \implies \ln(x+7)>1 \implies \frac{\ln(x+7)}{x} > \frac{1}{x} \implies \int_n^{n+1}\frac{\ln(x+7)}{x}\,dx > \int_n^{n+1}\frac{1}{x}\,dx > \frac{1}{n+1}$, and so

$$\sum_1^M \int_n^{n+1} \frac{\ln(x+7)}{x}\,dx > \sum_1^M \frac{1}{n+1}$$

which diverges as $M\to\infty$. Second, notice that because $\frac{\ln(x+7)}{x}$ is decreasing (its derivative is $\frac{1}{x^2}(\frac{x}{x+7}-\ln(x+7))$ which is <0 because $\frac{x}{x+7}<1$ and $\ln(x+7)>1$), we have that

$$\int_n^{n+1}\frac{\ln(x+7)}{x}\,dx \geq \frac{\ln(n+1+7)}{n+1} \geq \int_{n+1}^{n+2}\frac{\ln(x+7)}{x}\,dx$$

We also know by (2.12 Corollary) that $\lim_{n\to\infty}\frac{\ln(n+1+7)}{n+1} = 0$, and so we have an alternating series whose terms are decreasing and going to 0, and so by the alternating series test this converges conditionally.

Problem 14

$$\sum_1^\infty (-1)^{n-1}\left[e-\left(\frac{n+1}{n}\right)^n\right]$$

Recall that (by Taylor's theorem because $\ln x$ and e^x are both C^∞ on $(-1/2, 1/2)$ and $(1/2, 3/2)$ resp.), for all x in $(-1/2, 1/2)$, there is some $c \in (-1/2, 1/2)$ such that $\ln(1+x) = x - \frac{1}{2}x^2 + f^{(3)}(c)\frac{x^3}{3!}$. Thus for all $n > 2$, $\ln(1+\frac{1}{n}) = \frac{1}{n} - \frac{1}{2n^2} + f^{(3)}(c_n)\frac{1}{6n^3} \implies n\ln(1+\frac{1}{n}) = 1 - \frac{1}{2n} + f^{(3)}(c_n)\frac{1}{6n^2}$ and so e to the both sides yields $(1+\frac{1}{n})^n = e \cdot e^{-1/(2n)+f^{(3)}(c_n)\frac{1}{6n^2}}$. Note that the exponent $|-1/(2n) + f^{(3)}(c_n)\frac{1}{6n^2}|$ is bounded by some finite constant C because $f^{(3)}(c_n)$ is bounded by say C' (remember $f = \ln(1+x)$ on $(-1/2, 1/2)$).

Then, Taylor's theorem also gives us that for all $x \in (-C, C)$, there is some $d \in (-C, C)$ such that $e^x = 1 + x + e^d\frac{x^2}{2}$, and so

$$\left(1+\frac{1}{n}\right)^n = e \cdot e^{-1/(2n)+\frac{f^{(3)}(c_n)}{6n^2}} = e\left(1 - \frac{1}{2n} + \frac{f^{(3)}(c_n)}{6n^2} + \frac{e^{d_n}}{2}\left(-\frac{1}{2n} + \frac{f^{(3)}(c_n)}{6n^2}\right)^2\right)$$

and thus

$$e - \left(1+\frac{1}{n}\right)^n = e\left(\frac{1}{2n} - \frac{f^{(3)}(c_n)}{6n^2} - \frac{e^{d_n}}{2}\left(-\frac{1}{2n} + \frac{f^{(3)}(c_n)}{6n^2}\right)^2\right)$$

Recognize that $\sum_{n=1}^\infty \frac{f^{(3)}(c_n)}{6n^2}$ is absolutely convergent because $|\frac{f^{(3)}(c_n)}{6n^2}| \leq \frac{C'}{6n^2}$ which is absolutely convergent by the p-test. Similarly $\sum_{n=1}^\infty \frac{e^{d_n}}{2}\left(-\frac{1}{2n} + \frac{f^{(3)}(c_n)}{6n^2}\right)^2$ is absolutely convergent because the exponents of the n's in the denominators are all ≥ 2, and the constants hanging around don't change any of that. Lastly $\sum_{n=1}^\infty \frac{1}{2n}$ is NOT absolutely convergent, but IS conditionally convergent.

A sum of conditionally convergent series is still convergent (I proved something very similar last homework, it's just looking at partial sums and taking limits), and so $\sum_{n=1}^\infty e - \left(1+\frac{1}{n}\right)^n$ is conditionally convergent. However, it is not absolutely convergent, because if it were, then we could subtract the two other sums from the RHS and move them to the LHS to get that a sum of absolutely convergent series is equal to a series that is not absolutely convergent (i.e. $\sum_{n=1}^\infty \frac{1}{2n}$), which is a contradiction.

Problem 16

Suppose $\sum a_n$ is a convergent series and $\{b_n\}$ is decreasing sequence of positive numbers, we want to show that $\sum a_n b_n$ converges. Because $\lim b_n$ need not be zero, let $\lim b_n = c$. Then, we know that the altered sequence $\{b_n - c\}$ is a decreasing sequence of positive numbers that go to 0. Thus by Dirichlet's test (because $\sum a_n$ is convergent), $\sum a_n(b_n - c)$ is convergent, and because $\sum a_n \cdot c = c\sum a_n$ is also convergent, we know that $\sum a_n b_n$ is convergent (adding two convergent series is convergent).

Problem 17

If $\sum_1^\infty a_n$ converges, then $\sum_1^\infty n^{-p}a_n$ does too for any $p > 0$ by Dirichlet's test (because n^{-p} is decreasing to 0 and $\sum_1^\infty a_n$ converges). For $p > 1$, the series definitely absolutely converges (because $\sum_1^\infty a_n$ converges $\implies a_n \to 0 \implies |a_n| \leq C$ for some C, and so $\sum_1^\infty |n^{-p}a_n| \leq C\sum_1^\infty n^{-p}$ which converges.) For any $p \leq 1$, we can't say anything for sure; for example take $a_n = (-1)^n 1/(\ln n)$ which converges by the alternating series test, but for any $p \leq 1$ we have that $\sum_1^\infty |n^{-p}a_n| = \sum_1^\infty n^{-p}/(\ln n) \geq$

$\sum_1^\infty n^{-1}/(\ln n)$ which we know diverges by Problem 20 from Homework 5.

§6.5 - Problem 3

If we take $f(x) = (1-4x)^{-1/2}$, the first derivative is $f'(x) = -\frac{1}{2}(-4)(1-4x)^{-3/2}$ and so continuing the pattern we get that $f^{(n)}(x) = \left(\prod_{i=1}^n 4 \cdot \frac{2i-1}{2}\right)(1-4x)^{-(2n+1)/2}$. Evaluating at $x=0$ we get that $f^{(n)}(0) = 2\prod_{i=1}^n (2i-1) = 2^n \cdot \frac{(2n)!}{2^n n!} = \frac{(2n)!}{n!}$ and so the (formal) Taylor series around $x=0$ would be $\sum_0^\infty (2n)!x^n/(n!)^2$. This series converges absolutely for $|x| < \frac{1}{4}$ by the ratio test:

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\frac{(2n+2)!}{(n+1)!^2}|x^{n+1}|}{\frac{(2n)!}{(n!)^2}|x^n|} = \frac{(2n+2)(2n+1)}{(n+1)^2}|x| \implies \lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = 4|x| < 1$$

Taking for granted that the sum of this series actually is $(1-4x)^{-1/2}$ for $|x| < \frac{1}{4}$ (by the book's recommendation), we multiply the series by itself to get that (by geometric series, i.e theorem 6.3)

$$\frac{1}{1-4x} = 1 + 4x + (4x)^2 + \ldots = \sum_{j=0}^\infty \left(\sum_{m+n=j} a_m a_n\right) x^j = \sum_{j=0}^\infty \left(\sum_{n=0}^j a_n a_{j-n}\right) x^j$$

and so comparing x_j terms we get that

$$\sum_{n=0}^j \frac{(2n)!(2j-2n)!}{(n!)^2((j-n)!)^2} = 4^j$$

Problem 4

$\sum_0^\infty (-1)^n(n+1)^{-1/2}$ is conditionally convergent by the alternating series test (because $\frac{1}{\sqrt{n+1}}$ is decreasing to 0), but the Cauchy product of this series with itself diverges:

$$\left(\sum_0^M \frac{(-1)^n}{\sqrt{n+1}}\right)\left(\sum_0^M \frac{(-1)^n}{\sqrt{n+1}}\right) \geq \sum_{j=0}^M \left(\sum_{n=0}^j \frac{(-1)^n}{\sqrt{n+1}}\frac{(-1)^{j-n}}{\sqrt{j-n+1}}\right) = \sum_{j=0}^M (-1)^j \sum_{n=0}^j \frac{1}{\sqrt{(n+1)(j-n+1)}}$$

But the maximum of the simple parabola $f(x) = (x+1)(j-x+1)$ occurs at $x = \frac{1}{2}j$, and hence $(n+1)(j-n+1) \leq (\frac{1}{2}j+1)^2 \iff \frac{1}{\sqrt{(n+1)(j-n+1)}} \geq \frac{1}{1+j/2}$ for $n = 0, ..., j$, and so $\sum_{n=0}^j \frac{1}{\sqrt{(n+1)(j-n+1)}} \geq \frac{j}{1+j/2}$, which does not approach 0 as $j \to \infty$. Thus, the above sum can not possible converge because the terms don't have limit 0.

Problem 7

We want to show that $\sum_{m,n=1}^\infty \frac{1}{(m+n)^p}$ converges iff $p > 2$. We take the hint and use $\triangle$ partial sums:

$$\sum_{i=1}^k \sum_{m+n=i} \frac{1}{(m+n)^p} = \sum_{i=1}^k i \cdot \frac{1}{i^p} = \sum_{i=1}^k \frac{1}{i^{p-1}}$$

We know by the p-test that this sum converges if and only if $p > 2$.

335 Homework 5

Daniel Rui - 2/6/20

§6.2 - Problem 2

$\sum_{n=1}^{\infty} ne^{-n}$: note that $\frac{(n+1)e^{-(n+1)}}{ne^{-n}} = \frac{n+1}{n} \cdot \frac{1}{e}$, and furthermore that $\lim_{n\to\infty} \frac{n+1}{n}$ exists and equals 1. Thus, $\lim_{n\to\infty} \frac{(n+1)e^{-(n+1)}}{ne^{-n}} = \frac{1}{e} < 1$. Thus, by the ratio test, this series converges.

Problem 5

$\sum_{n=0}^{\infty} \frac{(2n+1)^{3n}}{(3n+1)^{2n}}$: note that $a_n^{1/n} = \frac{(2n+1)^3}{(3n+1)^2} = \frac{8n^3+c_1n^2+c_2n+c_3}{9n^2+c_4n+c_5}$ for some $c_1, \ldots, c_5 \in \mathbb{R}$. Dividing top and bottom by n^2, we get $\frac{8n+c_1+c_2\frac{1}{n}+c_3\frac{1}{n}}{9+c_4\frac{1}{n}+c_5\frac{1}{n^2}}$. As $n \to \infty$, this clearly goes to ∞ as well. Thus, by the root test, this being ≥ 1 for all sufficiently large n implies that the series diverges.

Problem 12

$\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$: note that $a_n^{1/n} = (\frac{n}{n+1})^n = (\frac{n+1}{n})^{-n} = \left((1+\frac{1}{n})^n\right)^{-1}$. Because $\lim_{n\to\infty}(1+\frac{1}{n})^n$ famously exists and equals e, the limit of $a_n^{1/n}$ exists and equals $\frac{1}{e} < 1$. Thus, by the root test, the series converges.

Problem 13

$\sum_{n=1}^{\infty}[1-\cos(1/n)]$: for any $x \in [0,1]$, (recalling that $\cos x$ is C^∞) we know by Taylor's theorem with Lagrange's remainder that $\cos(x) = 1 - \frac{x^2}{2} + \sin(c)\frac{x^3}{3!}$ for some $c \in (0,x)$. Thus, for all $x \in [0,1]$, $\cos(x) \geq 1 - \frac{x^2}{2}$ because $\sin x \geq 0$ on $[0,1]$. Then, $1-\cos(1/n) \leq 1 - (1 - \frac{(1/n)^2}{2}) = \frac{1}{2n^2}$. Famously (i.e. common knowledge; otherwise use integral test), $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges $\iff \sum_{n=1}^{\infty} \frac{1}{2n^2}$ converges. Because term by term the original series is less than $\frac{1}{2n^2}$, the original series must converge as well.

Problem 19

Suppose $a_n > 0$; we want to show that $\sum a_n$ converges $\implies$ $\sum a_n^p$ converges for any $p > 1$. We know that because $\sum a_n$ converges, $\lim_{n\to\infty} a_n = 0$, we know that there is some N such that $n \geq N \implies a_n < 1$. $a_n < 1 \implies a_n^p < a_n$ for any $p > 1$. Thus, because term by term the series $\sum_N^{\infty} a_n^p$ is less than the original series, it must converge also. Finally, $\sum_1^{N-1} a_n^p$ is finite, and so adding that to $\sum_N^{\infty} a_n^p$ still keeps $\implies \sum a_n^p$ convergent.

Problem 20

By the integral test, $\sum_2^{\infty} \frac{1}{n\log^p(n)}$ (obviously positive, decreasing because $n\log^p n$ obviously increases) converges if and only if $\int_2^{\infty} \frac{1}{x\log^p(x)}\,dx$ converges. Now taking $u = \log(x) \implies du = \frac{1}{x}\,dx$,

$$\int_2^M \frac{1}{x\log^p(x)}\,dx = \int_{\ln 2}^{\ln M} \frac{1}{u^p}\,du$$

We know that as $M \to \infty$, this latter integral converges if and only if $p > 1$ and diverges if and only if $p \leq 1$. Thus, the original series converges if and only if $p > 1$ and diverges if and only if $p \leq 1$.

Problem 21

By the integral test, $\sum_4^\infty \frac{1}{n\log(n)(\log\log n)^p}$ (obviously positive, decreasing because $n\log n(\log\log n)^p$ obviously increases) converges if and only if $\int_2^\infty \frac{1}{x\log x(\log\log x)^p}\,dx$ converges. Now taking $u = \log\log(x) \implies du = \frac{1}{x\log x}\,dx$,

$$\int_4^M \frac{1}{x\log x(\log\log x)^p}\,dx = \int_{\ln\ln 4}^{\ln\ln M} \frac{1}{u^p}\,du$$

We know that as $M \to \infty$, this latter integral converges if and only if $p > 1$ and diverges if and only if $p \leq 1$. Thus, the original series converges if and only if $p > 1$ and diverges if and only if $p \leq 1$.

Problem 23

$f(x) = \frac{x}{(x^2+1)^2} \implies f'(x) = \frac{(x^2+1)^2 - x(2x)2(x^2+1)}{(x^2+1)^2} = \frac{x^2+1-4x^2}{(x^2+1)} = \frac{-3x^2+1}{x^2+1}$. $f'(x)$ is negative for all $x > 1/\sqrt{3} \approx 0.577$. Because $f(x)$ is positive and decreasing after $1/\sqrt{3}$, the series is convergent (so we can write ∞ in our sums and integrals), and $\int_k^\infty f(x)\,dx \leq \sum_k^\infty f(n) \leq f(k) + \int_k^\infty f(x)\,dx$ for $k \geq 1$. Observe that

$$\int_k^\infty \frac{x}{(x^2+1)^2}\,dx = \int_{k^2+1}^\infty \frac{1}{2u^2}\,du = \frac{1}{2(k^2+1)}$$

and so

$$\sum_1^3 f(n) + \int_4^\infty f(x)\,dx \leq \sum_1^\infty f(n) \leq \sum_1^3 f(n) + f(4) + \int_4^\infty f(x)\,dx$$

and with explicit computation:

$$0.389 \approx \frac{1}{4} + \frac{2}{25} + \frac{3}{100} + \frac{1}{34} \leq \sum_1^\infty f(n) \leq \frac{1}{4} + \frac{2}{25} + \frac{3}{100} + \frac{4}{289} + \frac{1}{34} \approx 0.4033$$

Problem 24

We want to show that $c_k = 1 + \frac{1}{2} + \ldots + \frac{1}{k} - \log k$. Before we do anything else, let's prove that $\ln(x) \leq x-1$ with equality only when $x = 1$. The inequality is equivalent to $x \leq e^{x-1}$ with equality only when $x = 1$, and this inequality is true because (recalling that e^x is C^∞) we know by Taylor's theorem with Lagrange's remainder that $e^{(x-1)} = 1+(x-1)+\frac{(x-1)^2}{2!}\cdot e^c$ for some c between 1 and x. Now for any k, we want to prove that $0 < c_k < c_{k-1}$. First, positivity: $\sum_1^k \frac{1}{n} \geq \int_1^{k+1} \frac{1}{x}\,dx = \ln(k+1)$, which implies that $\sum_1^k \frac{1}{n} - \ln(k) \geq \ln(k+1) - \ln(k) > 0$. Second, decreasing: $c_{k-1} > c_{k-1} + \frac{1}{k} + \ln(k-1) - \ln(k) = c_k$ because $\frac{1}{k} + \ln(\frac{k-1}{k}) < 0$ by my little lemma $\ln(x) \leq x-1$ above. Thus, c_k converges to some number.

§6.3 - Problem 1

(a) $\sum_0^\infty x^n\cos(n\theta)$ is absolutely convergent for $|x| < 1, \theta \in \mathbb{R}$: observe that $|x^n\cos(n\theta)| = |x|^n|\cos(n\theta)| \leq |x|^n$ (because cos is always less than 1). We know that $\sum_0^\infty |x|^n$ converges for any $|x| < 1$ by

geometric series, and so the original series converges absolutely (by direct comparison term by term).

(b) $\sum_0^\infty n^{-2}\sin(n\theta)$ is absolutely convergent for $\theta \in \mathbb{R}$: again, observe that $|n^{-2}\sin n\theta| \leq |n^{-2}| = n^{-2}$. We know that $\sum_0^\infty n^{-2}$ converges, and so the original series converges absolutely (by direct comparison term by term).

(c) $\sum_1^\infty (-1)^n n^2 3^{1-n} x^n$ is absolutely convergent for $|x| < 3$: denote $a_n = |(-1)^n n^2 3^{1-n} x^n| = n^2 3^{1-n}|x|^n$. Now note that $\frac{a_{n+1}}{a_n} = \frac{(n+1)^2 3^{-n}|x|^{n+1}}{n^2 3^{1-n}|x|^n} = \frac{(n+1)^2}{n^2} \cdot \frac{1}{3} \cdot |x|$. The limit of this as $n \to \infty$ is just $\frac{|x|}{3}$, which we know by the given condition to be < 1. Thus, by the ratio test, this series is absolutely convergent.

Problem 3

We know from Example 2 that $\sum_1^\infty (-1)^{n+1}\frac{1}{n} = \ln 2$, and so $\sum_1^\infty (-1)^{n+1}\frac{1}{2n} = \frac{1}{2}\ln 2$ (each partial sum is exactly half of the corresponding partial sum in $\sum_1^\infty (-1)^{n+1}\frac{1}{n}$ and so the limit is exactly half, given that the first's limit exists). Thus, for every partial sum,

$$\sum_1^M (-1)^{n+1}\frac{1}{n} + \sum_1^M (-1)^{n+1}\frac{1}{2n}$$

Problem 4

Denote $A_k = \sum_0^k a_n$ and $B_k = \sum_0^k b_n$. If k is odd, then we are summing exactly the same things, so $A_k = B_k$ for k odd. If k is even, then $A_k = A_{k-1} + a_k$ while $B_k = A_{k-1} + a_{k+1}$ and so $A_k = B_k - a_{k+1} + a_k$. But as $k \to 0$, the a_k go to 0 as well, so the difference between A_k and B_k will get arbitrarily small. Thus, in both cases the partial sums get arbitrarily close to each other, so the limits must be the same.

Problem 5

Suppose that all $a_n > -1$.

(a) We want to show that $\sum a_n$ converges absolutely if and only if $\sum \ln(1+a_n)$ is absolutely convergent.

($\implies$) Because the sum converges, we know that past some point N, $|a_n| < \frac{1}{2}$. Recalling that $\ln(x)$ is C^∞ on $(0,\infty)$, Taylor's theorem tells us that for any $x \in (1/2, 3/2)$, $\ln(1+x) = x + \frac{1}{2}f^{(2)}(c)x^2$ for some $c \in (-1/2, 3/2)$. $|f^{(2)}(x)|$ is obviously bounded on this set, say by M, and so $|\ln(1+x)| = |x + \frac{1}{2}f^{(2)}(c)x^2| \leq |x| + Mx^2 \leq |x| + M|x|$ for all $x \in (-1/2, 3/2)$. Thus, for all a_n with $n \geq N$, $|\ln(1+a_n)| \leq (M+1)|a_n|$. By term by term comparison, $\sum_N^\infty |\ln(1+a_n)|$ is convergent as well, and adding back $\sum_1^{N-1} |\ln(1+a_n)|$ keeps it convergent.

($\impliedby$) Similar to above, past some point N, $|\ln(1+a_n)| < \frac{1}{2}$. Note that for any $x \in (-1/2, 1/2)$, $e^x - 1 = x + \frac{1}{2}f^{(2)}(c)x^2$ for some $c \in (-1/2, 1/2)$ (because e^x is C^∞ everywhere). $|f^{(2)}(x)|$ is obviously bounded on this set, say by M, and so $|e^x - 1| \leq |x| + M|x^2| \leq (M+1)|x|$. Thus, for

all $n \geq N$, $|a_n| = |e^{\ln(1+a_n)} - 1| \leq (M+1)|\ln(1+a_n)|$. By term by term comparison, $\sum_N^\infty |a_n|$ is convergent as well, and adding back $\sum_1^{N-1} |a_n|$ keeps it convergent.

(b) $\sum_1^\infty (-1)^{n+1}/\sqrt{n}$ is conditionally convergent: $1/\sqrt{n}$ is obviously decreasing as n increases, and its limit is 0, and so by the alternating series test (6.22), the given series is (conditionally) convergent. However $\sum_1^\infty \ln(1+a_n)$ is not: by Taylor's theorem, $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{6}f^{(3)}(c)x^3$ for $x \in (1/2, 3/2)$ (a set on which $\ln(\cdot)$ is C^∞). Now suppose that $\sum_N^\infty \ln(1+a_n) = \sum_N^\infty a_n - \frac{1}{2n} + f^{(3)}(c)a_n^3$ was conditionally convergent. (6.18) gives us then that

335 Homework 4

Daniel Rui - 1/23/20

§5.7 - Problem 2

Let us check the hypotheses of Stokes's theorem for the problem:

(1) $\mathbf{F}(x,y,z) = (y, y^2, x+2z)$ is obviously a C^1 vector field everywhere. The curl is of course $(0-0, -(1-0), 0-1) = (0,-1,-1)$.

(2) Let S_0 be the smooth surface $y+z=a$ and let S be the intersection of the filled ball $x^2+y^2+z^2 \leq a^2$ with $y+z=a$. Let the orientation be where $\hat{\mathbf{n}}$ points away from the center of the ball. Furthermore, from the equation of the plane we see that $\hat{\mathbf{n}} = \frac{1}{\sqrt{2}}(0,1,1)$

(3) Then $C := \partial S$ is exactly the intersection of the sphere $x^2+y^2+z^2=a^2$ and the $y+z=a$. Orient it so we traverse it counterclockwise (viewed from above). We will prove that C is a circle with center at $(0, a/2, a/2)$ by finding the distance from any point of the intersection to this center:

$$\sqrt{x^2+(y-\tfrac{a}{2})^2+(z-\tfrac{a}{2})^2} = \sqrt{x^2+y^2-ay+\tfrac{a^2}{4}+z^2-az+\tfrac{a^2}{4}} = \sqrt{a^2-a(y+z)+\tfrac{a^2}{4}} = \frac{a}{\sqrt{2}}$$

A circle is piecewise smooth.

We can now apply Stokes's theorem to get that

$$\int_C [y\,dx + y^2\,dy + (x+2z)\,dz] = \iint_S \frac{1}{\sqrt{2}}(-2)\,dA = -\sqrt{2}\cdot \text{Area}(S) = -\sqrt{2}\frac{\pi a^2}{2} = -\frac{\pi a^2}{\sqrt{2}}$$

Problem 5

Let us check the hypotheses of Stokes's theorem for the problem:

(1) $\mathbf{F}(x,y,z) = (2x, 2y, x^2+y^2+z^2)$ is obviously a C^1 vector field everywhere. The curl is of course $(2y-0, -(2x-0), 0-0) = (2y, -2x, 0)$.

(2) Let S_0 be the smooth surface $(x^2/4)+(y^2/9)+(z^2/27)=1$ and let S be the lower half of this ellipsoid. Orient it so that $\hat{\mathbf{n}}$ points toward the origin.

(3) Then ∂S is the intersection of the ellipsoid with the xy-plane, i.e. the ellipse $(x^2/4)+(y^2/9)=1$. This is clearly piecewise smooth. Orient it so we traverse it counterclockwise (viewed from above).

(4) Define D to be the filled ellipse $(x^2/4)+(y^2/9) \leq 1$, which is a subset of the smooth surface of the xy-plane. Notice that $\partial D = \partial S$. Orient it so that $\hat{\mathbf{n}}$ points up (and notice that $\hat{\mathbf{n}} = (0,0,1)$.

We can now apply Stokes's theorem to get that

$$\iint_S \text{curl}\,\mathbf{F}\cdot\hat{\mathbf{n}}\,dA = \int_{\partial S}\mathbf{F}\cdot\,d\mathbf{x} = \iint_D \text{curl}\,\mathbf{F}\cdot\hat{\mathbf{n}}\,dA = \iint_D (2y,-2x,0)\cdot(0,0,1)\,dA = 0$$

Problem 6

(a) We are given $\mathbf{F}(x,y,z) = \frac{1}{x^2+y^2}(-y,x,0)$. The curl is hence

$$\text{curl } \mathbf{F} = \left(0-0, -(0-0), \frac{(x^2+y^2)-x(2x)}{(x^2+y^2)^2} + \frac{(x^2+y^2)-y(2y)}{(x^2+y^2)^2}\right) = \mathbf{0}$$

(b) By direct calculation, over any circle C on any plane parallel to the xy-plane centered at the z-axis parameterized as $(x,y,z) = (R\cos\theta, R\sin\theta, z_0) \implies (dx,dy,dz) = (-R\sin\theta\, d\theta, R\cos\theta\, d\theta, 0)$,

$$\int_C \mathbf{F}\cdot\, d\mathbf{x} = \int_0^{2\pi} -\frac{R\sin\theta}{R^2}(-R\sin\theta) + \frac{R\cos\theta}{R^2}(R\cos\theta)\, d\theta = 2\pi$$

(c) This does not contradict Stokes's theorem because for any surface S that has boundary C, S must intersect the z-axis, but $\mathbf{F}$ is not even defined at the origin, so it is definitely not C^1 on some open set containing S.

Problem 7

C_r denotes a circle of radius r about the origin in the xz-plane, oriented counterclockwise as viewed from the positive y-axis. We are given $\mathbf{F}$ as a C^1 vector field on the complement of the y-axis in $\mathbb{R}^3$ s.t. $\int_{C_1}\mathbf{F}\cdot\, d\mathbf{x} = 5$ and curl $\mathbf{F}(x,y,z) = \left(\frac{z}{(x^2+z^2)^2}, 3, \frac{-x}{(x^2+z^2)^2}\right)$. Let D_r be the corresponding disks in the xz-plane s.t. $\partial D_r = C_r$, oriented so $\hat{\mathbf{n}} = (0,1,0)$.

$$\int_{C_r}\mathbf{F}\cdot\, d\mathbf{x} = \int_{\partial D_r}\mathbf{F}\cdot\, d\mathbf{x} = \int_{\partial(D_r\setminus D_1)}\mathbf{F}\cdot\, d\mathbf{x} + \int_{\partial D_1}\mathbf{F}\cdot\, d\mathbf{x} = \int_{\partial(D_r\setminus D_1)}\mathbf{F}\cdot\, d\mathbf{x} + 5$$

$\mathbf{F}$ is C^1 in an open set containing any $D_r \setminus D_1$, and $D_r \setminus D_1$ is again a subset of the xz-plane (a smooth surface) where ∂D_r is oriented counterclockwise as viewed from the positive y-axis and where ∂D_1 oriented clockwise as viewed from the positive y-axis. Thus, we can apply Stokes's theorem to get that

$$\int_{\partial(D_r\setminus D_1)}\mathbf{F}\cdot\, d\mathbf{x} = \iint_{D_r\setminus D_1} 3\, dA = 3\text{Area}(D_r\setminus D_1) = 3\pi r^2 - 3\pi$$

and so $\int_{C_r}\mathbf{F}\cdot\, d\mathbf{x} = 3\pi r^2 - 3\pi + 5$. For $r < 1$, we see that

$$\int_{C_1}\mathbf{F}\cdot\, d\mathbf{x} = \int_{\partial(D_1\setminus D_r)}\mathbf{F}\cdot\, d\mathbf{x} + \int_{\partial D_r}\mathbf{F}\cdot\, d\mathbf{x} = \iint_{D_1\setminus D_r} 3\, dA + \int_{C_r}\mathbf{F}\cdot\, d\mathbf{x}$$

and so again $\int_{C_r}\mathbf{F}\cdot\, d\mathbf{x} = 5 - (3\pi - 3\pi r^2) = 3\pi r^2 - 3\pi + 5$.

Problem 8

We are given that S is a smooth oriented surface in $\mathbb{R}^3$ with a piecewise smooth compatibly oriented boundary ∂S, and that f and g are C^1 and C^2 resp. on some open set containing S. Let us define $\mathbf{F} = f\nabla g$ which is C^1 on an open set containing S because of f was C^1 and g was C^2, and so by

(5.26) and (5.30) curl $\mathbf{F} = f\text{curl}\,(\nabla g) + \nabla f \times \nabla g = \nabla f \times \nabla g$ and so by Stokes's theorem,

$$\int_{\partial S} f\nabla g \cdot\, d\mathbf{x} = \iint_S (\nabla f \times \nabla g) \cdot \hat{\mathbf{n}}\, dA$$

§5.8 - Problem 1

(a) $\mathbf{G}(x,y) = (2xy + x^2, x^2 - y^2)$ is in fact a gradient of a function: in order for the x partial to be $2xy + x^2$, $f(x,y)$ must be of the form $x^2y + \frac{1}{3}x^3 + g(y)$. Then the y partial would be of the form $x^2 + g'(y)$ which needs to equal $x^2 - y^2$. Thus, $f(x,y) = x^2y + \frac{1}{3}x^3 - \frac{1}{3}y^3 + C$. We can verify easily that the answer is correct by computing the gradient.

(c) $\mathbf{G}(x,y) = (2e^{2x}\sin y - 3y + 5, e^{2x}\cos y - 3x)$ is in fact a gradient of a function: in order for the x partial to be $2e^{2x}\sin y - 3y + 5$, $f(x,y)$ must be of the form $e^{2x}\sin y - 3xy + 5x + g(y)$. Then the y partial would be of the form $e^{2x}\cos y - 3x + g'(y)$ which needs to equal $e^{2x}\cos y - 3x$. Thus, $f(x,y) = e^{2x}\sin y - 3xy + 5x + C$. We can verify easily that the answer is correct by computing the gradient.

(g) $\mathbf{G}(x,y,z,w) = (xw^2 + yzw, xzw + yz^2 - 2e^{2y+z}, xyw + y^2z - e^{2y+z} - w\sin zw, xyz + x^2w - z\sin zw)$ is in fact a gradient of a function: in order for the x partial to be $xw^2 + yxw, xzw + yz^2 - 2e^{2y+z}$, $f(x,y,z,w)$ must be of the form $\frac{1}{2}x^2w^2 + xyzw + g(y,z,w)$. Then the y partial would be of the form $xzw + \partial_y g(y,z,w)$ which means that $g(y,z,w) = \frac{1}{2}y^2z^2 - e^{2y+z} + h(z,w)$. Now we can write $f(x,y,z,w) = \frac{1}{2}x^2w^2 + xyzw + \frac{1}{2}y^2z^2 - e^{2y+z} + h(z,w)$ and so the z partial would be $xyw + y^2z - e^{2y+z} + \partial_z h(z,w)$, which means that $h(z,w) = \cos zw + j(w)$. Finally, taking the w partial yields that $j(w) = C$. Thus, $f(x,y) = \frac{1}{2}x^2w^2 + xyzw + \frac{1}{2}y^2z^2 - e^{2y+z} + \cos zw + C$. We can verify easily that the answer is correct by computing the gradient.

Problem 2

(a) We are given $\mathbf{G}(x,y,z) = (x^3 + yz, y - 3x^2y, 4y^2)$. From theorem 5.63 and the preceding commentary, we know that if $\text{div}(\mathbf{G}) \neq 0$ it can't possible be the curl of some vector field. Well, $\text{div}(\mathbf{G}) = 3x^2 + (1 - 3x^2) + 0 = 1 \neq 0$ so $\mathbf{G}$ is not the curl of a vector field.

(b) We are given $\mathbf{G}(x,y,z) = (xy + z, xz, -yz - x)$. First note that $\mathbf{G}$ is a C^1 vector field on all of $\mathbb{R}^3$ (because it's just multiplying and adding x, y, z). Furthermore, $\text{div}(\mathbf{G}) = y + 0 - y = 0$ and so theorem 5.63 applies. From the algorithm described in the "proof" of theorem 5.63, we can start with a vector field $\mathbf{F} = (F_1, F_2, 0) \implies \text{curl}(\mathbf{F}) = (-\partial_z F_2, \partial_z F_1, \partial_x F_2 - \partial_y F_1)$ implying that

$$F_2 = -\int_0^z xy + t\, dt + g(x,y) = -xyz - \frac{1}{2}z^2 + g(x,y)$$

and

$$F_1 = \int_0^z xt\, dt + h(x,y) = \frac{1}{2}xz^2 + h(x,y)$$

We then take Folland's suggestion of setting $h(x,y) = 0$ and setting $g(x,y) = \int_0^x G_3(t,y,0)\, dt = \int_0^x -t\, dt = -\frac{1}{2}x^2$, yielding $\mathbf{F}(x,y,z) = (\frac{1}{2}xz^2, -xyz - \frac{1}{2}z^2 - \frac{1}{2}x^2, 0)$ as one such vector field.

(c) We are given $\mathbf{G}(x,y,z) = (xe^{-x^2z^2} - 6x, 5y + 2z, z - ze^{-x^2z^2})$. First note that $\mathbf{G}$ is a C^1 vector field on all of $\mathbb{R}^3$ (because it's just multiplying and adding and exponentiating x, y, z). Furthermore, $\operatorname{div}(\mathbf{G}) = e^{-x^2z^2} - 2x^2z^2e^{-x^2z^2} - 6 + 5 + 1 - e^{-x^2z^2} + 2x^2z^2e^{-x^2z^2} = 0$ and so theorem 5.63 applies. Like above, we can see that we can write one such vector field as $\mathbf{F} = (F_1, F_2, 0)$ where

$$F_2 = -\int_0^z xe^{-x^2t^2} - 6x\,dt + g(x,y) = -x\int_0^z e^{-x^2t^2}\,dt + 6xz + g(x,y)$$

and

$$F_1 = \int_0^z 5y + 2t\,dt + h(x,y) = 5yz + z^2 + h(x,y)$$

We then take Folland's suggestion of setting $h(x,y) = 0$ and setting $g(x,y) = \int_0^x G_3(t,y,0)\,dt = \int_0^x 0\,dt = 0$, yielding $\mathbf{F}(x,y,z) = (5yz + z^2, -x\int_0^z e^{-x^2t^2}\,dt + 6xz, 0)$ as one such vector field.

Problem 3

Problem 4

§6.1 - Problem 1d

We know from theorem 6.3 that the geometric series $\sum_{n=0}^{\infty} a^n$ converges if and only if $|a| < 1$, in which case the sum is $\frac{1}{1-a}$. Thus in our case $(\sum_{n=0}^{\infty}(\log x)^n) - (\log x)^0$ is convergent if and only if $|\log x| < 1 \iff x \in (1/e, e)$, and it converges to $\frac{\log x}{1-\log x}$.

Problem 2

(b)

(c)

(d) From theorem 6.1c, if the series is convergent, $\lim_{n\to\infty} a_n = 0$. However, in this example, every odd indexed term is 1 and every even indexed term is $1/n$, and so the limit does not even exist. Thus, this series is divergent.

335 Homework 3

Daniel Rui - 1/23/20

§4.7 - Problem 3

$$\varphi(\mathbf{x}) = \iiint_{\mathbb{R}^3} \frac{\rho(\mathbf{x}-\mathbf{y})}{4\pi|\mathbf{y}|}\, d^3\mathbf{y}$$

We want to show that this integral is absolutely convergent if ρ is continuous and vanishes outside a bounded set. Because ρ vanishes outside a bounded set S, say where all $\mathbf{p} \in S, |\mathbf{p}| < M$, we know that for any $\mathbf{x}$, we can find $R := M + |\mathbf{x}|$ s.t. $|\mathbf{y}| > R \implies |\mathbf{x}-\mathbf{y}| > M$ (proof is just reverse triangle inequality). Thus, we can simplify our triple integral to be over a ball of radius R because $\rho = 0$ outside of that ball. Furthermore, because ρ is continuous (everywhere), it is bounded on $\bar{S}$, where the bound is say $|\rho| < C$. Then, the integral becomes

$$\iiint_{\mathbb{R}^3} \left|\frac{\rho(\mathbf{x}-\mathbf{y})}{4\pi|\mathbf{y}|}\right| d^3\mathbf{y} \le \frac{C}{4\pi} \iiint_{B_R} \frac{1}{|\mathbf{y}|}\, d^3y$$

which we know to converge by Proposition 4.65.

Problem 4

We have $f(x,y) = (x^2-y^2)(x^2+y^2)^{-2}$, and $S = [0,1]\times[0,1]$.

(a) Considering T to be the quarter circle $r \in [0,1], \theta \in [0,\pi/2]$, we see that $T \subset S \implies \iint_T |f|\, dA \le \iint_S |f|\, dA$, and so

$$\begin{aligned}\iint_S |f|\, dA \ge \iint_T |f|\, dA &= \int_0^{\pi/2}\int_0^1 \frac{r^2|\cos^2(\theta)-\sin^2(\theta)|}{r^4} r\, dr\, d\theta \\ &= \int_0^{\pi/2} |\cos^2(\theta)-\sin^2(\theta)| \int_0^1 \frac{1}{r}\, dr\, d\theta = \infty\end{aligned}$$

again by Proposition 4.65 (or just elementary computation and limit taking).

(b) Observe that

$$\int_0^1\int_0^1 f(x,y)\, dx\, dy = \int_0^1\int_0^1 \frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2}\, dx\, dy$$

and so we make the substitution $x = y\tan t \implies dx = y\sec^2 t\, dt$

$$\begin{aligned}&= \int_0^1\int_0^1 \left(\frac{1}{y^2\tan^2 t + y^2} - \frac{2y^2}{(y^2\tan^2 t + y^2)^2}\right) y\sec^2 t\, dt\, dy \\ &= \int_0^1\int_0^{\arctan(1/y)} \left(\frac{\cos^2(t)}{y^2} - 2\frac{\cos^4(t)}{y^2}\right) y\sec^2 t\, dt\, dy \\ &= \int_0^1 \frac{1}{y}\int_0^{\arctan(1/y)} 1 - 2\cos^2(t)\, dt\, dy = \int_0^1 \frac{1}{y}\left(-\sin(t)\cos(t)\right)\Big|_0^{\arctan(1/y)}\, dy\end{aligned}$$

$$= \int_0^1 -\frac{1}{y}\frac{1}{\sqrt{y^2+1}}\frac{y}{\sqrt{y^2+1}}\,dy = \int_0^1 -\frac{1}{y^2+1}\,dy$$
$$= -\arctan(y)|_0^1 = -\frac{\pi}{4}$$

While on the other hand for the second integral we can make the substitution $u = y \implies du = dy$ and $v = x \implies dv = dx$

$$\int_0^1\int_0^1 f(x,y)\,dy\,dx = \int_0^1\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\,dy\,dx$$
$$= \int_0^1\int_0^1 \frac{v^2-u^2}{(u^2+v^2)^2}\,du\,dv = \int_0^1\int_0^1 -\frac{u^2-v^2}{(u^2+v^2)^2}\,du\,dv$$
$$= -\int_0^1\int_0^1 f(x,y)\,dx\,dy = \frac{\pi}{4}$$

§5.6 - Problem 1

With a uniform distribution of mass on the sphere of radius r about the origin, the gravitational potential is

$$u(\mathbf{x}) = -\iiint_{\mathbb{R}^3} \frac{\rho(\mathbf{p})}{|\mathbf{p}-\mathbf{x}|}\,d^3p = -\iint_{S_r} \frac{c}{|\mathbf{p}-\mathbf{x}|}\,dA$$

Let us rotate the function such that $\mathbf{x}$ now lies on the positive z-axis — the integral stays the same because the sphere is symmetric across rotations. Further justification would require determining a rotation function and proving that it has Jacobian determinant = 1. Once $\mathbf{x}$ is of the form $(0,0,z)$ where $z = |\mathbf{x}|$, let us now move to spherical coordinates:

$$-c\int_0^\pi\int_0^{2\pi} \frac{1}{\sqrt{(r\cos\theta\sin\phi)^2+(r\sin\theta\sin\phi)^2+(r\cos\phi-z)^2}} r^2\sin\phi\,d\theta\,d\phi$$
$$= -2\pi c\int_0^\pi \frac{r^2\sin\phi}{\sqrt{r^2\sin^2\phi+r^2\cos^2\phi-2rz\cos\phi+z^2}}\,d\phi$$
$$= -2\pi c\int_0^\pi \frac{r^2\sin\phi}{\sqrt{r^2-2rz\cos\phi+z^2}}\,d\phi\,dr = -2\pi c\left(\frac{r}{z}\sqrt{r^2-2rz\cos\phi+z^2}\right)\Big|_0^\pi$$
$$= -2\pi c\left(\frac{r}{z}\sqrt{r^2+2rz+z^2} - \frac{r}{z}\sqrt{r^2-2rz+z^2}\right)$$
$$= -2\pi c\left(\frac{r}{z}|r+z| - \frac{r}{z}|r-z|\right)$$

Note that

(a) inside the sphere where $z = |\mathbf{x}| < r$, the potential is $u(\mathbf{x}) = -2\pi c\frac{r}{z}(r+z-(r-z)) = -4\pi cr$ (i.e. constant) and the gravitational field $[\nabla u](\mathbf{x}) = 0$ (i.e. it vanishes).

(b) outside the sphere where $z = |\mathbf{x}| > r$, the potential is $u(\mathbf{x}) = -2\pi c\frac{r}{z}(r+z-(z-r)) = -4\pi cr^2/z = -4\pi cr^2/|\mathbf{x}|$, and the field is $[\nabla u](\mathbf{x}) = -4\pi cr^2\mathbf{x}/|\mathbf{x}|^3$. Compare this to if the entire mass were located at the origin; we would have the whole mass $= 4\pi cr^2$ times $1/|\mathbf{0}-\mathbf{x}| = 1/|\mathbf{x}|$ for the potential and again be $-4\pi cr^2\mathbf{x}/|\mathbf{x}|^3$.

Problem 2

With uniform distribution of mass on the solid ball of radius R about the origin, show that the gravitational field at a point $\mathbf{x}$ is the same as if the mass closer to the origin than $\mathbf{x}$ were all located at the origin and the mass farther from the origin than $\mathbf{x}$ were absent. We perform the volume integral now (again rotating so that $\mathbf{x}$ now is $(0,0,z)$):

$$
\begin{aligned}
&-c\int_0^{\pi}\int_0^{2\pi}\int_0^{R}\frac{1}{\sqrt{(r\cos\theta\sin\phi)^2+(r\sin\theta\sin\phi)^2+(r\cos\phi-z)^2}}r^2\sin\phi\,dr\,d\theta\,d\phi\\
&=-2\pi c\int_0^{\pi}\int_0^{R}\frac{r^2\sin\phi}{\sqrt{r^2\sin^2\phi+r^2\cos^2\phi-2rz\cos\phi+z^2}}\,dr\,d\phi\\
&=-2\pi c\int_0^{\pi}\int_0^{R}\frac{r^2\sin\phi}{\sqrt{r^2-2rz\cos\phi+z^2}}\,dr\,d\phi\,dr=-2\pi c\int_0^{R}\left(\frac{r}{z}\sqrt{r^2-2rz\cos\phi+z^2}\right)\Big|_0^{\pi}\,dr\\
&=-2\pi c\int_0^{R}\left(\frac{r}{z}\sqrt{r^2+2rz+z^2}-\frac{r}{z}\sqrt{r^2-2rz+z^2}\right)\,dr\\
&=-2\pi c\int_0^{R}\frac{r}{z}|r+z|-\frac{r}{z}|r-z|\,dr\\
&=-2\pi c\left(\int_0^{R}\frac{r}{z}(r+z)\,dr-\left(\int_0^{\min\{R,z\}}\frac{r}{z}(z-r)\,dr+\int_{\min\{R,z\}}^{R}\frac{r}{z}(r-z)\,dr\right)\right)\\
&=-2\pi c\left(\frac{R^3}{3z}+\frac{R^2}{2}-\frac{\min\{R,z\}^2}{2}+\frac{\min\{R,z\}^3}{3z}-\frac{R^3-\min\{R,z\}^3}{3z}+\frac{R^2-\min\{R,z\}^2}{2}\right)\\
&=-2\pi c\left(R^2-\frac{\min\{R,z\}^2}{2}+\frac{\min\{R,z\}^3}{3z}+\frac{\min\{R,z\}^3}{3z}-\frac{\min\{R,z\}^2}{2}\right)\\
&=-2\pi c\left(R^2-\min\{R,z\}^2+\frac{2\min\{R,z\}^3}{3z}\right)
\end{aligned}
$$

If $z<R$, then we get that $u(\mathbf{x})=-2\pi c(R^2-z^2/3)$, which would mean that $[\nabla u](\mathbf{x})=$

335 Homework 2

Daniel Rui - 1/16/20

§5.5 - Problem 2

Let $\mathbf{F}(x,y,z) = (x^2+y^2+z^2)(x\hat{i}+y\hat{j}+z\hat{k})$. Luckily, the book provides for us that the outward unit normal to S_a is $\frac{\mathbf{x}}{a}$. Thus, $\mathbf{F}\cdot\mathbf{n} = (x^2+y^2+z^2)^2/a = a^3$ (because on S, $x^2+y^2+z^2=a^2$). Thus,

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dA = a^3\iint_S dA = 4\pi a^5$$

If we use the divergence theorem, we see that $\operatorname{div}\mathbf{F} = (3x^2+y^2+z^2)+(x^2+3y^2+z^2)+(x^2+y^2+3z^2) = 5(x^2+y^2+z^2)$ so the formula yields (taking R to be the region inside S):

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dA = \iiint_R 5(x^2+y^2+z^2)\,dV = 5\int_0^{2\pi}\int_0^{\pi}\int_0^a r^2\cdot r^2\sin\varphi\,dr\,d\varphi\,d\theta = 4\pi a^5$$

Problem 3

Let $\mathbf{F}(x,y,z) = x\hat{i}+y\hat{j}+z\hat{k}$ and R be a regular region in $\mathbb{R}^3$ with a piecewise smooth boundary. Then $\operatorname{div}\mathbf{F} = 3$ so

$$\frac{1}{3}\iint_{\partial R}\mathbf{F}\cdot\mathbf{n}\,dA = \frac{1}{3}\iiint_R \operatorname{div}\mathbf{F}\,dV = \iiint_R 1\,dV = \operatorname{vol}(R)$$

Problem 4

Let $\mathbf{F} = fg\hat{i}$, so $\mathbf{F}\cdot\mathbf{n} = fgn_x$ and $\operatorname{div}\mathbf{F} = \partial_x(fg) = f\partial_x g + g\partial_x f$. By the divergence theorem,

$$\iint_{\partial R} fgn_x\,dA = \iiint_R f\partial_x g\,dV + \iiint_R g\partial_x f\,dV$$

Problem 5

We have a regular region in $\mathbb{R}^3$ with a piecewise smooth boundary R and a function $f\in C^2(\mathbb{R}^2)$.

(a)
$$\iint_{\partial R}\frac{\partial f}{\partial n}\,dA = \iint_{\partial R}\nabla f\cdot\mathbf{n}\,dA = \iiint_R \operatorname{div}(\nabla f)\,dV = \iiint_R \nabla^2 f\,dV$$

(b) 5.28 gives that $\operatorname{div}(f\nabla f) = |\nabla f|^2 + f\nabla^2 f$, but we are given that $\nabla^2 f = 0$, so $\operatorname{div}(f\nabla f) = |\nabla f|^2$. Defining $\mathbf{F} = f\nabla f$, the divergence theorem gives

$$\iint_{\partial R} f\nabla f\cdot\mathbf{n}\,dA = \iint_{\partial R} f\frac{\partial f}{\partial n}\,dA = \iiint_R |\nabla f|^2 + f\nabla^2 f\,dV$$

Problem 6

Let $\mathbf{x} = (x,y,z)$ and $g(\mathbf{x}) = |\mathbf{x}|^{-1}$.

(a) Computing the gradient (for $\mathbf{x} \neq 0$):

$$\nabla g(\mathbf{x}) = \begin{bmatrix} \frac{-x}{(x^2+y^2+z^2)^{3/2}} \\ \frac{-y}{(x^2+y^2+z^2)^{3/2}} \\ \frac{-z}{(x^2+y^2+z^2)^{3/2}} \end{bmatrix} = -\frac{\mathbf{x}}{|\mathbf{x}|^3}$$

(b) From 5.4.2c from last week's homework, the Laplacian of $f(\mathbf{x}) = |\mathbf{x}|^a$ ($\mathbf{x} \in \mathbb{R}^n \backslash \{\mathbf{0}\}, a \in \mathbb{R}$) is

$$\nabla^2 f(\mathbf{x}) = a(a-1)r^{a-2} + (n-1)ar^{a-2} = a(a+n-2)r^{a-2}$$

Here we have $a = -1, n = 3$ so the formula gives 0. Note that $\operatorname{div}(\nabla g) = \nabla^2 g$.

(c) Again from the book's remark that the $\mathbf{n} = \frac{\mathbf{x}}{r}$ on the surface of a sphere of radius r around the origin, and the fact that $|\mathbf{x}| = r$ on the surface of that sphere,

$$\iint_S \frac{\partial g}{\partial n}\, dA = \iint_S -\frac{\mathbf{x}}{|\mathbf{x}|^3} \cdot \frac{\mathbf{x}}{r}\, dA = \iint_S -\frac{|\mathbf{x}|^2}{r|\mathbf{x}|^3}\, dA = \iint_S -\frac{1}{r|^2}\, dA = -\frac{1}{r^2}(4\pi r^2) = -4\pi$$

(d) The problem is that $g(\mathbf{x})$ is not a C^1 function on the region inside the sphere; it is unbounded around $(0,0,0)$. Thus the divergence theorem can not be used.

(e) We take the book's hint and consider $R \setminus B$, where B is some ball centered at the origin, but contained strictly within R (which we can do because we are given that the origin is in the interior of R). $R \setminus B$ is still a regular region with a piecewise smooth boundary, so we use the divergence theorem:

$$\begin{aligned} \iint_{\partial R} \frac{\partial g}{\partial n}\, dA &= \iint_{\partial R} \frac{\partial g}{\partial n}\, dA - \iint_{\partial B} \frac{\partial g}{\partial n}\, dA + \iint_{\partial B} \frac{\partial g}{\partial n}\, dA \\ &= \iint_{\partial(R\setminus B)} \frac{\partial g}{\partial n}\, dA - 4\pi \\ &= \iiint_{R\setminus B} 0\, dV - 4\pi = -4\pi \end{aligned}$$

§4.5 - Problem 1

We have

$$f(x,y) = \begin{cases} x^3 y^{-2} e^{-x^2/y} & y > 0 \\ 0 & y \leq 0 \end{cases}$$

(a) For any fixed $y_0 \leq 0$, $f(x, y_0) = 0$, obviously C^1. For any fixed $y_0 > 0$, $f(x, y_0) = x^3 y_0^{-2} e^{-x^2/y_0}$ is just a constant times a polynomial times an exponential, so obviously C^1. Similarly, for any fixed x_0,

$$f(x_0, y) = \begin{cases} c_1 y^{-2} e^{-c_2/y} & y > 0 \\ 0 & y \leq 0 \end{cases}$$

where $c_1 = x_0^3$ and $c_2 = x_0^2$. Obviously this is continuous for $y > 0$ and $y < 0$, so we just need to focus on $y = 0$. The limit from the left is of course 0, so we just need to see if the following limit exists and equals 0. If it exists, then

$$\lim_{y\to 0^+} \frac{c_1 e^{-c_2/y}}{y^2} = \lim_{u\to\infty} c_1 u^2 e^{-c_2 u} = 0$$

by Corollary 2.12. Reading the equations from right to left, we can safely say that the desired limit exists and equals 0. Similarly, we can verify that $f'(x_0, 0)$ exists and equals 0:

$$\lim_{y\to 0^+} \frac{\frac{c_1 e^{-c_2/y}}{y^2} - 0}{y - 0} = \lim_{y\to 0^+} \frac{c_1 e^{-c_2/y}}{y^3} = 0 = \lim_{y\to 0^-} 0$$

where the details are the same as above. For $y > 0$, $f'(x_0, y) = \dfrac{c_1 e^{-c_2/y}(c_2 - 2y)}{y^4}$ and for $y < 0$, $f'(x_0, y) = 0$. Our final step is to prove that the derivative is continuous at $y = 0$:

$$\lim_{y\to 0^+} \frac{c_1 e^{-c_2/y}(c_2 - 2y)}{y^4} = \lim_{y\to 0^+} \frac{c_1 c_2 e^{-c_2/y}}{y^4} - \lim_{y\to 0^+} \frac{2c_1 e^{-c_2/y}}{y^3} = 0 = \lim_{y\to 0^-} 0$$

and so all the $f(x_0, y)$ are C^1. However, e.g. taking a path along $y = x^2$, we get $f(x, x^2) = \frac{1}{ex}$ which is obviously unbounded as $x \to 0$. Thus, although taking slices along each x_0 and y_0 the function is C^1, it is still unbounded around 0.

(b) $F(x) = \int_0^1 f(x, y)\, dy = xe^{-x^2/y}\Big|_{y=0}^{y=1} = xe^{-x^2}$ (where we are doing limit calculations and using Corollary 2.12 in the background). Thus evaluating, we get $F'(0) = e^{-0^2} - 2(0^2)e^{-0^2} = 1$. However,

$$\int_0^1 [\partial_x f](0, y)\, dy = \int_0^1 (3(0^2)y - 2(0^4))y^{-2}e^{-0^2/y}\, dy = 0$$

Problem 4

Define $h(x) = \dfrac{1}{2}\displaystyle\int_0^x \sin(2(x - y))g(y)\, dy$. Then, from 4.51, we know that

$$h'(x) = \frac{1}{2}\sin 2(x - x)g(x) + \int_0^x \cos(2(x - y))g(y)\, dy = \int_0^x \cos(2(x - y))g(y)\, dy$$

and

$$h''(x) = \cos 2(x - x)g(x) - 2\int_0^x \sin(2(x - y))g(y)\, dy = g(x) - 4h(x)$$

and so moving the $-4h(x)$ to the left side, we get $h''(x) + 4h(x) = g(x)$.

Problem 5

Given $F(x) = \int_{\psi(x)}^{\varphi(x)}$, we want to find $F'(x)$. Like in Folland's proof of 4.51, we just differentiate with respect to each x in turn while treating the others as constants and summing (the first two terms use

u-sub and the FTC and for the second term use differentiation under the integral sign):

$$F'(x) = f(x, \varphi(x))\varphi'(x) - f(x, \psi(x))\psi'(x) + \int_{\psi(x)}^{\varphi(x)} \partial_x f(x, y)\, dy$$

Problem 6

We want to prove that

$$f^{[n]}(x) = \frac{1}{(n-1)!} \int_0^x (x-y)^{n-1} f(y)\, dy$$

is the nth-order antiderivative of f. We proceed by induction. For $n = 1$, $f^{[1]}(x) = \int_0^x f(y)\, dy$, which is obviously the first order anti-derivative of f. Now we assume that $f^{[1]}, \dots f^{[n-1]}$ are the 1st to $(n-1)$th order antiderivatives. Then,

$$\begin{aligned}(f^{[n]})'(x) &= \frac{1}{(n-1)!}(x-x)^{n-1} f(x) + \frac{n-1}{(n-1)!} \int_0^x (x-y)^{n-2} f(y)\, dy \\ &= \frac{1}{(n-2)!} \int_0^x (x-y)^{n-2} f(y)\, dy = f^{[n-1]}(x)\end{aligned}$$

So $f^{[n]}$ is the antiderivative of $f^{[n-1]}$, and because we know that $f^{[n-1]}$ is the $(n-1)$th order antiderivative of f, $f^{[n]}$ is the nth order antiderivative of f.

Problem 7

We are given a continuous function f on [0,1] and

$$u(x,t) = t^{-1/2} \int_0^1 e^{-(x-y)^2/4t} f(y)\, dy \quad \text{and} \quad v(x,t) = t \int_0^1 \frac{f(y)}{(x-y)^2 + t^2}\, dy$$

. For these computations, we just use lots of differentiating under the integral sign:

(a) We have

$$\partial_t u = -\frac{t^{-3/2}}{2} \int_0^1 e^{-(x-y)^2/4t} f(y)\, dy + t^{-1/2} \int_0^1 \frac{(x-y)^2}{4t^2} \cdot e^{-(x-y)^2/4t} f(y)\, dy$$

and

$$\partial_x u = t^{-1/2} \int_0^1 -\frac{(x-y)}{2t} \cdot e^{-(x-y)^2/4t} f(y)\, dy$$

so

$$\partial_x^2 u = t^{-1/2} \int_0^1 -\frac{1}{2t} \cdot e^{-(x-y)^2/4t} f(y) + \frac{(x-y)^2}{4t^2} \cdot e^{-(x-y)^2/4t}\, dy$$

which of course equals $\partial_t u$.

(b) The x partials:

$$\partial_x v = t \int_0^1 \frac{0 - f(y)[2(x-y)]}{[(x-y)^2 + t^2]^2}\, dy$$

and

$$\begin{aligned}\partial_x^2 v &= t\int_0^1 \frac{-2f(y)[(x-y)^2+t^2]^2 + f(y)[2(x-y)]\cdot 2(x-y)(2[(x-y)^2+t^2])}{[(x-y)^2+t^2]^4}\,dy \\ &= t\int_0^1 \frac{-2f(y)}{[(x-y)^2+t^2]^2} + \frac{8f(y)(x-y)^2}{[(x-y)^2+t^2]^3}\,dy\end{aligned}$$

The t partials:

$$\begin{aligned}\partial_t v &= \int_0^1 \frac{f(y)}{(x-y)^2+t^2}\,dy + t\int_0^1 \frac{0-f(y)(2t)}{[(x-y)^2+t^2]^2}\,dy \\ &= \int_0^1 \frac{f(y)}{(x-y)^2+t^2}\,dy - 2t^2\int_0^1 \frac{f(y)}{[(x-y)^2+t^2]^2}\,dy\end{aligned}$$

and

$$\begin{aligned}\partial_t^2 v &= \int_0^1 \frac{-f(y)(2t)}{[(x-y)^2+t^2]^2}\,dy - 4t\int_0^1 \frac{f(y)}{[(x-y)^2+t^2]^2}\,dy - 2t^2\int_0^1 \frac{-f(y)(2t)(2[(x-y)^2+t^2])}{[(x-y)^2+t^2]^4}\,dy \\ &= -6t\int_0^1 \frac{f(y)}{[(x-y)^2+t^2]^2}\,dy + 8t^3\int_0^1 \frac{f(y)}{[(x-y)^2+t^2]^3}\,dy\end{aligned}$$

and so finally

$$\begin{aligned}\partial_x^2 v + \partial_t^2 v &= \int_0^1 \frac{-2tf(y)}{[(x-y)^2+t^2]^2} + \frac{8tf(y)(x-y)^2}{[(x-y)^2+t^2]^3} + \\ &\qquad \frac{-6tf(y)}{[(x-y)^2+t^2]^2}\,dy + \frac{8t^3 f(y)}{[(x-y)^2+t^2]^3}\,dy \\ &= \int_0^1 \frac{-8tf(y)}{[(x-y)^2+t^2]^2} + \frac{8tf(y)[(x-y)^2+t^2]}{[(x-y)^2+t^2]^3}\,dy \\ &= 0\end{aligned}$$

§4.6 - Problem 3e

We split the integral up into two parts:

$$\int_0^\infty x^{-1/5}\sin\left(\frac{1}{x}\right)\,dx = \int_0^1 x^{-1/5}\sin\left(\frac{1}{x}\right)\,dx + \int_1^\infty x^{-1/5}\sin\left(\frac{1}{x}\right)\,dx$$

Firstly, sin is always ≤ 1 so

$$\int_0^1 x^{-1/5}\sin\left(\frac{1}{x}\right)\,dx \leq \int_0^1 x^{-1/5}\,dx = \frac{5}{4}x^{4/5}\Big|_0^1$$

so that converges obviously. For the other integral, take $u = \frac{1}{x} \implies du = -\frac{1}{x^2}\,dx \implies -x^2\,du = dx$, so

$$\int_1^\infty x^{-1/5}\sin\left(\frac{1}{x}\right)\,dx = -\int_1^0 x^{-1/5+2}\sin(u)\,du \leq \int_0^1 \frac{1}{u^{9/5}}u\,du = 5u^{1/5}\Big|_0^1$$

where the inequality holds because $\sin(u) \leq u$ on [0,1] (by the MVT, for every $x \in [0,1]$, $\dfrac{\sin x - 0}{x-0} = \cos c$ for some $c \in [0,x]$; $\cos \leq 1$ on [0,1], so $\sin x \leq x$). Thus, because both integrals converge, the

whole thing does.

Problem 4

We define $f_p(x) = \dfrac{1}{x(\log x)^p}$ for $(p > 0)$

(a) We want to prove that given any $p, \epsilon > 0$, $x^{-1-\epsilon} < f_p(x) < x^{-1}$ for x sufficiently large. First, note that because $\lim\limits_{x\to\infty} \frac{1}{\log^p x} = 0$, for x sufficiently large, $\frac{1}{\log^p x} < 1$. This proves the the right half of the sandwich inequality. For the left half, note that

$$\frac{1}{x \cdot x^\epsilon} < \frac{1}{x(\log x)^p} \iff (\log x)^p < x^\epsilon$$

but from Corollary 2.12 we know that the following limit exists and has value 0:

$$\lim_{x\to\infty} \frac{\log^p x}{x^\epsilon} = \left(\lim_{x\to\infty} \frac{\log x}{x^{\epsilon/p}}\right)^p = 0^p = 0$$

Thus, for sufficiently large x,

$$\left|\frac{\log^p x}{x^\epsilon}\right| < 1 \iff \log^p x < x^\epsilon$$

Combining the inequalities yields the desired result.

(b) Taking $u = \log x \implies du = \frac{1}{x}\,dx$,

$$\int_2^\infty \frac{1}{x(\log x)^p}\,dx = \int_{\log 2}^\infty \frac{1}{u^p}\,du = \left.\frac{u^{-p+1}}{-p+1}\right|_{\log 2}^\infty$$

which only converges if $p > 1$.

Problem 8

On the interval $[n\pi, (n+1)\pi]$, $\frac{1}{x} \geq \frac{1}{(n+1)\pi}$, so

$$\int_{n\pi}^{(n+1)\pi} \frac{|\sin x|}{x}\,dx \geq \frac{1}{(n+1)\pi}\int_{n\pi}^{(n+1)\pi} |\sin x|\,dx = \frac{2}{(n+1)\pi}$$

Thus,

$$\begin{aligned}
\int_1^\infty \frac{|\sin x|}{x}\,dx &= \int_1^\pi \frac{|\sin x|}{x}\,dx + \int_\pi^\infty \frac{|\sin x|}{x}\,dx \\
&= \int_1^\pi \frac{|\sin x|}{x}\,dx + \sum_{n=1}^\infty \int_{n\pi}^{(n+1)\pi} \frac{|\sin x|}{x}\,dx \\
&\geq \int_1^\pi \frac{|\sin x|}{x}\,dx + \sum_{n=1}^\infty \frac{2}{(n+1)\pi} = \infty
\end{aligned}$$

Problem 11

Let the Taylor expansion centered around 0 be $\varphi(x) = \varphi(0) + x\varphi'(0) + \frac{1}{2}x^2\varphi''(0) + R(x)$. Note that because $\varphi(x) \in C^3[-1,1]$, $\varphi'''(x)$ exists and is continuous. Thus, we can use Lagrange's remainder theorem, which says that

$$R(x) = \frac{\varphi'''(c)}{3!}x^3$$

Finally, $\varphi'''(x)$ is continuous (by C^3) on a compact interval $[-1,1]$ so it's bounded.

$$\begin{aligned} P.V.\int_{-1}^{1} \frac{\varphi(x)}{x^3}\,dx &= P.V.\int_{-1}^{1} \frac{\varphi(0)}{x^3}\,dx + P.V.\int_{-1}^{1} \frac{\varphi'(0)}{x^2}\,dx + P.V.\int_{-1}^{1} \frac{\varphi''(0)}{x}\,dx + \int_{-1}^{1} \frac{R(x)}{x^3}\,dx \\ &= P.V.\int_{-1}^{1} \frac{\varphi'(0)}{x^2}\,dx + \int_{-1}^{1} \frac{\varphi'''(c)}{3!}x^3\,dx \end{aligned}$$

where two of the principal values were 0 because $\frac{1}{x}$ and $\frac{1}{x^3}$ are odd. The second integral is of a bounded function on a compact interval, so it's properly Riemann integrable — no problem there. The principal value of $\int_{-1}^{1} \frac{1}{x^2}\,dx$ doesn't exist because it is an even function, so the only way for the original principal value to exist is if $\varphi'(0) = 0$.

335 Homework 1

Daniel Rui - 1/9/20

§5.2 - Problem 2

Using polar coordinates and the standard parameterization of circles:

$$\begin{aligned}
&\int_{\partial S} (xy^2\,dy - x^2y\,dx) \\
=&\int_0^{2\pi} (2\cos\theta)(4\sin^2\theta)(2\cos\theta\,d\theta) - (4\cos^2\theta)(2\sin\theta)(-2\sin\theta\,d\theta) \\
&-\int_0^{2\pi} (\cos\theta)(\sin^2\theta)(\cos\theta\,d\theta) - (\cos^2\theta)(\sin\theta)(-\sin\theta\,d\theta) \\
=&\int_0^{2\pi} 32\cos^2\theta\sin^2\theta\,d\theta - \int_0^{2\pi} 2\cos^2\theta\sin^2\theta\,d\theta \\
=&30\int_0^{2\pi} \cos^2\theta\sin^2\theta\,d\theta = \frac{30}{4}\int_0^{2\pi} \sin^2(2\theta)\,d\theta \\
=&\frac{15}{2}\int_0^{2\pi} \frac{1}{2} - \frac{\cos(4\theta)}{2}\,d\theta = \frac{15}{2}\pi
\end{aligned}$$

Using Green's theorem we get

$$\int_{\partial S} (xy^2\,dy - x^2y\,dx) = \iint_S (y^2+x^2)\,dx\,dy = \int_0^{2\pi}\int_1^2 r^2\cdot r\,dr\,d\theta = 2\pi\cdot\frac{15}{4} = \frac{15}{2}\pi$$

Problem 3

Let S be the region bound by C. By Green's theorem,

$$\int_C \left[y^3\,dx + \left(3x - x^3\right)\,dy\right] = \iint_S 3\left(1 - x^2 - y^2\right)\,dA$$

Looking at the graph of the integrand, it is > 0 on $S := \{(x,y) : x^2 + y^2 < 1\}$ and ≤ 0 outside. Thus to maximize the integral we should include all of S, and so C will be the unit circle parameterized by $(\cos\theta, \sin\theta), \theta \in [0, 2\pi)$.

Problem 6

From Corollary 5.17, we know that for the given regular region $S \subset \mathbb{R}^2$ with a piecewise smooth boundary, we have that

$$\int_{\partial S} \mathbf{F}\cdot\mathbf{n}\,ds = \iint_S \left(\frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2}\right)\,dA$$

Now in the problem, we are given $f, g \in C^2(\bar{S})$. So let

$$\mathbf{F}(x,y) = f(x,y)[\nabla g](x,y) = \left(f(x,y)\frac{\partial g}{\partial x}(x,y), f(x,y)\frac{\partial g}{\partial y}(x,y)\right)$$

Then, $\mathbf{F}\cdot\mathbf{n} = (f\nabla g)\cdot\mathbf{n} = f(\nabla g\cdot\mathbf{n}) = f\left(\dfrac{\partial g}{\partial \mathbf{n}}\right)$ by the definition of normal derivative (middle of page 227). Finally,

$$\partial_i F_i = \partial_i(f\partial_i g) = f\partial_{ii}g + (\partial_i f)(\partial_i g)$$

and so just plugging in we get that

$$\int_{\partial S} f\left(\frac{\partial g}{\partial \mathbf{n}}\right)\,ds = \iint_S \left[f(\partial_{xx}g + \partial_{yy}g) + \nabla f\cdot\nabla g\right]\,dA$$

§5.3 - Problem 2

The surface area integral is

$$\iint_S \sqrt{1+(\partial_x\varphi)^2+(\partial_y\varphi)^2}\,dx\,dy$$

S is of course the inside of the cylinder $x^2+y^2=a^2$, i.e. $\{(x,y) : x^2+y^2\le a^2\}$, and $z=\varphi(x,y) = x^2+y^2$ so the integral is

$$\begin{aligned}\iint_S \sqrt{1+4x^2+4y^2}\,dx\,dy &= \int_0^{2\pi}\int_0^a \sqrt{1+4r^2}\,r\,dr\,d\theta \\ &= \int_0^{2\pi} \frac{1}{8}\cdot\frac{2}{3}\left(1+4r^2\right)^{3/2}\Big|_0^a\,d\theta = \frac{1}{6}\pi\left[\left(1+4a^2\right)^{3/2}-1\right]\end{aligned}$$

Problem 4

The portion of the ellipsoid above the xy-plane can be described by $z=\varphi(x,y) = \dfrac{b}{a}\sqrt{a^2-x^2-y^2}$. By symmetry, the surface area is

$$\begin{aligned}&2\iint_{x^2+y^2\le a^2}\sqrt{1+\left(\frac{b}{a}\cdot\frac{x}{\sqrt{a^2-x^2-y^2}}\right)^2+\left(\frac{b}{a}\cdot\frac{y}{\sqrt{a^2-x^2-y^2}}\right)^2}\,dx\,dy \\ =&2\iint_{x^2+y^2\le a^2}\sqrt{1+\frac{b^2}{a^2}\cdot\frac{x^2+y^2}{a^2-x^2-y^2}}\,dx\,dy \\ =&2\int_0^a\int_0^{2\pi}\sqrt{\frac{a^4+(b^2-a^2)\,r^2}{a^2\,(a^2-r^2)}}r\,d\theta\,dr = 2\int_0^{2\pi}d\theta\int_0^a\sqrt{\frac{a^4+(b^2-a^2)\,r^2}{a^2\,(a^2-r^2)}}r\,dr \\ =&4\pi\int_0^a\sqrt{\frac{a^4+(b^2-a^2)\,r^2}{a^2\,(a^2-r^2)}}r\,dr\end{aligned}$$

Using the substitution $s=\dfrac{\sqrt{a^2-r^2}}{a} \implies ds = \dfrac{-r}{a\sqrt{a^2-r^2}}\,dr$ yields

$$4\pi\int_1^0\sqrt{a^4+(b^2-a^2)\,(a^2-a^2s^2)}(-\,ds) = 4\pi a\int_0^1\sqrt{b^2+(a^2-b^2)s^2}\,ds$$

Finally, we split into two cases (it makes a big difference if the coefficient in front of the s^2 is positive or negative!):

- if $a > b$, then we use $s = \dfrac{b\tan t}{\sqrt{a^2-b^2}} \implies ds = \dfrac{b\sec^2 t}{\sqrt{a^2-b^2}}\,dt$ and so the integral becomes

$$\begin{aligned}
&4\pi a \int_0^{\arctan\left(\frac{\sqrt{a^2-b^2}}{b}\right)} \sqrt{b^2+b^2\tan^2 t}\cdot\frac{b\sec^2 t}{\sqrt{a^2-b^2}}\,dt\\
=&\frac{4\pi ab^2}{\sqrt{a^2-b^2}}\int_0^{\arctan\left(\frac{\sqrt{a^2-b^2}}{b}\right)} \sec^3 t\,dt\\
=&\frac{4\pi ab^2}{\sqrt{a^2-b^2}}\int_0^{\arctan\left(\frac{\sqrt{a^2-b^2}}{b}\right)} \frac{\cos t}{\cos^4 t}\,dt\\
=&\frac{4\pi ab^2}{\sqrt{a^2-b^2}}\int_0^{\arctan\left(\frac{\sqrt{a^2-b^2}}{b}\right)} \frac{\cos t}{(1-\sin^2 t)^2}\,dt
\end{aligned}$$

We then let $u = \sin t \implies du = \cos t\,dt$. By drawing a right triangle with legs $\sqrt{a^2-b^2}, b$ and hypotenuse a, we see that $\sin\left(\arctan\left(\frac{\sqrt{a^2-b^2}}{b}\right)\right) = \frac{\sqrt{a^2-b^2}}{a}$ and so the integral becomes

$$\begin{aligned}
&\frac{4\pi ab^2}{\sqrt{a^2-b^2}}\int_0^{\frac{\sqrt{a^2-b^2}}{a}} \frac{1}{(1-u^2)^2}\,du\\
=&\frac{4\pi ab^2}{\sqrt{a^2-b^2}}\int_0^{\frac{\sqrt{a^2-b^2}}{a}} \frac{1}{(1+u)^2(1-u)^2}\,du\\
=&\frac{4\pi ab^2}{\sqrt{a^2-b^2}}\int_0^{\frac{\sqrt{a^2-b^2}}{a}} \frac{\frac{1}{4}}{u+1}+\frac{\frac{1}{4}}{(u+1)^2}-\frac{\frac{1}{4}}{u-1}+\frac{\frac{1}{4}}{(u-1)^2}\,du\\
=&\frac{\pi ab^2}{\sqrt{a^2-b^2}}\left(\ln(|u+1|)-(u+1)^{-1}-\ln(|u-1|)-(u-1)^{-1}\right)\Big|_0^{\frac{\sqrt{a^2-b^2}}{a}}\\
=&\frac{\pi ab^2}{\sqrt{a^2-b^2}}\left(\ln\left(\left|\frac{u+1}{u-1}\right|\right)-\left(\frac{1}{u+1}+\frac{1}{u-1}\right)\right)\Big|_0^{\frac{\sqrt{a^2-b^2}}{a}}\\
=&\frac{\pi ab^2}{\sqrt{a^2-b^2}}\left(\ln\left(\left|\frac{u+1}{u-1}\right|\right)-\frac{2u}{u^2-1}\right)\Big|_0^{\frac{\sqrt{a^2-b^2}}{a}}\\
=&\frac{\pi ab^2}{\sqrt{a^2-b^2}}\cdot\ln\left(\left|\frac{\sqrt{a^2-b^2}+a}{\sqrt{a^2-b^2}-a}\right|\right)-\frac{\pi ab^2}{\sqrt{a^2-b^2}}\cdot\left(\frac{2a\sqrt{a^2-b^2}}{a^2-b^2-a^2}\right)\\
=&2\pi a^2+\frac{\pi ab^2}{\sqrt{a^2-b^2}}\cdot\ln\left(\left|\frac{\sqrt{a^2-b^2}+a}{\sqrt{a^2-b^2}-a}\right|\cdot\left|\frac{\sqrt{a^2-b^2}+a}{\sqrt{a^2-b^2}+a}\right|\right)\\
=&2\pi a^2+\frac{\pi ab^2}{\sqrt{a^2-b^2}}\cdot\ln\left(\left|\frac{(\sqrt{a^2-b^2}+a)^2}{a^2-b^2-a^2}\right|\right)\\
=&2\pi a^2+\frac{\pi ab^2}{\sqrt{a^2-b^2}}\cdot\ln\left(\frac{(\sqrt{a^2-b^2}+a)^2}{b^2}\right)\\
=&2\pi a^2+\frac{2\pi ab^2}{\sqrt{a^2-b^2}}\cdot\ln\left(\frac{\sqrt{a^2-b^2}+a}{b}\right)
\end{aligned}$$

- if $b > a$, then we use $s = \dfrac{b\sin t}{\sqrt{b^2-a^2}} \implies ds = \dfrac{b\cos t}{\sqrt{b^2-a^2}}\,dt$ and so the integral becomes

$$\begin{aligned}
&4\pi a \int_0^{\arcsin\left(\frac{\sqrt{b^2-a^2}}{b}\right)} \sqrt{b^2 - b^2\sin^2 t}\cdot \frac{b\cos t}{\sqrt{b^2-a^2}}\,dt \\
=&\frac{4\pi ab^2}{\sqrt{b^2-a^2}} \int_0^{\arcsin\left(\frac{\sqrt{b^2-a^2}}{b}\right)} \cos^2 t\,dt \\
=&\frac{4\pi ab^2}{\sqrt{b^2-a^2}} \left(\frac{1}{2}t + \frac{\sin 2t}{4}\right)\Bigg|_0^{\arcsin\left(\frac{\sqrt{b^2-a^2}}{b}\right)} \\
=&\frac{4\pi ab^2}{\sqrt{b^2-a^2}} \left(\frac{1}{2}x + \frac{\sin t\cos t}{2}\right)\Bigg|_0^{\arcsin\left(\frac{\sqrt{b^2-a^2}}{b}\right)} \qquad \text{(draw a triangle to find } \cos(\ldots)\text{)} \\
=&\frac{2\pi ab^2}{\sqrt{b^2-a^2}}\cdot \arcsin\left(\frac{\sqrt{b^2-a^2}}{b}\right) + \frac{2\pi ab^2}{\sqrt{b^2-a^2}}\cdot\frac{\sqrt{b^2-a^2}}{b}\cdot\frac{a}{b} \\
=&2\pi a^2 + \frac{2\pi ab^2}{\sqrt{b^2-a^2}}\cdot \arcsin\left(\frac{\sqrt{b^2-a^2}}{b}\right)
\end{aligned}$$

Oh and of course, if $a = b$, the integral is trivially $4\pi ab = 4\pi a^2$.

Problem 7

Because the sphere is symmetric,

$$\iint_S x^2\,dA = \iint_S y^2\,dA = \iint_S z^2\,dA$$

Visually these three integrals are just rotated versions of each other (for any function $\mathbb{R}^3 \to \mathbb{R}$, think of coloring each point $\mathbf{x} \in \mathbb{R}^3$ according to $f(\mathbf{x})$; brighter for higher values, darker for values close to 0 and think about what the functions $f(x,y,z) = x^2, y^2, z^2$ look like when colored in the manner I described — you'll see the that the coloring for $f = x^2$ and $f = y^2$ are the same, just rotated), and because the sphere looks the same under any rotation, all the surface integrals are also the same. This means that

$$\iint_S (x^2 + y^2 - 2z^2)\,dA = \iint_S x^2\,dA + \iint_S y^2\,dA - 2\iint_S z^2\,dA = 0$$

§5.4 - Problem 2c

We want to find the Laplacian of $f(\mathbf{x}) = |\mathbf{x}|^a$ ($\mathbf{x} \in \mathbb{R}^n \backslash \{\mathbf{0}\}, a \in \mathbb{R}$). From Exercise 2.6.9 (from Math 334), we know that letting $f(\mathbf{x}) = g(r)$ where $r = |\mathbf{x}|$, we have that

$$\nabla^2 f(\mathbf{x}) = g''(r) + (n-1)r^{-1}g'(r)$$

and so letting $g(r) = r^a$, the formula yields

$$\nabla^2 f(\mathbf{x}) = a(a-1)r^{a-2} + (n-1)ar^{a-2} = a(a+n-2)r^{a-2}$$

Problem 3

We have $\mathbf{F}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$$\begin{aligned}
\operatorname{curl}(\mathbf{a}\times\mathbf{F}) &= \operatorname{curl}\Big[(a_2F_3 - a_3F_2)\hat{i} + (a_3F_1 - a_1F_3)\hat{j} + (a_1F_2 - a_2F_1)\hat{k}\Big] \\
&= \Big(\partial_2(a_1F_2 - a_2F_1) - \partial_3(a_3F_1 - a_1F_3)\Big)\hat{i} + \Big(\partial_3(a_2F_3 - a_3F_2) - \partial_1(a_1F_2 - a_2F_1)\Big)\hat{i} \\
&+ \Big(\partial_1(a_3F_1 - a_1F_3) - \partial_2(a_2F_3 - a_3F_2)\Big)\hat{i} \\
&= 2a_1\hat{i} + 2a_2\hat{j} + 2a_3\hat{k} = 2\mathbf{a}
\end{aligned}$$

$$\begin{aligned}
\operatorname{div}[(\mathbf{a}\cdot\mathbf{F})\mathbf{a}] &= \operatorname{div}\Big[(a_1F_1 + a_2F_2 + a_3F_3)\mathbf{a}\Big] \\
&= \partial_1(a_1F_1 + a_2F_2 + a_3F_3)a_1 + \partial_2(a_1F_1 + a_2F_2 + a_3F_3)a_2 \\
&= \partial_3(a_1F_1 + a_2F_2 + a_3F_3)a_3 \\
&= a_1^2 + a_2^2 + a_3^2\hat{k} = |\mathbf{a}|
\end{aligned}$$

$$\begin{aligned}
\operatorname{div}[(\mathbf{a}\times\mathbf{F})\times\mathbf{a}] &= \mathbf{a}\cdot\operatorname{curl}(\mathbf{a}\times\mathbf{F}) - (\mathbf{a}\times\mathbf{F})(\operatorname{curl}\,\mathbf{a}) \quad \text{from 5.29} \\
&= \mathbf{a}\cdot\operatorname{curl}(\mathbf{a}\times\mathbf{F}) = \mathbf{a}\cdot 2\mathbf{a} = 2|\mathbf{a}|^2
\end{aligned}$$

Problem 8

The formula $\operatorname{div}(\mathbf{F}\times\mathbf{G}) = \mathbf{G}\cdot(\operatorname{curl}\mathbf{F}) + \mathbf{F}\cdot(\operatorname{curl}\mathbf{G})$ must be wrong because this formula is symmetric in $\mathbf{F}$ and $\mathbf{G}$, i.e. we would have

$$\operatorname{div}(\mathbf{F}\times\mathbf{G}) = \operatorname{div}(\mathbf{G}\times\mathbf{F})$$

but $\mathbf{F}\times\mathbf{G} = -\mathbf{G}\times\mathbf{F}$ and $\operatorname{div}(a\cdot\mathbf{F}) = a\cdot\operatorname{div}(\mathbf{F})$ for scalar a, so contradiction.

Problem 9

From 5.29 and 5.30 from the book, repeated below:

$$\operatorname{div}(\mathbf{F}\times\mathbf{G}) = \mathbf{G}\cdot(\operatorname{curl}\mathbf{F}) - \mathbf{F}\cdot(\operatorname{curl}\mathbf{G})$$

$$\operatorname{curl}(\operatorname{grad} f) = (\partial_2\partial_3 f - \partial_3\partial_2 f)\,\hat{i} + (\partial_3\partial_1 f - \partial_1\partial_3 f)\,\hat{j} + (\partial_1\partial_2 f - \partial_2\partial_1 f)\,\hat{k} = \mathbf{0}$$

we know that for any $f, g \in C^2$

$$\operatorname{div}(\operatorname{grad} f\times\operatorname{grad} g) = \operatorname{grad} g\cdot(\operatorname{curl}(\operatorname{grad} f)) - \operatorname{grad} f\cdot(\operatorname{curl}(\operatorname{grad} g)) = 0$$