Integrals dependent on parameters • Let $F(\mathbf{x}) = \int \cdots \int_S f(\mathbf{x},\mathbf{y}) d^n\mathbf{y}$ where $S \subset \mathbb{R}^n$ is compact and $T \subset \mathbb{R}^m$ is either compact or open, and S is measurable. Thus $F : T \to \mathbb{R}$. If $f(\mathbf{x},\mathbf{y})$ is continuous on $T \times S$ ($\mathbf{x} \in T$ and $\mathbf{y} \in S$), F is continuous on T. • Also, suppose S is compact and measurable, and T is open. If f and $\nabla_{\mathbf{x}} f$ are continuous on $T \times S$, then F is a C^1 function on T, and $\partial_{x_j} F(\mathbf{x}) = \int \cdots \int_S \partial_{x_j} f(\mathbf{x},\mathbf{y}) d^n\mathbf{y}$. • BCT: Suppose $S \subset \mathbb{R}^n$ is measurable and $\{f_i\}$ is a sequence of measurable functions that converges to some measurable f. Also that $|f_j| \le C$ for all j and $\mathbf{y} \in S$. Then, $\lim_{n\to\infty} \int \cdots \int_S f_n d^n\mathbf{y} = \int \cdots \int_S f d^n\mathbf{y}$. • Suppose a measurable $S \subset \mathbb{R}^n$ and any $T \subset \mathbb{R}^m$, and $f(\mathbf{x},\mathbf{y})$ is a function on $T \times S$ that is integrable as a function of $\mathbf{y} \in S$ for each $\mathbf{x} \in T$. Then, 1) if f is continuous as a function of $\mathbf{x} \in T$ for each $\mathbf{y} \in S$ and is bounded, F is continuous on T; 2) If T is open, f is C^1 as a function of $\mathbf{x} \in T$ for each $\mathbf{y} \in S$, and $|\nabla_{\mathbf{x}} f|$ is bounded, F is a C^1 on T, and $\partial_{x_j} F(\mathbf{x}) = \int \cdots \int_S \partial_{x_j} f(\mathbf{x},\mathbf{y}) d^n\mathbf{y}$. • For $F(x) = \int_a^{\varphi(x)} f(x,y)\,dy$, if f and $\partial_x f$ are continuous and φ is of class C^1, then $F'(x) = f(x,\varphi(x))\varphi'(x) + \int_a^{\varphi(x)} \frac{\partial f}{\partial x}(x,y)\,dy$.

Improper integrals • Type I: $\int_a^\infty f := \lim_{b\to\infty} \int_a^b f\,dx$ where f is integrable on every finite $[a,b]$. Converges if the limit exists. • Type II: $f(b) = \pm\infty$, and $\int_a^b f := \lim_{c\to b^-} \int_a^c f dx$ where f is integrable for all $[a,c]$ $\forall c$. Converges if the limit exists. • For all tests, suppose f and g be integrable on S. • Comparison Test: if $0 \le f \le g$ holds, f diverges so does g; if g converges so does f; the inequality only needed for sufficiently large x for type I and for sufficiently close to singularity for type II • Absolute Convergence Test: if $\int_a^\infty |f|$ converges and $\int_a^b f$ exists $\forall b$, then $\int_0^\infty f$ converges • Ratio Test $L := \lim_{x\to\infty} \frac{f}{g}$ if $0 < L < \infty$ then either both converge or both diverge, if $L = 0$ and g converges the so does f, if $L = \infty$ and g diverges so does f. • If $0 \le f(x) \le Cx^{-p}$ for some large x and $p > 1$, $\int_a^\infty f(x)dx$ converges; if the inequality is in the other direction, the integral diverges. • Cauchy principal value, for $c \in [a,b]$ is a singularity $PV\int_a^b f := \lim_{\epsilon\to 0^+} \int_a^{c-\epsilon} f + \int_{c+\epsilon}^b f$ as long as the integrals and limit exist $PV\int_{-1}^1 \frac{1}{x} = \lim_{\epsilon\to 0^+} \int_{-1}^{-\epsilon} \frac{1}{x} + \int_\epsilon^1 \frac{1}{x} = 0$. • Suppose $a < 0 < b$, and $\phi(x)$ continuous on $[a,b]$ and differentiable at 0. Then, $PV\int_a^b \frac{\phi(x)}{x} dx$ exists. • For multivariables, $\int \cdots \int_S f dV^n = \lim_{n\to\infty} \int \cdots \int_{U_n} f dV^n$, where $\{U_i\}$ is an increasing sequence of sets that converges to S. • For $p > 0$, the integral $\int \cdots \int \frac{1}{|\mathbf{x}|^p} d^n\mathbf{x}$ over a ball $B((0,0),a)$ is finite iff $p < n$. The integral over the complement is finite iff $p > n$.

Line integral: • Rectifiable: Given a $1-1$ cont. mapping $g : [a,b] \to \mathbb{R}^n$ whose range is C and partition $P = \{t_0, \cdots, t_J\}$ of $[a,b]$, define $L_P(C) = \sum_{i=1}^J |g(t_i) - g(t_{i-1})|$. If $\mathcal{L}$, a set of $L_P(C)$ is bounded, C is called rectifiable, and the arclength is defined by the sup of $L_P(C)$. • If $g \in C^1$, then C is rectifiable and $L(C) = \int_a^b |g'(t)|dt$.

Surface Area/Integrals • Scalar functions: if S is graph of function $z = \varphi(x,y)$, $(x,y) \in W$, $\iint_S f\,dA = \iint_W f(x,y,\varphi(x,y))\sqrt{1 + (\partial_x\varphi)^2 + (\partial_y\varphi)^2}\,dx\,dy$ • if S parameterized by $\mathbf{x} = \mathbf{G}(u,v)$, $(u,v) \in W$, W measurable, then $\iint_S f\,dA = \iint_W f(\mathbf{G}(u,v)) |\frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v}|\,du\,dv$. • Vector valued functions (following same hypotheses as above): $\iint_S \mathbf{F} \cdot \mathbf{n}\,dA = \iint_W \mathbf{F}(x,y,\varphi(x,y)) \cdot (-\partial_x\varphi, -\partial_y\varphi, 1)\,du\,dv$ and $\iint_S \mathbf{F} \cdot \mathbf{n}\,dA = \iint_W \mathbf{F}(\mathbf{G}(u,v)) \cdot (\frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v})\,du\,dv$. Note: $\cdot \mathbf{n} dA = \cdot(\partial_u \mathbf{G} \times \partial_v \mathbf{G})\,du\,dv$

Green's Theorem • Suppose S is a regular region in $\mathbb{R}^2$ with piecewise smooth boundary ∂S. Suppose also that $\mathbf{F}$ is a C^1 vector field on $\bar{S}$. Then $\int_{\partial S} \mathbf{F} \cdot d\mathbf{x} = \int_{\partial S} (P,Q) \cdot d\mathbf{x} = \iint_S (Q_x - P_y)\,dA$. • Regular region: $\overline{S^{int}} = S$, or $\forall x \in \partial S, \forall r > 0, B_r(x) \cap S^{int} \neq \varnothing$. • $\int_{\partial S} \mathbf{F} \cdot \hat{\mathbf{n}}\,ds = \iint_S \partial_{x_1} F_1 + \partial_{x_2} F_2\,dA = \int_0^1 \mathbf{F}(\mathbf{g}(t)) \cdot \hat{\mathbf{n}} |\mathbf{g}'(t)|\,dt$, where $\mathbf{g}(t) = (x(t), y(t))$ for $t \in [0,1]$. • Normal derivative: $\int_{\partial S} \frac{\partial f}{\partial n}\,ds := \int_{\partial S} \nabla f \cdot \hat{\mathbf{n}}\,ds = \iint_S (\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2}) dA$. • Green's Formula: If $R \subset \mathbb{R}^3$ is a regular region with a piecewise smooth boundary, and $f, g \in C^2$ on $\overline{R}$, $\iint_{\partial R} f \nabla g \cdot \mathbf{n} dA = \iiint_R (\nabla f \cdot \nabla g + f\nabla^2 g) dV$ and $\iint_{\partial R} (f\nabla g - g\nabla f) \cdot \mathbf{n} dA = \iiint_R (f\nabla^2 g - g\nabla^2 f) dV$.

Divergence Theorem • Suppose R is a regular region in $\mathbb{R}^3$ with piecewise smooth boundary, oriented so that $\hat{\mathbf{n}}$ points out of R, and suppose $\mathbf{F}$ is C^1 on R. Then $\iint_{\partial R} \mathbf{F} \cdot \hat{\mathbf{n}}\,dA = \iiint_R \operatorname{div} \mathbf{F}\,dV = \iiint_R \partial_{x_1}\mathbf{F} + \partial_{x_2}\mathbf{F} + \partial_{x_3}\mathbf{F}\,dV$. • Green's Formulas: Let R be as above and f and g be C^2 functions on $\overline{R}$. Then $\iint_{\partial R} f \nabla g \cdot \mathbf{n}\,dA = \iiint_R (\nabla f \cdot \nabla g + f\nabla^2 g)\,dV$ and $\iint_{\partial R} (f\nabla g - g\nabla f) \cdot \mathbf{n}\,dA = \iiint_R (f\nabla^2 g - g\nabla^2 f)\,dV$. • Poisson's Equation: Suppose ρ is a function of class C^1 (5.46 + remarks) on $\mathbb{R}^3$ that vanishes outside a bounded set, and let $u(\mathbf{x}) = \iiint_{\mathbb{R}^3} \frac{\rho(\mathbf{x}+\mathbf{y})}{|\mathbf{y}|} d^3\mathbf{y}$. Then u is a C^2 function and $\nabla^2 u = -4\pi\rho$.

Stokes's Theorem • Let S_0 be a smooth surface in $\mathbb{R}^3$, and let S be a region in S_0 bounded by a piecewise smooth boundary $\partial S \subset S_0$, and S is oriented, and ∂S is oriented compatibly. Then let $\mathbf{F}$ be a C^1 vector field defined on some neighborhood of S. Then $\int_{\partial S} \mathbf{F} \cdot d\mathbf{x} = \iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n} dA$. • If S is a closed surface (it has no boundary) in $\mathbb{R}^3$ and $\mathbf{F}$ is a C^1 vector field on S then $\iint_S (\operatorname{curl} \mathbf{F}) \cdot \hat{\mathbf{n}}\,dA = 0$.

Formulas for grad, curl, div, Laplacian • $\operatorname{curl}(f) = \nabla \times f$ • $\operatorname{div}(f) = \nabla \cdot f$ • $\operatorname{grad}(fg) = f \operatorname{grad} g + g \operatorname{grad} f$ • $\operatorname{curl}(f\mathbf{G}) = f \operatorname{curl} \mathbf{G} + (\operatorname{grad} f) \times \mathbf{G}$
• $\operatorname{div}(f\mathbf{G}) = f \operatorname{div} \mathbf{G} + (\operatorname{grad} f) \cdot \mathbf{G}$ • $\operatorname{div}(\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\operatorname{curl} \mathbf{F}) - \mathbf{F} \cdot (\operatorname{curl} \mathbf{G})$ • by equality of MIXED PARTIALS (i.e. when C^2), $\operatorname{curl}(\operatorname{grad} f) = 0$ and $\operatorname{div}(\operatorname{curl} \mathbf{F}) = 0$.
• $\operatorname{div}(\operatorname{grad} f) = \partial_1^2 f + \cdots + \partial_n^2 f = \nabla^2 f =: \Delta f$ (the Laplacian) • $\operatorname{grad}(\operatorname{div} \mathbf{F}) - \operatorname{curl}(\operatorname{curl} \mathbf{F}) = \nabla^2 \mathbf{F} = (\nabla^2 F_1, \nabla^2 F_2, \nabla^2 F_3)$

Integrating vector derivatives • If $\mathbf{G}$ is continuous vector field on open $R \in \mathbb{R}^n$, the following are equivalent: (A) If C_1 and C_2 are any two oriented curves in R with the same initial point and the same final point, then $\int_{C_1} \mathbf{G} \cdot d\mathbf{x} = \int_{C_2} \mathbf{G} \cdot d\mathbf{x}$; (B) If C is any closed curve in R, $\int_C \mathbf{G} \cdot d\mathbf{x} = 0$. (C) $\mathbf{G}$ is the gradient of some C^1 function f on R. If a vector field satisfies these conditions, it is conservative. • Suppose R is a convex set in $\mathbb{R}^n$ and $\mathbf{G}$ is a C^1 vector field on R. If $\forall j \neq k : \partial_k G_j - \partial_j G_k = 0$ (equivalently in $\mathbb{R}^3 : \operatorname{curl} \mathbf{G} = 0$), then $\mathbf{G}$ is the gradient of some C^2 function f on $\mathbb{R}$. • Suppose R is a convex set in $\mathbb{R}^3$ and $\mathbf{G}$ is a C^1 vector field on R. If $\operatorname{div} \mathbf{G} = 0$ on R, then $\mathbf{G}$ is the curl of a C^2 vector field on R.

Applicable Problems

• 1. If $f(x,t) = \frac{\sin xt}{t}$ if $t \neq 0$, and x if $t = 0$. Define auxiliary $h(y) = \frac{\sin y}{y}$ for $t \neq 0$ and 1 for $t = 0$. Then $f(x,t) = xh(xt)$ (must check this: if $t = 0, xh(0) = x = f(x,0)$ ✓; if $t \neq 0, x = 0$, $0 \cdot h(0) = 0 = f(0,t)$ ✓; if $t, x \neq 0$, $\frac{x \sin xt}{xt} = \frac{\sin xt}{t}$ ✓). So $f(x,t)$ is composition of continuous functions, so it's continuous. Going back to definition, $\partial_x f(x,t)$ is $\cos xt$ for $t \neq 0$, and 1 for $t = 0$. This is clearly continuous. Set $g(x) = \int_0^{\pi/2} f(x,t)\,dt$. $t \in [0, \pi/2]$ COMPACT, and $x \in \mathbb{R}$ is OPEN. We can now do diff. under integral: $g'(x) = \int_0^{\pi/2} \cos xt\,dt$ and so $g'(0) = \frac{\sin xt}{x}|_0^{\pi/2} = \sin(x\pi/2)/x$ for $x \neq 0$ and $\pi/2$ for $x = 0$.
• 2. let $\mathbf{F} = (\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2})$ and let C be unit disk. Then $\int_{x^2+y^2=1} \frac{-y\,dx + x\,dy}{x^2+y^2} = \int_{\partial C} F_1\,dx + F_2\,dy = \int_0^{2\pi} -r\sin\theta(-r\sin\theta\,d\theta) + r\cos\theta(r\cos\theta\,d\theta) = 2\pi$. Let E be filled ellipse $\frac{x^2}{4} + \frac{y^2}{9} \le 1$. Let S be region $E \setminus C$. S is bounded (i.e. relatively compact) and regular, the boundary is piecewise smooth, and $\mathbf{F}$ is C^1 on open set around S. Then use Green's theorem: $\int_{\partial S} F_1\,dx + F_2\,dy = \iint_S \partial_x F_2 - \partial_y F_1\,dA = \iint_S 0\,dA = 0$. Adding back the integral over the circle we see that $\int_{\partial E} \mathbf{F} \cdot d\mathbf{x} = 2\pi$.
• 3. S is region where $\theta \in [\alpha, \beta], r \in [0, f(\theta)]$, f is C^1. WTS area of S is $\int_\alpha^\beta \frac{1}{2} f^2(\theta)\,d\theta$. $\iint_S 1\,dA$ is also the area. To use Green's theorem, want $\partial_x F_2 - \partial_y F_1 = 1$, so take $\mathbf{F} = (-y/2, x/2)$. The region is bounded, regular; boundary is piecewise smooth, and $\mathbf{F}$ is C^1. So $\iint_S 1\,dA = \int_{\partial S} \mathbf{F} \cdot d\mathbf{x}$. ∂S splits into three parts; the first 2 are $\mathbf{g}_1(t) = tf(\alpha)(\cos\alpha, \sin\alpha) \implies \mathbf{g}_1'(t) = f(\alpha)(\cos\alpha, \sin\alpha)$, $t \in [0,1]$. $\mathbf{F}$ is then $\frac{1}{2} f(\alpha)(-\sin\alpha, \cos\alpha)$, and so $\mathbf{F} \cdot d\mathbf{x} = 0$. $\mathbf{g}_2$ is similar, and its integral is also 0. The last curve C is $\mathbf{r}(\theta) = f(\theta)(\cos\theta, \sin\theta) \implies \mathbf{r}'(\theta) = (f'(\theta)\cos\theta - f(\theta)\sin\theta, f'(\theta)\sin\theta + f(\theta)\cos\theta)$, $\theta \in [\alpha, \beta]$. Then $\int_C \mathbf{F} \cdot d\mathbf{x} = \int_\alpha^\beta \mathbf{F}(\mathbf{r}(\theta)) \cdot \mathbf{r}'(\theta)\,d\theta$ and so doing the computation you get the desired result.
• 4. For any $g(x)$ such that $\int_a^\infty \frac{1}{g(x)} dx = \infty$, take $f(x) = \frac{1}{2g(x)}$ or $\frac{1}{1+g(x)}$ to show that $|f(x)g(x)| < 1$ does not imply $\int_a^\infty f(x)\,dx$ converges.
• 5. Let C be the curve of intersection of $y + z = 0$ and $x^2 + y^2 = a^2$ oriented counterclockwise. Since the intersection of $y + z = 0$ and $x^2 + y^2 = a^2$ is a piecewise smooth boundary is an oriented regular surface, and $\mathbf{F} = (xz + 1, yz + 2)$ is C^1 everywhere, Stokes is applicable. By Stokes's, $\int_{\partial S} F \cdot d\mathbf{x} = \iint_S \operatorname{curl} F \cdot \mathbf{n} dA$. Let $(x,y,z) = (r\cos(\theta), r\sin(\theta), -r\sin(\theta))$. Then, $\int_S \operatorname{curl} F \cdot \mathbf{n} dA = \int_0^{2\pi} \int_0^a \operatorname{curl} F \cdot |\partial_r G \times \partial_\theta G|\,dr\,d\theta = \int_0^{2\pi} \int_0^a (-r\sin(\theta), r\cos(\theta), 2) \cdot (0, r, r) dA = \int_0^{2\pi} \int_0^a r^2\cos(\theta) + 2r dr d\theta = 2\pi a^2$.
• 6. If $\phi(x) = \int_0^{\pi/2} \cos(x\sin(t))dt$, $x\phi''(x) + \phi'(x) + x\phi(x) = 0$: Note that $t \in [0, \frac{\pi}{2}]$, and $x \in (-\infty, \infty)$, $\cos(x\sin(t))$ is continuous everywhere, and $\partial_x \cos(x\sin(t)) = -\sin(t)\sin(x\sin(t))$ is too, so we can differentiate under the integral w.r.t. x, and we can do this twice because $\partial_{xx} \cos(x\sin(t)) = -\sin^2(t)\cos(x\sin(t))$ is continuous. Now we compute that $x\phi''(x) + \phi'(x) + x\phi(x) = \int_0^{\frac{\pi}{2}} -x\sin^2(t)\cos(x\sin(t)) - \sin(t)\sin(x\sin(t)) + x\cos(x\sin(t))\,dt =$
$\int_0^{\frac{\pi}{2}} x(1 - \sin^2(t))\cos(x\sin(t)) - \sin(t)\sin(x\sin(t))\,dt = \int_0^{\pi/2} x\cos^2(t)\cos(x\sin(t)) - \sin(t)\sin(x\sin(t))\,dt$. Use integration by parts on the second term of the integrand with $f = \sin(x\sin(t))$, $g' = \sin(t) \implies fg|_0^{\pi/2} = 0$, and $-\int_0^{\pi/2} f'g\,dt = \int_0^{\pi/2} x\cos t(\sin(x\sin t))(-\cos t)\,dt$ evaluates to the first term of the integrand, so it cancels to 0.
• 7. Prove $\int_C \frac{x\,dx + y\,dy}{x^2+y^2}$ is independent of path on $\mathbb{R}^2 \setminus \mathbf{0}$ (find a function that the integrand is the gradient of). The field is the gradient of $\frac{\ln(x^2+y^2)}{2}$. $\frac{\ln(x^2+y^2)}{2}$ is C^1 on $\mathbb{R}^2 \setminus \mathbf{0}$, and $\mathbb{R}^2 \setminus \mathbf{0}$ is open, being gradient of C^1 function $\iff$ path independent, this proves path independence.
• 8. WTS that $\int_1^\infty \cos(x^2)\,dx$ converges: $\int_1^R \cos(x^2)\,dx$; $u = x^2 \implies du = 2x\,dx$ so $\int_1^{R^2} \cos(u)/(2\sqrt{u})\,du$. Integrating by parts this is $\frac{1}{2}((\sin u/\sqrt{u})|_1^{R^2} + \frac{1}{2}\int_1^{R^2} \sin u/(u^{3/2})\,du)$. Now note that $|\sin u/(u^{3/2})| \le |1/u^{3/2}| = 1/u^{3/2}$, and we know $\int_1^\infty u^{-3/2}\,du$ is convergent, so $\int_1^\infty |\sin u/(u^{3/2})|\,du$ is convergent, and so $\int_1^\infty \sin u/(u^{3/2})\,du$ is too. Looking at our mess from integration by parts, both parts converge so $\int_1^\infty \cos x^2\,dx$ converges. • To prove that this diverges in absolute value, use the u-sub from above to again get to $\frac{1}{2}\int_1^\infty |\cos u/\sqrt{u}|\,du$. Then consider $\int_{k\pi+\pi/2}^{(k+1)\pi+\pi/2} |\cos u|/\sqrt{u}\,du \ge 1/\sqrt{(k+1)\pi + \pi/2} \int_{k\pi+\pi/2}^{(k+1)\pi+\pi/2} |\cos u|\,du = 2\sqrt{\frac{2}{\pi(2k+3)}}$ and so summing we diverge.
• 9. a) $\int_{-\infty}^\infty x^2 e^{-|x|}\,dx$ This converges, because by the book's example, $\lim_{x\to\infty} x^a e^{-x} \to 0$ for any positive a, so $e^{-x} < \frac{1}{x^4}$ for all x bounded below by some large enough c, which gives $\int_0^\infty x^2 e^{-x} \le \int_0^c x^2 e^{-x} dx + \int_c^\infty \frac{1}{x^2} dx = \int_0^c x^2 e^{-x} dx - \frac{1}{c} < \infty$. b) $\int_0^{\frac{\pi}{2}} \frac{dx}{(\cos(x))^{2/3}}$ converges, because for $x \in [0, \frac{\pi}{2}]$, $\sin(x) \ge \frac{2x}{\pi}$ (use Rolle's and contradiction; there can be only two solutions of $\sin(x) = \frac{2x}{\pi}$ for the domain), it follows that, with $x = \frac{\pi}{2} - u$, $\int_0^{\frac{\pi}{2}} \frac{1}{(\cos(x))^{2/3}} = \int_{\pi/2}^0 -\frac{1}{(\sin(u))^{2/3}} = \int_0^{\pi/2} \frac{1}{(\sin(u))^{2/3}} \le \int_0^{\frac{\pi}{2}} (\frac{\pi}{2u})^{2/3} < \infty$
c) $\int_1^\infty \frac{\sin(1/x)}{x}\,dx$ converges, because by taking $u = \frac{1}{x}$, $= \int_0^1 \sin(u)/u\,du \le \int_0^1 1\,du$, since $\sin(x) \le x$ for $x \in [0, 2\pi]$.
• 10. Cauchy Inequality: let $h(t) = \int (f - tg)^2 = \int (f^2 - 2tfg + t^2 g^2)$. Then, $h'(t) = \int (2tg^2 - 2fg)$, which is 0 when $t = \frac{\int fg}{\int g^2}$, and $h''(t) > 0$, so it is a minimum. So minimum of h is $\int (f - \frac{\int fg}{\int g^2} g)^2 = \int f^2 - 2\frac{(\int fg)^2}{\int g^2} + \frac{(\int fg)^2}{\int g^2} \ge 0 \iff \int f^2 - \frac{(\int fg)^2}{\int g^2} \ge 0 \iff \int f^2 \int g^2 \le (\int fg)^2$.
• 12. Let S be the surface obtained by rotating the circle $(x-2)^2 + z^2 = 1$ around the z–axis. Compute the integral $\int_S \mathbf{F} \cdot \mathbf{n} dA$, where
$\mathbf{F} = (x + f_1(y,z), y + f_2(x,z), z + +f_3(x,y))$. Since $\operatorname{div}(\mathbf{F}) = 3$, by the divergence theorem, we see that this integral is equal to 3 times the volume of the Torus, which is given by $2\pi R r^2 = 4\pi$ so we get 12π, where r is radius of the circle rotating and R is radius from the rotation's center to the circle's center.
• 13. Suppose $w''(x) + w(x) = 0 \forall x$, $w(0) = 0$, $w'(0) = 1$, and $f(x) = \int_0^x (w(x-y))h(y)dy$. We see that because $w''(x)$ exists $\forall x$, w is C^1. But $w''(x) = -w(x)$, and so $w''(x)$ is C^1. We keep going to see that w is C^∞, so we can differentiate under the integral. Then, $f'(x) = \int_0^x w'(x-y)h(y)dy + w(x-x)h(x) = \int_0^x w'(x-y)h(y)dy$, and $f''(x) = \int_0^x w''(x-y)h(y)dy + w'(x-x)h(x) = \int_0^x w''(x-y)h(y)dy + h(x)$, giving $f(x) + f''(x) = \int_0^x w(x-y)h(y) + w''(x-y)h(y)dy + h(x) = h(x) + \int_0^x (w(x-y) + w''(x-y))h(y)dy$. But $w'' + w = 0$, so the integral $= 0$, so $f''(x) + f(x) = h(x)$. Finally, $f(0) = \int_0^0 (w(x-y))h(y)dy = 0$ and $f'(0) = \int_0^0 w'(x-y)h(y)\,dy = 0$.

Homework Problems • Show that $\int_{\partial S} f \frac{\partial g}{\partial \mathbf{n}} ds = \iint_S f(\partial_{xx} g + \partial_{yy} g) + \nabla f \cdot \nabla g\,dA$. Note that $f\frac{\partial g}{\partial \mathbf{n}} = (f\nabla g) \cdot n$. By Green's, we know $\int_{\partial S} \mathbf{F} \cdot \mathbf{n}\,ds = \iint_S \partial_x F_1 + \partial_y F_2\,dA$. But $F_1 = fg_x$ and $F_2 = fg_y$, so just doing the product rule with each x_i proves it. • Given $F(\mathbf{x}) = f(r) := f(|\mathbf{x}|)$. By chain rule, $\partial_i F = \frac{x_i f'(r)}{r}$. Then, again by the chain rule, $\partial_i^2 F = \frac{f'(r)}{r} - \frac{x_i^2 f'(r)}{r^3} + \frac{x_i^2 f''(r)}{r^2}$. Finally, $\sum_{i=1}^n \partial_i^2 F = \sum_{i=1}^n \frac{f'(r)}{r} - \frac{x_i^2 f'(r)}{r^3} + \frac{x_i^2 f''(r)}{r^2} = n\frac{f'(r)}{r} - \frac{r^2 f'(r)}{r^3} + \frac{r^2 f''(r)}{r^2} = (n-1)\frac{f'(r)}{r} + f''(r)$. • S is a smooth oriented surface with a piecewise smooth, compatibly oriented boundary ∂S, and $f \in C^1$, $g \in C^2$ on some open set containing S, so Stokes's theorem is applicable. Then, using Stokes's theorem and $\operatorname{curl}(f\nabla g) = f \cdot \operatorname{curl}(\nabla g) + \nabla f \times \nabla g$ and $\operatorname{curl}(\nabla g) = 0$, $\int_{\partial S} f\nabla g \cdot d\mathbf{x} = \iint_S \operatorname{curl}(f\nabla g) \cdot \mathbf{n}\,dA = \iint_S f \operatorname{curl}(\nabla g) + \nabla f \times \nabla g\,dA =$
$\iint_S f \cdot \mathbf{0} + \nabla f \times \nabla g\,dA = \iint_S \nabla f \times \nabla g\,dA$. • One can find a C^2 function f such that $\operatorname{div}(H) = \nabla^2 f$ by the assumption on the website. Then, it follows that $0 = \operatorname{div}(H) - \nabla^2 f = \operatorname{div}(H - \nabla f)$, which implies that $\exists G \in C^2$ such that $\operatorname{curl}(G) = H - \nabla f$; to use theorem check that R is convex & open, and $H - \nabla f \in C^1$.