336 Final

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Problem 1

Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function, and suppose that $|f(z^2)| \leq |f(z)|$ for all $z \in \mathbb{C}$. We want to show that f must be constant. Liouville's theorem says that any bounded entire function must be constant, so it suffices to show that f must be bounded.

Because f is entire, it is bounded, say by M, on the compact set $\{z \in \mathbb{C} : |z| \leq 2\}$. I claim that if $|f| \leq M$ on $\overline{D}_r := \{z \in \mathbb{C} : |z| \leq r\}$ (where $r \in \mathbb{R}$), then $|f| \leq M$ on $\overline{D}_{r^2} := \{z \in \mathbb{C} : |z| \leq r^2\}$. This is because for any $z_0 \in \overline{D}_{r^2}$, we can write it as $z_0 = r_0 e^{i\theta_0}$ where $r_0 \in [0, r^2]$ and $\theta_0 \in [0, 2\pi)$, allowing us to define $z_0' = \sqrt{r_0} e^{i\theta_0/2}$, which satisfies $(z_0')^2 = z_0$. However, because $r_0 \in [0, r^2]$, $\sqrt{r_0} \in [0, r]$, and hence $z_0' \in \overline{D}_r$, and so $|f(z_0')| \leq M$. From $|f(z^2)| \leq |f(z)|$, we see that $|f(z_0)| \leq |f(z_0')| \leq M$. Remember, z_0 was arbitrarily chosen in $\overline{D}_{r^2}$, and so $|f| \leq M$ on all of $\overline{D}_{r^2}$.

With our initial claim that $|f| \leq M$ on $\overline{D}_2$, we can now use this to prove that $|f| \leq M$ on $\overline{D}_4$, and then $\overline{D}_8$ and so on. Because every $z \in \mathbb{C}$ will lie in $\overline{D}_{2^n}$ for some n large enough, this shows that $|f| \leq M$ on all of $\mathbb{C}$, and hence f is bounded (which we said suffices to show that f is constant).

Problem 2

Let $f : \mathbb{D} \to \mathbb{C}$ (where $\mathbb{D} := \{z : |z| < 1\}$) be an analytic function such that there is a real number $a \in (0, 1)$ and a constant C such that $|f(z) - f(w)| \leq C|w - z|^a$ for all $z, w \in \mathbb{D}$. We want to prove that $|f'(z)| \leq C(1 - |z|)^{a-1}$ for all $z \in \mathbb{D}$.

Fix any $z_0 \in \mathbb{D}$. Because $\mathbb{D}$ is open, for all $\epsilon > 0$ sufficiently small, $|z_0| < 1 - \epsilon$. Now define

$$g(z) = \begin{cases} \frac{f(z) - f(z_0)}{z - z_0} & \text{for } z \neq z_0 \\ f'(z_0) & \text{for } z = z_0 \end{cases}$$

Because $g(z)$ is analytic on $\mathbb{D} \backslash \{z_0\}$ (quotient of analytic functions where the denominator is non-zero), and is continuous at z_0 (by the definition of the complex derivative, where $f'(z_0)$ exists because f is analytic), the Riemann removable singularity theorem says that $g(z)$ is analytic on $\mathbb{D}$. The Hölder condition for f gives us that for $z \neq z_0$,

$$|g(z)| = \frac{|f(z) - f(z_0)|}{|z - z_0|} \leq C|z - z_0|^{a-1} \leq C(|z| - |z_0|)^{a-1}$$

where the last inequality is by the reverse triangle inequality. Now for any $\epsilon > 0$ sufficiently small we

have that $z_0 \in D_{(1-\epsilon)}$ and for any $z \in \partial D_{(1-\epsilon)}$,

$$|g(z)| \leq C(1 - \epsilon - |z_0|)^{a-1}$$

and so by the maximum modulus principle,

$$|g(z_0)| \leq C(1 - \epsilon - |z_0|)^{a-1}.$$

Because this holds for any $\epsilon > 0$ sufficiently small, it holds for the limit as $\epsilon \searrow 0$ ($(1 - \epsilon - |z_0|)^{a-1}$ decreases as $\epsilon \searrow 0$), and so

$$|g(z_0)| = |f'(z_0)| \leq C(1 - |z_0|)^{a-1}.$$

Because $z_0 \in \mathbb{D}$ was arbitrary, we have that for all $z \in \mathbb{D}$, $|f'(z)| \leq C(1 - |z|)^{a-1}$.

Problem 3

Let $u : W \to \mathbb{R}$ be a non-constant harmonic function (where W is a domain; i.e. an open connected set in $\mathbb{C}$). We want to show that u maps open subsets of W to open subsets of $\mathbb{R}$.

Say we have some open set $U \subseteq W$. Because U is open, we know that for any $z \in U$, there is $\delta_z > 0$ such that $B(z, \delta_z) \subset U$, and so we can write $U = \bigcup_{z \in U} B(z, \delta_z)$. We know by the identity theorem for harmonic functions that u is non-constant on any open subset of W, and hence u is non-constant on any of the $B(z, \delta_z)$. Moreover, on each $B(z, \delta_z)$, there is some harmonic function v_z such that $f_z = u + iv_z$ is analytic on $B(z, \delta_z)$. Obviously, f_z is non-constant because u is non-constant on $B(z, \delta_z)$, so by the open mapping theorem, $f_z(B(z, \delta_z))$ is open. Because u is the real part of f_z, $u(B(z, \delta_z)) = \{\text{Re}(z') : z' \in f_z(B(z, \delta_z))\}$. Call this set the "projection of the set $f_z(B(z, \delta_z))$ on the real line". We now show that the projection $\text{Re}(V)$ of any open set $V \subseteq \mathbb{C}$ to the real line is also open.

Suppose that the projection $\text{Re}(V)$ is not open. Then, there is some $x_0 \in \text{Re}(V)$ such that for all $r > 0$, there is some $x_r \in B(x_0, r)$ that is outside of $\text{Re}(V)$. First, because $x_0 \in \text{Re}(V)$, there must be some $y_0 \in \mathbb{R}$ such that $x_0 + iy_0 \in V$. But because $x_r \notin \text{Re}(V)$, it must be that $x_r + iy_0 \notin V$ for all $r > 0$. That means that although $z_0 := x_0 + iy_0 \in V$, for any $r > 0$, $B(z_0, r)$ contains $x_r + iy_0 \notin V$, and so V is not open.

Thus, for any $z \in U$, $f_z(B(z, \delta_z))$ is open and hence $u(B(z, \delta_z))$ is open. Lastly,

$$\begin{aligned} u(U) = \{u(z) : z \in U\} &= \left\{ u(z') : z' \in \bigcup_{z \in U} B(z, \delta_z) \right\} \\ &= \bigcup_{z \in U} \{u(z') : z' \in B(z, \delta_z)\} = \bigcup_{z \in U} u(B(z, \delta_z)) \end{aligned}$$

and the union (uncountable or otherwise) of open sets is open, and so $u(U)$ is open.

Problem 4

Any real valued function $u(x, y)$ harmonic on $\mathbb{R}^2$ such that $u^2(x, y)$ must be constant: as per the definition of harmonicity, $0 = \nabla^2 u^2 = \partial_x(2uu_x) + \partial_y(2uu_y) = 2u_x u_x + 2uu_{xx} + 2u_y u_y + 2uu_{yy} = 2(u_x^2 + u_y^2) + 2u(u_{xx} + u_{yy})$. But from the harmonicity of u, $u_{xx} + u_{yy} = 0$, and so we have that $u_x^2 + u_y^2 = 0$ on all of $\mathbb{R}^2$. Because u is real valued, its partial derivatives are too, and the only way for the sum of the squares of two real values to be 0 is if both values are 0. Thus, $u_x = u_y = 0$ everywhere on $\mathbb{R}^2$, which implies that u is constant on $\mathbb{R}^2$.

Problem 5

Let f, g be analytic on a relatively compact open set $W \subset \mathbb{C}$ and continuous on the closure of W. We want to prove that the maximum of $h(z) = |f(z)| + |g(z)|$ occurs on ∂W.

Suppose that the maximum of $h(z)$ does not occur on ∂W, so there is some point $z_0 \in W$ such that $h(z_0) > h(z)$ for all $z \in \partial W$. Define the constants

$$a = \frac{|f(z_0)|}{f(z_0)} \quad \text{and} \quad b = \frac{|g(z_0)|}{g(z_0)}$$

(unless of course if $f(z_0) = 0$ of $g(z_0) = 0$, in which case set a or b to be 1). Then, in all cases, $|a| = |b| = 1$. Next, define $s(z) = af(z) + bg(z)$ so that $s(z_0) = |f(z_0)| + |g(z_0)| = h(z_0)$. The good thing about s (unlike h for example) is that we know that s is analytic, and so we can use the maximum modulus principle which tells us that s attains its maximum at some $z_1 \in \partial W$. But by the triangle inequality, we have that $|s(z)| \leq |f(z)| + |g(z)| = h(z)$ for every $z \in W$, and so

$$h(z_0) = s(z_0) \leq |s(z_0)| \leq |s(z_1)| \leq h(z_1) < h(z_0)$$

where $s(z_0)$ was real (it's equal to $h(z_0)$ which is real) so we could use it in an inequality like that. This is clearly a contradiction, so it must be that h takes its maximum value on the boundary ∂W.

Problem 6

The Schwarz-Pick theorem says that for analytic $f : \mathbb{D} \to \mathbb{D}$,

$$\left| \frac{f(z_1) - f(z_2)}{1 - \overline{f(z_1)} f(z_2)} \right| \leq \left| \frac{z_1 - z_2}{1 - \overline{z_1} z_2} \right|$$

for any $z_1, z_2 \in \mathbb{D}$. Plugging in the conditions from the problem, $f(\frac{1}{2}) = \frac{2}{3}$ and $f(\frac{1}{4}) = \frac{1}{3}$, we see that

$$\left| \frac{\frac{2}{3} - \frac{1}{3}}{1 - \frac{1}{3}\frac{2}{3}} \right| \leq \left| \frac{\frac{1}{2} - \frac{1}{4}}{1 - \frac{1}{2}\frac{1}{4}} \right| \iff \left| \frac{1}{3} \cdot \frac{9}{7} \right| \leq \left| \frac{1}{4} \cdot \frac{8}{7} \right| \iff \frac{3}{7} \leq \frac{2}{7}$$

is NOT true, and so no analytic $f : \mathbb{D} \to \mathbb{D}$ can possibly satisfy $f(\frac{1}{2}) = \frac{2}{3}$ and $f(\frac{1}{4}) = \frac{1}{3}$.

Problem 7

Let (non-identically zero) f, g be entire functions such that $|f(z)| \leq |g(z)|$ when $|z| \geq 2020$. We want to prove that $\frac{f}{g}$ is a ratio of polynomials. First, note that because the zeroes of an analytic function are isolated, there can only be finitely many zeroes $\{z_1, \ldots, z_n\}$ (counting multiplicities) in $D_{2020} \subset \overline{D_{2020}}$, because otherwise there would be an accumulation point by Bolzano-Weierstrass (taking $\overline{D_{2020}}$ to be the compact set in question). Then, there is an analytic function h, non-zero inside D_{2020} such that $g(z) = (z - z_1) \cdots (z - z_n)h(z)$ for all $z \in \mathbb{C}$ (the argument for this is found on the end of pg. 154 in Gamelin). Then, because for all $z \in \mathbb{C} \setminus D_{2020} = \{z \in \mathbb{C} : |z| \geq 2020\}$, $|f(z)| \leq |g(z)|$, and so for all $z \in \mathbb{C} \setminus D_{2020}$ such that $g(z) \neq 0$,

$$\left|\frac{f(z)}{g(z)}\right| \leq 1 \implies \left|\frac{f(z)}{h(z)}\right| \leq |(z - z_1) \cdots (z - z_n)|$$

(recall that none of the $\{z_1, \ldots, z_n\}$ are in $\mathbb{C} \setminus D_{2020}$ and so we can move the $(z - z_i)$ terms around). f and h are both entire, and so $\frac{f}{h}$ is analytic everywhere except at the zeroes of h (which all lie in $\mathbb{C} \setminus D_{2020}$). But for at all the zeroes of h, the above inequality holds in a neighborhood around those zeroes, and because the RHS of that inequality is bounded in any neighborhood around those zeroes, the Riemann removable singularity theorem gives that $\frac{f}{h}$ can be extended analytically to get an entire function; we now let q denote this extended version of the old $\frac{f}{h}$. We can now use the result we proved in the homework problem §IV.5.4, which I will now copy here:

- Homework problem §IV.5.4: for any entire function f such that $f(z)/z^n$ is bounded for $|z| \geq R$ must be a polynomial of degree at most n.

- We suppose that $f(z)$ is an entire function s.t. $g(z) = f(z)/z^n$ is bounded (by say B) for $|z| \geq R$. This implies that for any z_0, and for any big enough R_{z_0}, $|z - z_0| < R_{z_0}$ contains the disk $|z| < R$, which means that all $z \in \mathbb{C}$ such that $|z - z_0| \geq R_{z_0}$ satisfy $|g(z)| \leq B$. In particular, $|g(z)| \leq B$ for $|z - z_0| = R_{z_0}$, a circle contained in the region $|z| \leq |z_0| + R_{z_0}$, and so $|f(z)| = |g(z)||z^n| \leq B(|z_0| + R_{z_0})^n$ for $|z - z_0| = R_{z_0}$. $f(z)$ is of course analytic inside this region, as it is entire, and so now we may apply Cauchy's estimates to get that:

  $$\left|f^{(n+1)}(z_0)\right| \leq \frac{(n+1)!}{R_{z_0}^{n+1}} B(|z_0| + R_{z_0})^n$$

  Letting R_{z_0} go to ∞ (because recall that I said this works for ANY big enough R_{z_0}), we see that $\left|f^{(n+1)}(z_0)\right| = 0 \implies f^{(n+1)}(z_0) = 0$. Thus all derivatives of order $(n+1)$ and higher are 0, EVERYWHERE on $\mathbb{C}$ (because z_0 was arbitrary). Now because f is analytic in a neighborhood of z_0, it equals its power series there, which we just showed is a finite degree polynomial. From the identity theorem, it must be that f is that finite degree polynomial $a_0 + a_1 z + \ldots + a_n z^n$ on all of $\mathbb{C}$.

Because of the inequality involving $|\frac{f}{h}|$ above (q must satisfy the same inequality), we see that q/z^n is in fact bounded on $\mathbb{C} \setminus D_{2020}$ (reciprocals tend to 0 as $|z| \to \infty$). Thus by the homework problem,

q is a polynomial. Lastly,

$$\frac{f(z)}{g(z)} = \frac{f(z)}{h(z)(z-z_1)\cdots(z-z_n)} = \frac{q(z)}{(z-z_1)\cdots(z-z_n)}$$

is a ratio of polynomials.

Problem 8

Let (non-constant) f be analytic on the domain $W := \mathbb{C} \setminus \{z_1, z_2, \ldots, z_n\}$. We want to show that the image of f is dense in $\mathbb{C}$.

Suppose not; that is, suppose that there is some $w_0 \in \mathbb{C}$ such that there is some $\epsilon > 0$ for which there are no $z \in W$ that satisfy $|f(z) - w_0| < \epsilon$. In other words if the image of f is not dense in $\mathbb{C}$, there is some $w_0 \in \mathbb{C}$ and $\epsilon > 0$ such that $|f(z) - w_0| \geq \epsilon$ for all $z \in W$. That means that for all $z \in W$,

$$g(z) := \frac{1}{f(z) - w_0} \leq \frac{1}{\epsilon}$$

and so g is bounded and analytic on W. By the Riemann removable singularity theorem (because g is bounded on a neighborhood around each of the z_i), g can be extended analytically over all the z_i to get an entire function. Now we have that g is bounded and entire, so by Liouville's theorem, g is constant, implying that $f(z) - w_0$ is constant, and hence f is constant on $\mathbb{C}$, which contradicts our initial statement that f was non-constant.

Problem 9

The one property we know that is equivalent to harmonic on any domain W (not just star shaped ones or anything) is that a continuous function h on a domain W is harmonic if and only if it satisfies the mean value property, where the mean value property is that $z_0 \in W$, there is $\epsilon > 0$ such that

$$h(z_0) = \frac{1}{2\pi} \int_0^{2\pi} h(z_0 + re^{i\theta})\, d\theta$$

for all $r \in (0, \epsilon)$. Moreover, as long as $\epsilon_{z_0} > 0$ is small enough so that $B(z_0, \epsilon_{z_0})$ is contained inside W (remember that W is open), the mean value property at z_0 holds for all $r \in (0, \epsilon_{z_0})$. Thus if we fix some $z_0 \in W$ (and δ_0 such that $B(z_0, \delta_{z_0}) \subset W$, and if we have a sequence u_n of harmonic functions on a domain W that converge uniformly on compact sets, every single one of them satisfies the mean value property for $r \in (0, \delta_{z_0})$. Furthermore, on the compact set $\overline{B(z_0, \delta_{z_0}/2)}$, the u_n converge uniformly to some u and so we can pull the limit outside the integral as such: for any $r \in (0, \delta_{z_0}/2) \subset (0, \delta_{z_0})$,

$$\frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta})\, d\theta = \lim_{n\to\infty} \frac{1}{2\pi} \int_0^{2\pi} u_n(z_0 + re^{i\theta})\, d\theta = \lim_{n\to\infty} u_n(z_0) = u(z_0)$$

and so u satisfies the mean value property. Lastly, u is continuous because the the convergence is uniform and the u_n are harmonic and hence continuous, and so by the "continuous $\implies$ (harmonic

$\iff$ mean value property)" theorem mentioned above, u must be harmonic.

Problem 10

We have $f(z) = \frac{z-a}{1-\bar{a}z}$ where $|a| < 1$. As before, let $\mathbb{D}$ denote the open unit disk.

(a) As in page 51 of Gamelin, we have that

$$\det J_f(z) = |f'(z)|^2 \text{ where } J_f = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$$

We then note that f is a bijective map from $\mathbb{D}$ to $\mathbb{D}$ (pg. 263 of Gamelin). Thus, by the change of variables formula for double integrals,

$$\iint_{\mathbb{D}} |f'(z)|\, dA = \iint_{\mathbb{D}} \det J_f(x,y)\, dA = \iint_{f(\mathbb{D})} dA = \text{Area}(f(\mathbb{D})) = \text{Area}(\mathbb{D}) = \pi$$

or rearranging:

$$\frac{1}{\pi} \iint_{\mathbb{D}} |f'(z)|\, dA = 1$$

(b) For this, we don't have the luxury of substituting the Jacobian because the $|f'|$ is not squared, so we have to actually compute f' (or rather $|f'|$):

$$|f'(z)| = \left| \frac{(1-\bar{a}z)1 - (z-a)(-\bar{a})}{(1-\bar{a}z)^2} \right| = \left| \frac{1-|a|^2}{(1-\bar{a}z)^2} \right| = \frac{1-|a|^2}{|1-\bar{a}z|^2}$$

From pg. 276 of Gamelin, the Poisson kernel $P_r(\theta)$ for $z = re^{i\theta} \in \mathbb{D}$ is

$$P_r(\theta) = \frac{1-|z|^2}{|1-z|^2}$$

Thus, writing $|f'(z)|$ in such a way to involve the Poisson kernel, we see (for $z = re^{i\varphi} \in \mathbb{D}$) that denoting $\bar{a}z = \bar{a}re^{i\varphi} = |a|re^{i(\varphi-\theta)}$ (where $a = |a|e^{i\theta}$),

$$|f'(z)| = \frac{1-|\bar{a}z|^2}{|1-\bar{a}z|^2} \cdot \frac{1-|a|^2}{1-|\bar{a}z|^2} = P_{|a|r}(\varphi-\theta) \cdot \frac{1-|a|^2}{1-|\bar{a}z|^2} = P_{|a|r}(\varphi-\theta) \cdot \frac{1-|a|^2}{1-|a|^2r^2}$$

and so

$$\begin{aligned} \frac{1}{\pi} \int_{\mathbb{D}} |f'(z)|\, dx\, dy &= \frac{1}{\pi} \int_0^1 \int_{-\pi}^{\pi} |f'(re^{i\varphi})| \cdot r\, d\varphi\, dr \\ &= \frac{1}{\pi} \int_0^1 \int_{-\pi}^{\pi} P_{|a|r}(\varphi-\theta) \cdot \frac{1-|a|^2}{1-|a|^2r^2} \cdot r\, d\varphi\, dr \\ &= \frac{1}{\pi} \int_0^1 \frac{r(1-|a|^2)}{1-|a|^2r^2} \left(\int_{-\pi}^{\pi} P_{|a|r}(\varphi-\theta)\, d\varphi \right) dr \\ &= \frac{1}{\pi} \int_0^1 \frac{r(1-|a|^2)}{1-|a|^2r^2} \left(\int_{-\pi}^{\pi} P_{|a|r}(\varphi)\, d\varphi \right) dr \end{aligned}$$

where the last equality used the 2π-periodicity of the Poisson kernel (the same way Gamelin used

in going from (1.2a) to (1.2b) on pg. 275 of Gamelin). (1.5) on pg. 276 of Gamelin gives that the integral in the parentheses above is 2π, and so

$$\begin{aligned}
\frac{1}{\pi}\int_{\mathbb{D}} |f'(z)|\,dx\,dy &= \frac{1}{\pi}\cdot 2\pi\cdot\int_0^1 \frac{r(1-|a|^2)}{1-|a|^2r^2}\,dr \\
&= 2\cdot\frac{1-|a|^2}{-2|a|^2}\cdot\int_0^1 \frac{-2|a|^2r}{1-|a|^2r^2}\,dr \\
&= -\frac{1-|a|^2}{|a|^2}\Big(\log(1-|a|^2r^2)\Big)\Big|_{r=0}^{r=1} \\
&= -\frac{1-|a|^2}{|a|^2}\log(1-|a|^2) = \frac{1-|a|^2}{|a|^2}\log\left(\frac{1}{1-|a|^2}\right)
\end{aligned}$$

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§VII.3 - Problem 2

Following Gamelin's example in the section, we let $z = e^{i\theta} \implies d\theta = \frac{1}{iz}\,dz$, and so

$$\int_0^{2\pi} \frac{1}{a + b\sin\theta}\,d\theta = \oint_{|z|=1} \frac{1}{a + b(\frac{1}{2i}(z + 1/z))}\frac{dz}{iz} = 2\oint_{|z|=1} \frac{1}{bz^2 + 2iaz - b}\,dz$$

The poles of the integrand are the two zeros of $bz^2 + 2iaz - b$ which are

$$\frac{-2ai \pm \sqrt{-4a^2 + 4b^2}}{2b} = i\left(\frac{-a \pm \sqrt{a^2 - b^2}}{b}\right)$$

But notice that

$$\begin{aligned} a > b > 0 &\implies a^2 - 2b(a-b) < a^2 < a^2 + b^2 \implies a^2 - 2ab + b^2 < a^2 - b^2 < a^2 \\ &\implies a - b < \sqrt{a^2 - b^2} < a \implies -1 < \frac{-a + \sqrt{a^2 - b^2}}{b} < 0 \end{aligned}$$

and

$$a > b > 0 \implies \frac{a}{b} > 1 \implies \frac{a + \sqrt{a^2 - b^2}}{b} > 1 \implies \frac{-a - \sqrt{a^2 - b^2}}{b} < -1$$

and so only the first root is inside the unit circle. Thus, by Rule 3,

$$\operatorname{Res}\left[\frac{2}{bz^2 + 2iaz - b}, i\left(\frac{-a + \sqrt{a^2 - b^2}}{b}\right)\right] = \frac{2}{2bi\left(\frac{-a+\sqrt{a^2-b^2}}{b}\right) + 2ai} = \frac{1}{i\sqrt{a^2 - b^2}}$$

and so by the residue theorem, (and for $a > b > 0$),

$$\int_0^{2\pi} \frac{1}{a + b\sin\theta}\,d\theta = 2\oint_{|z|=1} \frac{1}{bz^2 + 2iaz - b}\,dz = \frac{2\pi i}{i\sqrt{a^2 - b^2}} = \frac{2\pi}{\sqrt{a^2 - b^2}}$$

§VIII.1 - Problem 1

Consider $p(z) = z^4 + 2z^2 - z + 1$. p does not have any zeroes on the real or imaginary axes: For $x \le 0$, $p(x)$ is obviously > 0. For $x > 0$, $x^2 - x + 1$ is obviously positive. Now for the imaginary axis: $p(iy) = y^4 - 2y^2 - iy + 1$ is clearly non-zero for $y \neq 0$ because of the non-zero imaginary part, and for $y = 0$, $p(iy) = 1$.

We do as Gamelin does in the example for the section (page 227): consider the quarter disk D_R with radius R, whose boundary has three parts: a line segment on the real axis (going right), a quarter circle Γ_R, and a line segment on the imaginary axis (going down). Along the segment on the real axis, the argument is 0 because everything stays real. Along Γ_R, for large enough R, z^4 dominates so

$\arg p(z) \approx 4 \arg z$, and so because along $\Gamma_R, \arg z$ goes from 0 to $\frac{\pi}{2}$, we have that $\arg p(z)$ along Γ_R increases $\approx 4(\frac{\pi}{2}) = 2\pi$. Now at the beginning of the line segment on the imaginary axis, $p(iR)$ has real part about $\approx R^4$ and imaginary part $\approx -R$ so $p(iR)$ lies in the fourth quadrant and has argument ≈ 0. Finally, notice that as z goes from iR to 0, $p(z)$ does not cross the real axis (the imaginary part is only zero for $y = 0$), and so the argument stays ≈ 0. Thus, the total increase of $\arg p(z)$ around ∂D_R is approximately 2π (and so it must be exactly 2π by the argument principle), which then (again by the argument principle) means that there is exactly 1 zero in the 1st quadrant.

The coefficients of p are real, so the roots must come in conjugates, which tells us that there is a root in the 4th quadrant. By the fundamental theorem of algebra, there are 2 other roots, which by root-conjugacy means that there is one root in quadrant 2 and one root in quadrant 3, giving us one root in each quadrant.

Problem 2

Consider $p(z) = z^4 + z^3 + 4z^2 + 3z + 2$. p does not have any zeroes on the real or imaginary axes: For $x \geq 0$, $p(x)$ is obviously > 0. For $x < 0$, $x^4 + x^3 + 1 > 0$ and $3x^2 + 3x + 1 > 0$ so $p(x)$ is positive. Now for the imaginary axis: $p(iy) = y^4 - iy^3 - 4y^2 + 3iy + 2$ is clearly non-zero for $y \neq 0, \pm\sqrt{3}$ because of the non-zero imaginary part, and for $y = 0, \pm\sqrt{3}$, the real part of $p(iy)$, i.e. $y^4 - 4y^2 + 2$ is positive.

We do as Gamelin does in the example for the section (page 227): consider the quarter disk D_R with radius R, whose boundary has three parts: a line segment on the real axis (going right), a quarter circle Γ_R, and a line segment on the imaginary axis (going down). Along the segment on the real axis, the argument is 0 because everything stays real. Along Γ_R, for large enough R, z^4 dominates so $\arg p(z) \approx 4 \arg z$, and so because along $\Gamma_R, \arg z$ goes from 0 to $\frac{\pi}{2}$, we have that $\arg p(z)$ along Γ_R increases $\approx 4(\frac{\pi}{2}) = 2\pi$. Now at the beginning of the line segment on the imaginary axis, $p(iR)$ has real part about $\approx R^4$ and imaginary part $\approx -R^3$ so $p(iR)$ lies in the fourth quadrant and has argument ≈ 0. Finally, notice that as z goes from iR to 0, $p(z)$ crosses the real axis exactly once at $y = \sqrt{3}$, where $p(i\sqrt{3}) = 9 - 12 + 2 = -1$, and at $y = 0$, $p(0) = 2$ and so the argument goes from just below the real axis to one clockwise rotation around ending on the real axis (so a total of -2π). Thus, the total increase of $\arg p(z)$ around ∂D_R is approximately $2\pi - 2\pi = 0$ (and so it must be exactly 0 by the argument principle), which then (again by the argument principle) means that there is exactly 0 zeros in the 1st quadrant.

By the fundamental theorem of algebra, there are 4 roots, which by root-conjugacy means that there is two roots in quadrant 2 and two roots in quadrant 3 (and none in quadrants 1 and 4).

§VIII.2 - Problem 1

Let $p(z) = 2z^5 + 6z - 1$. Let $h(z) = 2z^5 - 1$ and let $f(z) = 6z$ (obviously all of these functions are analytic everywhere on $\mathbb{C}$). Then for z on the unit circle, $|h(z)| \leq 2 + 1 < 6 = |6z| = |f(z)|$, and so by Rouché's theorem, $f(z)$ and $p(z) = f(z) + h(z)$ have the same number of zeros (counting multiplicities)

within the unit circle, where $f(z)$ clearly has a zero with multiplicity 1 at $z = 0$. Thus $p(z)$ has one zero on the unit disk, but it has real coefficients, so complex roots must come in conjugates, and if there is only one root in the unit disk, that must mean that it is on the real axis. Because for $x \leq 0$, $p(x)$ is obviously negative, that must mean that $p(x)$ has a zero for $x \in (0, 1)$.

Now let $h(z) = 6z - 1$ and $f(z) = 2z^5$ (obviously all of these functions are analytic everywhere on $\mathbb{C}$). Then for z s.t. $|z| = 2$, $|h(z)| \leq 12 + 1 < 2(2^5) = |f(z)|$, and so by Rouché's theorem, $f(z)$ and $p(z) = f(z) + h(z)$ have the same number of zeros (counting multiplicities) within the unit circle, where $f(z)$ clearly has a zero with multiplicity 5 at $z = 0$. Thus $p(z)$ has five zeros in $\{|z| < 2\}$. We know there's only one root in the unit circle, and there are no zeros on the unit circle (one could repeat the argument from the first paragraph with a circle with radius r slightly bigger than 1, while still having $2r^5 + 1 < 6r$), and so all four other roots must lie in $\{1 < |z| < 2\}$.

Problem 3

We are given $p(z) = 1 + z + \frac{z^2}{2!} + \ldots + \frac{z^m}{m!} + 3z^n$. Take $h(z)$ to be $p(z) - 3z^n$, and let $f(z) = 3z^n$ so that on the unit circle,

$$|h(z)| \leq 1 + |z| + \frac{|z^2|}{2!} + \ldots + \frac{|z^m|}{m!} \leq \sum_{k=0}^{\infty} \frac{1}{k!} = e < 3 = |f(z)|$$

which means that we can apply Rouché's theorem which says that p and f must have the same number of zeroes (counting multiplicity) in the unit circle, where f clearly has a zero at $z = 0$ of multiplicity n. That is, $p(z)$ has n zeroes counting multiplicity in the unit circle.

§VIII.3 - Problem 1

Because f is analytic, its zeroes are isolated. That means that there is small enough $\rho > 0$ such that $|f(z)| > 0$ on $\{|z - z_0| = \rho\} \subset D$. Furthermore, this is a compact set, so it attains its minimum, say $\alpha > 0$; i.e. on $\{|z - z_0| = \rho\}$, $|f(z)| \geq \alpha$. Now because convergence of $f_k \to f$ on D means uniform convergence on every compact subset of D, we have that for all k large enough, $|f_k(z) - f(z)| < \alpha$ for z in $\{|z - z_0| = \rho\}$. Thus, by Rouché's theorem, $p(z) = (f_k(z) - f(z)) + f(z)$ has the exactly the same number of zeroes counting multiplicity in $\{|z - z_0| < \rho\}$ as f, that number being of course N.

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§VI.1 - Problem 5

Suppose that f is analytic on the puncture plan $D = \mathbb{C} \setminus \{0\}$. Show that there's a constant c such that $f(z) - c/z$ has a primitive on D (and give a formula for c in terms of an integral of $f(z)$).

Applying the Laurent Series Expansion theorem with $\sigma = \infty$ and $\rho = 0$ and $z_0 = 0$, we get that

$$f(z) = \sum_{k=-\infty}^{\infty} a_k z^k \text{ where } a_n = \frac{1}{2\pi i} \oint_{|z|=r} \frac{f(z)}{z^{n+1}}\, dz$$

for all $z \in D$ and $r \in (0, \infty)$ (where the convergence is uniform on every closed subannulus in D). Recall from Homework 5, §IV.2.5, that we proved that $f(z)$ has a primitive in D if and only if $\int_\gamma f(z)\, dz = 0$ for every closed path γ in D. Now consider $g(z) = f(z) - \frac{a_{-1}}{z} = \sum_{k \neq -1} a_k z^k$: for any γ, it lies in sum subannulus of D (and hence we have the sum is uniformly convergent), so we can exchange the sum and integral via the Theorem on page 135:

$$\int_\gamma g(z)\, dz = \sum_{k \neq -1} a_k \int_\gamma z^k\, dz = 0$$

where the integral around any γ of z^k for $k \neq -1$ is 0 because any z^k for $k \neq 1$ has a primitive (the same as the real antiderivative). Thus, $g(z)$ has a primitive on D, where

$$c = a_{-1} = \frac{1}{2\pi i} \oint_{|z|=r} f(z)\, dz$$

Problem 7

Show that a harmonic function u on an annulus $D := \{A < |z| < B\}$ has a unique expansion

$$u\left(re^{i\theta}\right) = \sum_{n=-\infty}^{\infty} a_n r^n \cos(n\theta) + \sum_{n \neq 0} b_n r^n \sin(n\theta) + c \log r$$

which is uniformly convergent on each circle in the annulus. Show that for each $r, A < r < B$, the coefficients a_n, b_n, and c satisfy

$$\begin{aligned}
a_n r^n + a_{-n} r^{-n} &= \frac{1}{\pi} \int_{-\pi}^{\pi} u\left(re^{i\theta}\right) \cos(n\theta) d\theta, \quad n \neq 0 \\
b_n r^n - b_{-n} r^{-n} &= \frac{1}{\pi} \int_{-\pi}^{\pi} u\left(re^{i\theta}\right) \sin(n\theta) d\theta, \quad n \neq 0 \\
a_0 + c \log r &= \frac{1}{2\pi} \int_{-\pi}^{\pi} u\left(re^{i\theta}\right) d\theta
\end{aligned}$$

Hint. Use a decomposition of the form $u = \operatorname{Re} f + c \log |z|$, where f is analytic on the annulus. (See Exercise III.3.4.)

By problem §III.3.4, any harmonic function u has some c for which $u(z) - c \log |z|$ is harmonic on D/is the real part of an analytic function f on D. This means that $u = \operatorname{Re} f + c \log |z|$ on D. Now because f is analytic on D, we use the Laurent Series Expansion theorem to see that f has a Laurent series $\sum_{n=-\infty}^{\infty} (a_n - ib_n) z^n$ that converges absolutely everywhere on D, and uniformly on any subannulus. Then, with all this in mind, we see that

$$\begin{aligned}
\operatorname{Re}(f(re^{i\theta})) &= \operatorname{Re}\left(\sum_{n=-\infty}^{\infty} (a_n - ib_n) r(\cos\theta + i \sin\theta) \right) = \sum_{n=-\infty}^{\infty} \operatorname{Re}\Big((a_n - ib_n) r^n (\cos n\theta + i \sin n\theta) \Big) \\
&= \sum_{n=-\infty}^{\infty} a_n r^n \cos n\theta + b_n r^n \sin n\theta = \sum_{n=-\infty}^{\infty} a_n r^n \cos n\theta + \sum_{n=-\infty}^{\infty} b_n r^n \sin n\theta \\
&= \sum_{n=-\infty}^{\infty} a_n r^n \cos n\theta + \sum_{n \neq 0} b_n r^n \sin n\theta
\end{aligned}$$

and so we have the representation

$$u(re^{i\theta}) = \sum_{n=-\infty}^{\infty} a_n r^n \cos n\theta + \sum_{n \neq 0} b_n r^n \sin n\theta + c \log r$$

Furthermore, this is unique, because the c coefficient for the log term is unique (there's an integral formula for it based on u in III.3.4), and harmonic conjugates are unique (see page 55 of Gamelin), and the coefficients of Laurent series are unique (see Laurent Series Expansion theorem, page 168 Gamelin). Moreover, due to uniform convergence on subannuli (allowing interchange of sums and integrals!), we have for $n \neq 0$ that

$$\begin{aligned}
\int_{-\pi}^{\pi} u(re^{i\theta}) \cos(n\theta)\, d\theta = \sum_{k=-\infty}^{\infty} a_k r^k \int_{-\pi}^{\pi} \cos k\theta \cos n\theta \, d\theta \quad &+ \sum_{k \neq 0} b_k r^k \int_{-\pi}^{\pi} \sin k\theta \cos n\theta \, d\theta \\
&+ c \log r \int_{-\pi}^{\pi} \cos n\theta \, d\theta \\
= a_n r^n \pi + a_n \pi r^{-n}&
\end{aligned}$$

where the integrals (one can verify on WolframAlpha) are:

$$\int_{-\pi}^{\pi} \cos k\theta \cos n\theta \, d\theta = \begin{cases} 0 & \text{if } k \neq n \\ \pi & \text{if } k = n \end{cases} \quad \text{and} \quad \int_{-\pi}^{\pi} \sin k\theta \cos n\theta \, d\theta = 0 \quad \text{and} \quad \int_{-\pi}^{\pi} \cos n\theta \, d\theta = 0$$

The sine integral follows similarly. The final formula is verified as follows:

$$\begin{aligned}\int_{-\pi}^{\pi} u(re^{i\theta})\,d\theta &= \sum_{n=-\infty}^{\infty} a_n r^n \int_{-\pi}^{\pi} \cos n\theta\,d\theta + \sum_{n\neq 0} b_n r^n \int_{-\pi}^{\pi} \sin n\theta\,d\theta + c\log r \int_{-\pi}^{\pi} d\theta \\ &= 2\pi a_0 + 2\pi C\log r\end{aligned}$$

§VI.2 - Problem 10

We use Homework 7, §V.4.13 N times (because the derivatives of analytic functions are analytic, and because N being the order of the zero means that $f'(z_0) = \ldots = f^{(N-1)}(z_0) = 0$) to get that

$$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \lim_{z\to z_0} \frac{f^{(N)}(z)}{g^{(N)}(z)} = \frac{f^{(N)}(z_0)}{g^{(N)}(z_0)}$$

Problem 11

Suppose that $f(z) = \sum a_k z^k$ is analytic for $|z| < R$, and suppose that $f(z)$ extends to be meromorphic for $|z| < R + \epsilon$ with one pole z_0 on the circle $|z| = R$. Show that $a_k/a_{k+1} \to z_0$ as $k \to \infty$.

Looking at the discussion at the end of page 172 and start of page 173 for Gamelin, let

$$f(z) - \sum_{j=1}^{m} \frac{b_j}{(z-z_0)^j} = \sum c_k z^k$$

be analytic in $|z| < R + \epsilon$ (i.e. we are subtracting the principle part of f). We know that for $|z| < |z_0| = R$,

$$\frac{1}{z-z_0} = -\frac{1/z_0}{1-\frac{z}{z_0}} = -\frac{1}{z_0}\sum_{n=0}^{\infty}\left(\frac{z}{z_0}\right)^n$$

and so taking derivatives (we can take derivatives of power series, and those power series will have the same radius of convergence) we see that

$$f^{(1)}(z) = (-1)\frac{1}{(z-z_0)^2} = -\frac{1}{z_0}\sum_{n=1}^{\infty}(z_0^{-n})nz^{n-1} = -\frac{1}{z_0}\sum_{n=0}^{\infty}(z_0^{-n-1})(n+1)z^n$$

and in general:

$$f^{(j-1)}(z) = (-1)(-2)\cdots(-(j-1))\frac{1}{(z-z_0)^j} = -\frac{1}{z_0}\sum_{n=0}^{\infty}(z_0^{-n-(j-1)})(n+1)\cdots(n+(j-1))z^n$$

or via rearrangement:

$$\frac{1}{(z-z_0)^j} = \frac{(-1)^j}{(j-1)!}\frac{1}{z_0}\sum_{n=0}^{\infty}\frac{(n+1)\cdots(n+(j-1))}{z_0^{n+(j-1)}}z^n$$

and so looking at the coefficients of z^k, we see that

$$a_k = c_k + \sum_{j=1}^{m} \frac{(-1)^j}{(j-1)!z_0} b_j \frac{(k+j-1)\dots(k+1)}{z_0^{k+j-1}}$$
$$a_{k+1} = c_{k+1} + \sum_{j=1}^{m} \frac{(-1)^j}{(j-1)!z_0} b_j \frac{(k+j)\dots(k+2)}{z_0^{k+j}}.$$

Dividing and multiplying top and bottom by z_0^{k+1} we get

$$\begin{aligned}\frac{a_k}{a_{k+1}} &= z_0 \frac{c_k z_0^k + \sum_{j=1}^{m} \frac{(-1)^j}{(j-1)!} b_j (k+j-1)\dots(k+1) z_0^{-j}}{c_{k+1} z_0^{k+1} + \sum_{j=1}^{m} \frac{(-1)^j}{(j-1)!} b_j (k+j)\dots(k+2) z_0^{-j}} \\ &= z_0 \frac{c_k z_0^k + \frac{(-1)^m}{(m-1)!} b_m z_0^{-m} k^{m-1} + \dots}{c_{k+1} z_0^{k+1} + \frac{(-1)^m}{(m-1)!} b_m z_0^{-m} k^{m-1} + \dots}.\end{aligned}$$

Since $\sum c_k z_0^k$ converges, $c_k z_0^k \to 0$ as $k \to \infty$. Now let $k \to \infty$ and see that

$$\frac{a_k}{a_{k+1}} \to z_0.$$

Problem 12

Suppose that z_0 is an isolated singularity of $f(z)$ that is not removable. Then it is an isolated singularity of e^f at z_0.

We split into two cases:

- The isolated singularity of f at z_0 is a pole (say of order N): then using the theorem on page 173, $f(z) = g(z)/(z-z_0)^N$, where $g(z)$ is analytic at z_0 and $g(z_0) \neq 0$. Denote $g(z_0) = r_0 e^{i\theta_0}$. We prove that e^f has an essential singularity at z_0 by find two sequences that converge to z_0:
  - Define the sequence $z_n = z_0 + \frac{1}{n} e^{i\theta_0/n}$, so that

    $$f(z_n) = g(z_n) n^n e^{-i\theta_0}$$

    and so as $n \to \infty$, $e^{f(z_n)} = e^{g(z_n) e^{-i\theta_0} n^n}$ goes to infinity because $g(z_n) \to r_0 e^{i\theta_0}$ where $r_0 > 0$. This shows that e^f is not bounded near z_0, and hence by Riemann's removable singularity theorem, the singularity of e^f at z_0 can't be removable.
  - Define $z_n = z_0 + \frac{1}{n} e^{i(\theta_0+\pi)/n}$ so that

    $$f(z_n) = g(z_n) n^n e^{-i(\theta_0+\pi)}$$

and so as $n \to \infty$, $e^{f(z_n)} = e^{g(z_n)e^{-i\theta_0}e^{-i\pi}n^n} = e^{-g(z_n)e^{-i\theta_0}n^n} \to 0$ (because $g(z_n) \to r_0 e^{i\theta_0}$ where $r_0 > 0$). With these two sequences, we've shown that $\lim_{z\to z_0} |e^f| = \infty$ is false, and so the singularity of e^f at z_0 cannot be a pole, implying of course that it's an essential singularity.

- The isolated singularity of f at z_0 is essential: well, by the Casorati-Weierstrass Theorem, for every $w_0 \in \mathbb{C}$, there is a sequence $z_n \to z_0$ such that $f(z_n) \to w_0$. Thus we can find two sequences, 1 for $w_0 = 0$ and the other for w_0 as big as we desire, meaning (similar to above) that the singularity can't be removable (because there's a sequence $\{z_n\}$ s.t. $f(z_n)$ converges to arbitrarily large w_0), and it can't be a pole (because there's a sequence s.t. $f(z_n)$ converges to 0), which means that e^f has an essential singularity at z_0.

§VII.1 - Problem 3c

The zeroes of $\cos z$ are all real and simple, and are at $\frac{\pi}{2} + k\pi$. Thus in the set $\{z : |z| < 2\}$ we have that $z/\cos z$ has two simple poles at $\pm\frac{\pi}{2}$. Thus using the residue theorem and applying rule 3,

$$\begin{aligned}\oint_{|z|=2} \frac{z}{\cos z}\, dz &= 2\pi i \left(\operatorname{Res}\left[\frac{z}{\cos z}, \pi/2\right] + \operatorname{Res}\left[\frac{z}{\cos z}, -\pi/2\right]\right) \\ &= 2\pi i \left(\frac{\pi/2}{-\sin\frac{\pi}{2}} + \frac{-\pi/2}{-\sin(-\frac{\pi}{2})}\right) = 2\pi i \left(-\tfrac{\pi}{2} - \tfrac{\pi}{2}\right) = -2\pi^2 i\end{aligned}$$

Problem 4

Suppose $P(z)$ and $Q(z)$ be polynomials such that the zeros of $Q(z)$ are simple zeros at the points $z_1, \ldots, z_m$ and $\deg P(z) < \deg Q(z)$. We want to find the partial fraction decomposition. Rule 3 says that

$$\operatorname{Res}\left[\frac{P(z)}{Q(z)}, z_j\right] = \frac{P(z_j)}{Q'(z_j)}$$

The theorem on the middle of page 180 and the following paragraph says that in this situation (i.e. $\deg P < \deg Q$), that $\frac{P}{Q}$ is the sum of the principal parts at each of its poles, which in this case are $\frac{P(z_j)}{Q'(z_j)(z-z_j)}$ at each simple pole z_j (because the residue $= a_{-1}$, and we only have a_{-1} because the poles are simple), which when all put together yields

$$\frac{P(z)}{Q(z)} = \sum_{j=1}^{m} \frac{P(z_j)}{Q'(z_j)(z - z_j)}$$

§VII.2 - Problem 5

We proceed as Gamelin does in his examples for this section: let D_R be the half-disk in the upper half-plane bounded by the interval $[-R, R]$ on the real axis and the semicircular contour Γ_R of radius R in the upperhalf-plane. The function we are considering, $\frac{z^2}{z^4+1}$ has two simple poles inside D_R at

$\frac{1+i}{\sqrt{2}}$ and $\frac{-1+i}{\sqrt{2}}$. Rule 3 gives that

$$\operatorname{Res}\left[\frac{z^2}{z^4+1}, \frac{1+i}{\sqrt{2}}\right] = \frac{(\frac{1+i}{\sqrt{2}})^2}{4(\frac{1+i}{\sqrt{2}})^3} = \frac{\sqrt{2}}{4(1+i)} \quad \text{and} \quad \operatorname{Res}\left[\frac{z^2}{z^4+1}, \frac{-1+i}{\sqrt{2}}\right] = \frac{\sqrt{2}}{4(-1+i)}$$

and the residue theorem gives that,

$$\int_{D_R} \frac{z^2}{z^4+1}\,dz = 2\pi i\left(\frac{\sqrt{2}}{4(1+i)} + \frac{\sqrt{2}}{4(-1+i)}\right) = 2\pi i\left(\frac{\sqrt{2}(1-i)}{8} - \frac{\sqrt{2}(1+i)}{8}\right) = \frac{\pi}{\sqrt{2}}$$

For $z \in \Gamma_R$ we have that

$$\left|\frac{z^2}{z^4+1}\right| = \frac{|z^2|}{|z^4+1|} \leq \frac{|z^2|}{|z^4|-|1|} = \frac{R^2}{R^4-1}$$

so by the ML-estimate we have that

$$\left|\int_{\Gamma_R} \frac{z^2}{z^4+1}\,dz\right| \leq \frac{R^2}{R^4-1}\pi R$$

which goes to 0 as $R \to \infty$. Thus, combining everything we get that

$$\frac{\pi}{\sqrt{2}} = \int_{D_R} \frac{z^2}{z^4+1}\,dz = \int_{-R}^{R} \frac{x^2}{x^4+1}\,dx + \int_{\Gamma_R} \frac{z^2}{z^4+1}\,dz$$

and so finally we have that

$$\int_0^\infty \frac{x^2}{x^4+1}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2}{x^4+1}\,dx = \frac{1}{2}\lim_{R\to\infty}\int_{-R}^{R} \frac{x^2}{x^4+1}\,dx = \frac{\pi}{2\sqrt{2}}$$

Problem 8

We proceed as Gamelin does in his examples for this section: let D_R be the half-disk in the upper half-plane bounded by the interval $[-R, R]$ on the real axis and the semicircular contour Γ_R of radius R in the upperhalf-plane. The function we are considering, $\frac{e^{iz}}{(1+z^2)^2}$ has one double pole inside D_R at i. Because it's a double pole, we use Rule 2, which gives that

$$\operatorname{Res}\left[\frac{e^{iz}}{(1+z^2)^2}, i\right] = \lim_{z\to i}\frac{d}{dz}\frac{e^{iz}}{(z+i)^2} = \lim_{z\to i}\frac{(z+i)^2 ie^{iz} - e^{iz}(2(z+i))}{(z+i)^4} = -\frac{i}{2e}$$

Then, the residue theorem gives that,

$$\int_{D_R} \frac{e^{iz}}{(1+z^2)^2}\,dz = \frac{\pi}{e}$$

For $z \in \Gamma_R$ we have that

$$\left|\frac{e^{iz}}{(z^2+1)^2}\right| \leq \frac{1}{(R^2-1)^2}$$

so by the ML-estimate we have that (recalling from page 201 of Gamelin that $|e^{iz}| \leq 1$ on the upper half plane)

$$\left| \int_{\Gamma_R} \frac{e^{iz}}{(z^2+1)^2}\, dz \right| \leq \frac{1}{(R^2-1)^2} \pi R$$

which goes to 0 as $R \to \infty$. Thus, combining everything we get that

$$\frac{\pi}{e} = \int_{D_R} \frac{e^{iz}}{(z^2+1)^2}\, dz = \int_{-R}^{R} \frac{e^{ix}}{(x^2+1)^2}\, dx + \int_{\Gamma_R} \frac{e^{iz}}{(z^2+1)^2}\, dz$$

and so finally we have that

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+1)^2}\, dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{e^{ix}}{(x^2+1)^2}\, dx = \frac{\pi}{e} \implies \int_{-\infty}^{\infty} \frac{\cos(x)}{(x^2+1)^2}\, dx = \frac{\pi}{e}$$

by looking at real parts.

Problem 9

Note that the real part of $\frac{1}{2}(1 - e^{2ix})$ is $\sin^2 x$. We proceed as Gamelin does in his examples for this section: let D_R be the half-disk in the upper half-plane bounded by the interval $[-R, R]$ on the real axis and the semicircular contour Γ_R of radius R in the upperhalf-plane. The function we are considering, $\frac{1-e^{2iz}}{2(z^2+1)}$ has one simple pole inside D_R at i. Rule 3 gives that

$$\operatorname{Res}\left[\frac{1-e^{2iz}}{2(z^2+1)}, i\right] = \frac{1-e^{2i(i)}}{4(i)} = \frac{1-e^{-2}}{4i}$$

and the residue theorem gives

$$\int_{D_R} \frac{1-e^{2iz}}{2(z^2+1)}\, dz = \frac{\pi}{2}(1 - \tfrac{1}{e^2})$$

For $z \in \Gamma_R$ we have that

$$\left| \frac{1-e^{2ix}}{2(z^2+1)} \right| \leq \frac{|1| + |e^{2iz}|}{|z^4+1|} \leq \frac{|z^2|}{|z^4|-|1|} = \frac{R^2}{R^4-1}$$

so by the ML-estimate we have that (recalling from page 201 of Gamelin that $|e^{iz}| \leq 1$ on the upper half plane)

$$\left| \int_{\Gamma_R} \frac{1-e^{2iz}}{2(z^2+1)}\, dz \right| \leq \frac{|1| + |e^{2iz}|}{2(|z^2|-|1|)} \leq \frac{2}{2(R^2-1)} \pi R$$

which goes to 0 as $R \to \infty$. Thus, combining everything we get that

$$\frac{\pi}{2}(1 - \tfrac{1}{e^2}) = \int_{D_R} \frac{1-e^{2iz}}{2(z^2+1)}\, dz = \int_{-R}^{R} \frac{1-e^{2ix}}{2(x^2+1)}\, dx + \int_{\Gamma_R} \frac{1-e^{2iz}}{2(z^2+1)}\, dz$$

and so finally we have that

$$\int_{-\infty}^{\infty} \frac{1-e^{2ix}}{2(x^2+1)}\, dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{1-e^{2ix}}{2(x^2+1)}\, dx = \frac{\pi}{2}(1 - \tfrac{1}{e^2}) \implies \int_{-\infty}^{\infty} \frac{\sin^2(x)}{x^2+1}\, dx = \frac{\pi}{2}(1 - \tfrac{1}{e^2})$$

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§V.3 - Problem 3

(a) The series $\sum_{n=0}^{\infty} z^{3^n}$ (i.e. $a_k = 1$ if $k = 3^n$ for some n, and 0 otherwise) has radius of convergence 1 by the Cauchy-Hadamard formula:

$$R = \frac{1}{\limsup_{k\to\infty} |a_k|^{1/k}} = \frac{1}{1} = 1$$

where $\limsup_{k\to\infty} |a_k|^{1/k} = \limsup_{k\to\infty} a_k$ (because a_k only takes on 0 or 1, and 0 or 1 to the $1/k$ power is still 0 or 1), and the limsup of a_k is 1 because there are infinitely many 1's.

(b) The series $\sum_{p \text{ prime}} z^p$ (i.e. $a_k = 1$ if k is prime) has radius of convergence 1 by the Cauchy-Hadamard formula:

$$R = \frac{1}{\limsup_{k\to\infty} |a_k|^{1/k}} = \frac{1}{1} = 1$$

where $\limsup_{k\to\infty} |a_k|^{1/k} = \limsup_{k\to\infty} a_k$ (because a_k only takes on 0 or 1, and 0 or 1 to the $1/k$ power is still 0 or 1), and the limsup of a_k is 1 because there are infinitely many primes (by the famous Euclid proof by contradiction).

Problem 4

The series $f(z) = \sum_{n=0} z^{n!}$ (i.e. $a_k = 1$ if $k = n!$) has radius of convergence 1 by the Cauchy-Hadamard formula:

$$R = \frac{1}{\limsup_{k\to\infty} |a_k|^{1/k}} = \frac{1}{1} = 1$$

where $\limsup_{k\to\infty} |a_k|^{1/k} = \limsup_{k\to\infty} a_k$ (because a_k only takes on 0 or 1, and 0 or 1 to the $1/k$ power is still 0 or 1), and the limsup of a_k is 1 because there are infinitely many factorials. Thus by the theorem on page 140, $f(z)$ is analytic on the open disk $\{z : |z| < 1\}$.

Let λ denote any root of unity (that is to say, there is N s.t. $\lambda^N = 1$). That means, that for $n \geq N$, $n!$ has an "N", and so $\lambda^{n!} = 1$. That means that

$$f(rz) = \sum_{n=0}^{N-1} (r\lambda)^n + \sum_{n=N}^{\infty} r^n$$

Now as $r \nearrow 1$, $\sum_{n=N}^{\infty} r^n$ is unbounded, because for every $M \in \mathbb{N}$, there is $r_M < 1$ such that $r_M^{N+2M} \geq \frac{1}{2}$ (because the limit as $r \nearrow 1$ of r^{N+2M} is 1), and so (because $r_M < 1$ and $r_M^{n_1} \geq r_M^{n_2}$ for $n_1 \leq n_2$)

$$\sum_{n=N}^{N+2M} r_M^n \geq \sum_{n=N}^{N+2M} r_M^{N+2M} \geq 2M\frac{1}{2} = M$$

All the terms are positive (because we are only considering $r < 1$ close to 1), $\sum_{n=N}^{\infty} r^n$ also $\geq M$ at r_M, and so the modulus of $\sum_{n=0}^{\infty}(r\lambda)^n$ is $\geq M$ at r_M, and so as $r \nearrow 1$, $|f(rz)| \to +\infty$.

§V.4 - Problem 13

We want to prove that if $f(z)$ and $g(z)$ are analytic, $f(z_0) = g(z_0) = 0$, and $g(z)$ is not identically 0, then

$$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \lim_{z\to z_0} \frac{f'(z)}{g'(z)}$$

in the sense that either both limits are finite and equal, or both are infinite.

Due to analyticity of f and g (we are only concerned with analyticity at z_0), we have that

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} b_n(z - z_0)^n$$

but because $f(z_0) = g(z_0) = 0$, $a_0 = b_0 = 0$. Then, for z close to z_0 (but not equal), and $g(z) \neq 0$,

$$\frac{f(z)}{g(z)} = \frac{(z - z_0)(a_1 + \sum_{n=1}^{\infty} a_{n+1}(z - z_0)^n)}{(z - z_0)(b_1 + \sum_{n=1}^{\infty} b_{n+1}(z - z_0)^n)} = \frac{a_1 + \sum_{n=1}^{\infty} a_{n+1}(z - z_0)^n}{b_1 + \sum_{n=1}^{\infty} b_{n+1}(z - z_0)^n}$$

If $b_1 \neq 0$, then close to z_0 the denominator will not be 0, and the limit as $z \to z_0$ of the expression will just be $\frac{a_1}{b_1}$. If $b_1 = 0$, then the expression will be unbounded as $z \to z_0$. On the other hand, using the theorem of page 140, we get that f' and g' are analytic and are

$$f'(z) = \sum_{n=1}^{\infty} n a_n(z - z_0)^{n-1} \quad \text{and} \quad g'(z) = \sum_{n=1}^{\infty} n b_n(z - z_0)^{n-1}$$

around a neighborhood of z_0. Then,

$$\frac{f'(z)}{g'(z)} = \frac{a_1 + \sum_{n=2}^{\infty} n a_n(z - z_0)^{n-1}}{b_1 + \sum_{n=2}^{\infty} n b_n(z - z_0)^{n-1}}$$

If $b_1 \neq 0$, then close to z_0 the denominator will not be 0, and the limit as $z \to z_0$ of the expression will just be $\frac{a_1}{b_1}$. If $b_1 = 0$, then the expression will be unbounded as $z \to z_0$. Thus, either both limits are finite and equal $\frac{a_1}{b_1}$, or both are infinite, and so the claim is proven.

Problem 14

Let f be a continuous function on the unit circle $T = \{|z| = 1\}$ Show that f can be approximated uniformly on T by a sequence of polynomials in z if and only if f has an extension F that is continuous on the closed disk $\{|z| \leq 1\}$ and analytic on the interior $\{|z| < 1\}$. Hint. To approximate such an F, consider dilates $F_r(z) = F(rz)$

($\implies$) We have some polynomials $\{p_k\}$ that converge uniformly to f on T. Thus, these polynomials are uniformly Cauchy: for m, n large enough, $|p_m - p_n| < \epsilon$ on T. But polynomials are analytic on $\mathbb{C}$.

By the maximum principle on page 88, we have that $|p_m - p_n| \leq \epsilon$ on the boundary of the unit disk, and so $|p_m - p_n| \leq \epsilon$ on the unit disk. Uniformly Cauchy is equivalent to being uniformly convergent to something, so the p_k converge to some function F on the closed unit disk. Uniform convergence of continuous functions on a compact interval results in a continuous function, so F is continuous. Finally, by the theorem in the middle of page 136, analytic functions (such as the $\{p_k\}$) on a domain converge uniformly to an analytic function on the domain; and so F is analytic on the open unit disk. ($\Longleftarrow$) we suppose that there is F continuous on the closed unit disk and analytic on the open unit disk, that is extended to the continuous f on T. Analyticity gives that

$$F(z) = \sum_{n=0}^{\infty} a_n z^n \quad |z| < 1$$

Define (for $r < 1$) $F_r(z) = F(rz) = \sum_{n=0}^{\infty} a_n r^n z^n$ (which is analytic on the open disk of radius $\frac{1}{r}$. All the F_r (as in the infinite sums) converge uniformly on the closed unit disk (via the Weierstrass M-test), and F is uniformly continuous on the closed unit disk (which is compact), and so we get that for r close enough to 1, $|F(z) - F(rz)|$ is uniformly bounded by an arbitrarily small number, and so we get that for large enough N,

$$\left| F(z) - \sum_{n=0}^{N} a_n r^n z_n \right| \leq \left| F_r(z) - \sum_{n=0}^{N} a_n r^n z_n \right| + |F(z) - F_r(z)| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

so we have found a sequence of polynomials uniformly approximating f on T.

§V.6 - Problem 6

Show that the coefficients of a power series "depend continuously" on the function they represent, in the following sense. If $\{f_m(z)\}$ is a sequence of analytic functions that converges uniformly to $f(z)$ for $|z| < \rho$, and

$$f_m(z) = \sum_{k=0}^{\infty} a_{k,m} z^k, \quad f(z) = \sum_{k=0}^{\infty} a_k z^k$$

then for each $k \geq 0$, we have $a_{k,m} \to a_k$ as $m \to \infty$

Fix some $k \geq 0$. We have by Cauchy's integral formula (see 4.3 on page 144) that for any $r \in (0, \rho)$, say $r = \rho/2$,

$$a_{k,m} = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f_m(z)}{(z-z_0)^{k+1}}\, dz \quad \text{and} \quad a_k = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{k+1}}\, dz$$

Now because we have uniform convergence of the f_m to f, for any fixed $\epsilon > 0$, I can find M s.t. for $m \geq M$, $|f_m(z) - f(z)| < \epsilon r^k$ for all $|z| < \rho$. Thus, by 4.4 of the same page,

$$|a_{k,m} - a_k| \leq \frac{\epsilon r^k}{r^k} = \epsilon$$

Thus by the definition of limit, we have that $a_{k,m}$ approaches a_k.

§V.7 - Problem 5

Show that

$$\int_{-\infty}^{\infty} e^{-zt^2+2wt}dt = \sqrt{\frac{\pi}{z}}e^{w^2/z}, \quad z, w \in \mathbb{C}, \operatorname{Re} z > 0$$

where we take the principal branch of the square root.

We follow the hint and prove that the integral is analytic in z and w: first we sketch the proof of the theorem on page 121 for the case where the integral dt is improper: Gamelin already has that

$$\int_{\partial R}\int_{-a}^{a} h(t,z)\,dt\,dz = 0$$

for all fixed $a > 0$. Then,

$$\int_{\partial R} H(z)\,dz = \int_{\partial R} H(z) - \int_{-a}^{a} h(t,z)\,dt\,dz$$

if we can get that $|H(z) - \int_{-a}^{a} h(t,z)\,dt|$ is bounded uniformly by a small constant dependent on a (i.e. that the improper integral converges uniformly), then everything will work out. I do not think it is very hard to find the improperly integrable $g(t)$ s.t. $|h(t,z)| \leq g(t)$ given the bounds that z be on ∂R, so I will leave it be so as to attend to material that is more 336 than 335. Summarizing, the improper integral extension to the theorem on page 121 gives that fixing one of z, w and letting the other be the variable, the function

$$F(z,w) = \int_{-\infty}^{\infty} e^{-zt^2+2wt}\,dt$$

is analytic on the set D given in the problem. Now consider the subset of D where z is real (and > 0), and w is real (this set obviously has non-isolated points). Making the u-sub $u = t - w/z \implies du = dt$ (the issue regarding the improper bounds is trivial as we are just shifting the variable in consideration by the "constant" w/z; rigorous proof is again something I would do in 335, so I would not like to spend too much time here on that), we get that

$$F(z,w) = \int_{-\infty}^{\infty} e^{-z(t-(w/z))^2+w^2/z}\,dt = e^{w^2/z}\int_{-\infty}^{\infty} e^{-zu^2}\,du$$

As we know from our previous experience with the Gaussian integral (after making a variable change $v = u\sqrt{z} \implies dv = \sqrt{z}\,du$; again multiplying by a factor does not change the improper integral bounds), we have that

$$F(z,w) = e^{w^2/z}\int_{-\infty}^{\infty} e^{-zu^2}\,du = e^{w^2/z}\int_{-\infty}^{\infty} e^{-v^2}\frac{1}{\sqrt{z}}\,dv = e^{w^2/z}\frac{\sqrt{\pi}}{\sqrt{z}}$$

We can then extend this relation by the principle of permanence of function equations to all of D.

Problem 7

Show that if $u(x, y)$ is a harmonic function on a domain D such that all the partial derivatives of $u(x, y)$ vanish at the same point of D then $u(x, y)$ is constant on D.

Let us look only at a neighborhood N around $z_0 \in D$ (so a star shaped domain in D), and so by page 83, we have a harmonic conjugate v s.t. $f = u + iv$ is analytic on N. Now because the complex derivative must equal the limit as approached parallel to the real axis, we have that $f'(z) = u_x + iv_x = u_x - iu_y$ (by the C-R equations). Due to analyticity, f is infinitely differentiable on N, and so we have that $f''(z) = u_{xx} - iu_{yx}$ and $f'''(z) = u_{xxx} - iu_{yxx}$ and so on. As all the partials of u are 0 at z_0, that means that all derivatives of f at z_0 are 0. Thus by the theorem on page 144, on a neighborhood of z_0, $f(z)$ is equal to its Taylor series, which we've just shown is $f(z_0)$ (as all derivatives are 0). Well, this open set has points that are not isolated (literally by the definition of open set), and so by the the identity/uniqueness theorem, $f(z) = f(z_0)$ on all of D.

Problem 10

Show that if $f(z)$ is a continuous function on a domain D such that $f(z)^N$ is analytic on D for some integer N, then $f(z)$ is analytic on D.

(If f is identically 0, then trivial, so assume not). First let us consider the z for which $f(z) \neq 0$. Let $g(z) = f(z)^N$. Then,

$$\frac{g(z+h) - g(z)}{h} = \frac{f(z+h) - f(z)\Big(f(z+h)^{N-1} + f(z+h)^{N-2}f(z) + \ldots + f(z+h)f(z)^{N-2} + f(z)^{N-1}\Big)}{h}$$

Dividing through the stuff in parenthesis (which is not zero for small enough h because its limit is $Nf(z)^{N-1}$ which we are considering not be 0), we get that

$$f'(z) = \frac{g'(z)}{Nf(z)^{N-1}}$$

and so $f(z)$ is analytic on $D \setminus Z$ where Z is the set of z for which $f(z) = 0$. Moreover, $g(z) = f(z)^N = 0 \iff z \in Z$, and so because we assumed f was not identically 0, we have by the theorem on page 156 that the points of Z are isolated. Thus choosing any $z \in Z$, we get a neighborhood N around it such that we know that f is analytic on $N \setminus \{z\}$, and because $f(z)$ is continuous and hence bounded on say a compact disk of half the radius of N centered at z, we have by the Riemann removable singularity theorem that $f(z)$ is analytic at z. Thus, we've proved that f is analytic on $D \setminus Z$ and at the points of Z, so we are done.

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§IV.4 - Problem 2

We know from page 83 of Gamelin that all harmonic functions on a star shaped domain D has a harmonic conjugate v on D, which means that if we have a harmonic function u (the book is unclear, but I will assume it to be given that u is harmonic on all of $\mathbb{C}$), then there is some v such that $f = u + iv$ is analytic (on $\mathbb{C}$). The corollary on page 115 gives that f is infinitely differentiable, with all derivatives analytic (on $\mathbb{C}$), and so the partials of u must also be infinitely differentiable. Spelling out the details, we have that

$$\begin{aligned}[\partial_x u](x,y) &= \lim_{h\to 0} \frac{\operatorname{Re}(f(x+h+iy)) - \operatorname{Re}(f(x+iy))}{h} = \lim_{h\to 0} \frac{\operatorname{Re}(f(x+h+iy) - f(x+iy))}{h} \\ &= \operatorname{Re}\left(\lim_{h\to 0} \frac{f(x+h+iy) - f(x+iy)}{h}\right) = \operatorname{Re}(f'(x+iy))\end{aligned}$$

and

$$\begin{aligned}[\partial_y u](x,y) &= \lim_{h\to 0} \frac{\operatorname{Re}(f(x+iy+ih)) - \operatorname{Re}(f(x+iy))}{h} = \lim_{h\to 0} \frac{\operatorname{Re}(f(x+iy+ih) - f(x+iy))}{h} \\ &= \operatorname{Re}\left(i\lim_{h\to 0} \frac{f(x+iy+ih) - f(x+iy)}{ih}\right) = \operatorname{Re}(if'(x+iy))\end{aligned}$$

where all the f' are analytic, and so the 1st partials of u are continuous. For higher partials, we do one example:

$$\begin{aligned}[\partial_y[\partial_x u]](x,y) &= \lim_{h\to 0} \frac{\operatorname{Re}(f'(x+iy+ih)) - \operatorname{Re}(f'(x+iy))}{h} = \lim_{h\to 0} \frac{\operatorname{Re}(f'(x+iy+ih) - f'(x+iy))}{h} \\ &= \operatorname{Re}\left(i\lim_{h\to 0} \frac{f'(x+iy+ih) - f'(x+iy)}{ih}\right) = \operatorname{Re}(if''(x+iy))\end{aligned}$$

and so on.

§IV.5 - Problem 1

As above, u is harmonic on $\mathbb{C}$ means that there is some v s.t. $f = u + iv$ is analytic on $\mathbb{C}$. We want to show that if u is harmonic and bounded (say by B), then u is constant. Consider $e^f = e^u e^{iv} = e^u(\cos v + i\sin v)$ which is analytic on $\mathbb{C}$ (i.e. entire) because f is and e^x is as well (and composition preserves analyticity). e^f is clearly bounded because $|e^f| = |e^u(\cos v + i\sin v)| = |e^u| = e^u \le e^B$. Thus by Liouville's theorem on page 118, e^f is constant, which means that f must be constant (because e^x doesn't take on the same value for two different x), which of course implies that u is constant.

Problem 2

Let's say that the (open) disk is centered at z_0 with radius $\epsilon > 0$. The assumption is that f (an entire function) does not take values inside this disk; that is, $|f(z) - z_0| \geq \epsilon$ for all $z \in \mathbb{C}$. Then consider $g(z) = \frac{1}{f(z)-z_0}$, which is entire because the denominator is entire and never 0. Furthermore, $|g(z)| = \frac{1}{|f(z)-z_0|} \leq \frac{1}{\epsilon}$. So $g(z)$ is bounded, so Liouville's theorem gives that $g(z)$ is constant, which of course implies that $f(z)$ is constant ($C = \frac{1}{f(z)-z_0} \implies f(z) = \frac{1}{C+z_0}$).

Problem 4

We suppose that $f(z)$ is an entire function s.t. $g(z) = f(z)/z^n$ is bounded (by say B) for $|z| \geq R$. This implies that for any z_0, and for any big enough R_{z_0}, $|z - z_0| < R_{z_0}$ contains the disk $|z| < R$, which means that all $|z - z_0| \geq R_{z_0}$ satisfy $|g(z)| \leq B$. In particular, $|g(z)| \leq B$ for $|z - z_0| = R_{z_0}$, a circle contained in the region $|z| \leq |z_0| + R_{z_0}$, and so $|f(z)| \leq B(|z_0| + R_{z_0})^n$ for $|z - z_0| = R_{z_0}$. $f(z)$ is of course analytic inside this region, as it is entire, and so now we may apply Cauchy's estimates to get that:

$$\left|f^{(n+1)}(z_0)\right| \leq \frac{(n+1)!}{R_{z_0}^{n+1}} B(|z_0| + R_{z_0})^n$$

Letting R_{z_0} go to ∞ (because recall that I said this works for ANY big enough R_{z_0}), we see that $\left|f^{(n+1)}(z_0)\right| = 0 \implies f^{(n+1)}(z_0) = 0$. Thus all derivatives of order $(n+1)$ and higher are 0, EVERYWHERE on $\mathbb{C}$ (because z_0 was arbitrary). Now because f is analytic in a neighborhood of z_0, it equals its power series there, which we just showed is a finite degree polynomial. From the identity theorem, it must be that f is that finite degree polynomial $a_0 + a_1 z + \ldots + a_n z^n$ on all of $\mathbb{C}$.

Finally, if $g(z)$ is bounded on the entire complex plane, then it must be that $f(z)/z^n = a_n + a_{n-1}/z + \ldots + a_0/z^n$ is bounded at 0, which can only happen if $a_{n-1} = \ldots = a_0 = 0$. Thus, bounded everywhere $\implies f(z) = a_n z^n$.

§IV.6 - Problem 2

Let $h(t)$ be a continuous function on $[a, b]$. Show that the Fourier transform

$$H(z) = \int_a^b h(t) e^{-itz}\, dt$$

is an entire function that satisfies $|H(z)| \leq Ce^{A|y|}, z = z + iy \in \mathbb{C}$, for constants $A, C > 0$. Well, because h is continuous on $[a, b]$ and e^{-itz} is analytic on $z \in \mathbb{C}$, $h(t, z) = h(t)e^{-itz}$ is continuous complex valued function for $t \in [a, b]$ and $z \in \mathbb{C}$, and is an analytic function of z for every fixed $t \in [a, b]$, and so we apply the first theorem on page 121, which yields that $H(z)$ is analytic.

As for the bound, notice that $h(t)$ is a continuous function on a compact interval $[a, b]$, so it is bounded

by say B. Then:

$$|H(z)| = \int_a^b |h(t)||e^{-itz}|\,dt \leq \int_a^b B|e^{-itx+ty}|\,dt = \int_a^b Be^{ty}\,dt$$

where e^{ty} is bounded by $e^{A|y|}$ for $A = \max\{|a|, |b|\}$, which means that the whole thing is bounded by $B(b-a)e^{A|y|}$ (and so taking $C = B(b-a)$ we get the desired bound.

§V.2 - Problem 7

Let a_n be bounded sequence of complex numbers. We want to show that for every $\epsilon > 0$, the series $\sum_{n=1}^{\infty} a_n n^{-z}$ converges uniformly for $\operatorname{Re} z \geq 1 + \epsilon$.

We proceed by the Weierstrass M-test: if we find M_n s.t. $|a_n n^{-z}| \leq M_n$ for all $z \in \{\operatorname{Re} z \geq 1 + \epsilon\}$ and $\sum M_n < \infty$, then the claim is proven. Well, the a_n are bounded by say B, so we have that

$$|a_n n^{-z}| = B|n^{-z}| = B|e^{-z\log n}| = B|e^{-x\log n - iy\log n} = B|e^{-x\log n}| = Bn^{-x} = \frac{B}{n^x} \leq \frac{B}{n^{1+\epsilon}} =: M_n$$

where of course $\sum M_n$ converges for any $\epsilon > 0$ because of the p-test. Thus, as advertised, the Weierstrass M-test gives that $\sum_{n=1}^{\infty} a_n n^{-z}$ converges uniformly on $\{\operatorname{Re} z \geq 1 + \epsilon\}$.

336 Homework 5

Daniel Rui - 5/4/20

§IV.1 - Problem 4

For a bounded domain D with smooth boundary, we know from the definition of line integral presented on page 102:

$$\begin{aligned}\int_{\partial D} \bar{z}\, dz &= \int_{\partial D} (x - iy)\, dx + (x - iy)i\, dy = \int_{\partial D} x\, dx + y\, dy + i \int_{\partial D} -y\, dx + x\, dy \\ &= \iint_D (0 - 0)\, dA + i \iint_D (1 - (-1))\, dA = 2i\text{Area}(D)\end{aligned}$$

where we used Green's theorem twice where obviously the P and Q in each application had continuous 1st-partials, and the domain is given to have a smooth boundary.

Problem 8

Suppose the continuous function $g(\theta) = f(e^{i\theta})$ on the unit circle satisfies $|g(\theta)| \leq M$, and $|\int_{|z|=1} f(z)\, dz| = 2\pi M$. From the middle of page 103, we can parameterize the unit circle as $z(\theta) = e^{i\theta}$ and get that $dz = i(\cos\theta + i\sin\theta)\, d\theta = ie^{i\theta}\, d\theta$, and so the condition above says that

$$2\pi M = \left|\int_{|z|=1} f(z)\, dz\right| = \left|\int_0^{2\pi} f(e^{i\theta}) ie^{i\theta}\, d\theta\right| = \left|i\int_0^{2\pi} g(\theta)e^{i\theta}\, d\theta\right| \leq |i| \int_0^{2\pi} |g(\theta)||e^{i\theta}|\, d\theta = \int_0^{2\pi} |g(\theta)|\, d\theta$$

We know that $|g(\theta)| \leq M$, but the equation above tells us that it must be equal to M for all $\theta \in [0, 2\pi)$, because if it were $< M$ at some θ_0, then by continuity, we would have that the integral $\int_0^{2\pi} |g(\theta)| - M\, d\theta < 0 \iff \int_0^{2\pi} |g(\theta)|\, d\theta < 2\pi M$ (we proved a theorem like this on Homework 10 of 334), which contradicts the inequality in the above display. Thus, there is some continuous function h such that $g(\theta) = Me^{ih(\theta)}$, and so

$$2\pi M e^{i\vartheta} = \int_{|z|=1} f(z)\, dz = \int_0^{2\pi} g(\theta) ie^{i\theta}\, d\theta = \int_0^{2\pi} Me^{ih(\theta)} ie^{i\theta}\, d\theta = Mi \int_0^{2\pi} e^{ih(\theta)+i\theta}\, d\theta$$

where $\vartheta \in [0, 2\pi)$ is some constant. Dividing by M (if M is zero, obviously $f(z) = 0$ on the unit circle), the above equation is equivalent to

$$-2\pi i = \int_0^{2\pi} e^{i(h(\theta)+\theta-\vartheta)}\, d\theta$$

Comparing imaginary parts, we see that $-2\pi = \int_0^{2\pi} \sin(h(\theta) + \theta - \vartheta)\, d\theta$. We know that $\sin(\theta)$ takes values in $[-1, 1]$, and if at any point it's not -1, due to continuity the integral will fail to be -2π. Thus, we know that for all $\theta \in [0, 2\pi)$, $\sin(h(\theta) + \theta - \vartheta) = -1$, or $h(\theta) + \theta - \vartheta = \frac{3}{2}\pi + 2k\pi$. Plugging

this back in, we have that

$$f(e^{i\theta}) = g(\theta) = Me^{ih(\theta)} = Me^{i(2k\pi+3\pi/2+\vartheta-\theta)} = M \cdot 1 \cdot (-i) \cdot e^{i\vartheta} \cdot e^{-i\theta} = -iMe^{i\vartheta}\overline{e^{i\theta}}$$

and thus on $\{z : |z| = 1\}$, we have that $f(z) = c\bar{z}$ where $c = -iMe^{i\vartheta} \implies |c| = M$.

Problem 9

We have $H(w) = \int_\gamma \frac{h(z)}{z-w}\,dz = \int_\gamma \frac{h(z)\,dx}{z-w} + i\int_\gamma \frac{h(z)\,dy}{z-w}$ for $z \in \gamma$ and $w \in \mathbb{C} \setminus \gamma$ where h is a continuous function on the curve γ. Then we have

$$\begin{aligned}
\frac{H(w) - H(w_0)}{w - w_0} &= \frac{1}{w-w_0}\left(\int_\gamma \frac{h(z)}{z-w} - \frac{h(z)}{z-w_0}\,dx + i\int_\gamma \frac{h(z)}{z-w} - \frac{h(z)}{z-w_0}\,dy\right) \\
&= \frac{1}{w-w_0}\left(\int_\gamma \frac{h(z)\cdot(w-w_0)}{(z-w)(z-w_0)}\,dx + i\int_\gamma \frac{h(z)\cdot(w-w_0)}{(z-w)(z-w_0)}\,dy\right) \\
&= \int_\gamma \frac{h(z)}{(z-w)(z-w_0)}\,dx + i\int_\gamma \frac{h(z)}{(z-w)(z-w_0)}\,dy
\end{aligned}$$

which goes to $\int_\gamma \frac{h(z)}{(z-w)^2}\,dx + i\int_\gamma \frac{h(z)}{(z-w)^2}\,dy$ as $w \to w_0$ because the integrands converge uniformly (and limits can be pulled inside integrals when we have uniform convergence); we just look at the real part now (the imaginary part is the same): $\sup_{z\in\gamma,|w-w_0|\le r} |\frac{h(z)}{(z-w)(z-w_0)} - \frac{h(z)}{(z-w_0)^2}| = \sup_{z\in\gamma,|w-w_0|\le r} |\frac{h(z)}{(z-w_0)}|\frac{(w-w_0)}{(z-w)(z-w_0)}| = (w-w_0)\sup_{z\in\gamma,|w-w_0|\le r} |\frac{h(z)}{(z-w_0)(z-w)(z-w_0)}|$ but sup of a continuous function over a compact set (we choose r small enough that the denominator is never 0, which we can do because $\mathbb{C}\setminus\gamma$ is open) is finite, so we can bound the above by just $(w-w_0)\cdot C$, which goes to 0 uniformly as $w \to w_0$. (Compare this to problem II.2.6 from Homework 2 and problem III.1.6 from Homework 3). Thus, $H(\omega)$ is analytic on $\mathbb{C}\setminus\gamma$ and

$$H'(\omega) = \int_\gamma \frac{h(z)}{(z-w)^2}\,dx + i\int_\gamma \frac{h(z)}{(z-w)^2}\,dy = \int_\gamma \frac{h(z)}{(z-w)^2}\,dz$$

§IV.2 - Problem 5

We want to show that $f(z)$ has a primitive in D if and only if $\int_\gamma f(z)\,dz = 0$ for every closed path γ in D.

($\implies$) From the theorem at the beginning of the section on page 107, we have that $\int_A^B f(z)\,dz = F(B) - F(A)$ where F is the primitive of f (which we are assuming to exist), and where the integral can be taken over any path from A to B in D. Because $A = B$ in a closed path, we see that this theorem gives that the integral over any closed path is 0.

($\impliedby$) on page 82, it says that independent of path $\iff$ exact. But we know that independent of path $\iff$ integral over any closed path is 0, and so because we have that the integral over any closed $\gamma \in D$ is 0, it must be that $f(z)\,dx + if(z)\,dy$ is exact; that is, there is F s.t. $dF = \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy = f(z)\,dz + if(z)\,dy$, so we have

$$\frac{\partial F}{\partial x} = f(z) \quad \text{and} \quad \frac{\partial F}{\partial y} = if(z)$$

and so if $F = u + iv$, we have

$$\partial_x u + i\partial_y v = f(z) = \operatorname{Re} f + i \operatorname{Im} f \quad \text{and} \quad \partial_y u + i\partial_y v = if(z) = i \operatorname{Re} f - \operatorname{Im} f$$

and comparing real and imaginary parts, we see that $\partial_x u = \partial_y v$ and $\partial_x v = -\partial_y u$ and so the C-R equations are satisfied, hence F is analytic on D, and is a primitive of $f(z)$ (it satisfies the formula on the middle of page 107).

§IV.3 - Problem 1

Integrating $e^{-z^2/2}$ around a rectangle $D_{R,t}$ with vertices $\pm R, it \pm R$, we have

$$\begin{aligned}\int_{\partial D_{R,t}} e^{-z^2/2}\,dz &= i\int_0^t e^{-(R+iy)^2/2}\,dy + \int_R^{-R} e^{-(x+it)^2/2}\,dx + i\int_t^0 e^{-(-R+iy)^2/2}\,dy + \int_{-R}^R e^{-(x+i0)^2/2}\,dx \\ &= i\int_0^t e^{-R^2/2}e^{-iRy}e^{y^2/2}\,dy + \int_R^{-R} e^{-x^2/2}e^{-ixt}e^{t^2/2}\,dx + i\int_t^0 e^{-R^2/2}e^{iRy}e^{y^2/2}\,dy + \int_{-R}^R e^{-x^2/2}\,dx\end{aligned}$$

where the dz goes to either dx or dy, because along the boundaries of the rectangle, either x or y stays constant. Now for any fixed t, and as R goes to ∞, the integrals along vertical line segments go to 0, because $e^{-R^2/2}$ goes to 0 and $|e^{-iRy}e^{y^2/2}|$ is bounded on $y \in [0,t]$. The horizontal line segment integrals converge to

$$-e^{t^2/2}\int_{-\infty}^{\infty} e^{-x^2/2}e^{-ixt}\,dx \quad \text{and} \quad \int_{-\infty}^{\infty} e^{-x^2/2}\,dx$$

respectively, with the latter equal to $\sqrt{2\pi}$. Because $D_{R,t}$ is a bounded domain with piecewise smooth boundary, and $e^{-z^2/2}$ is an analytic function on $D_{R,t}$ (with smooth extension to ∂D), we can use Cauchy's theorem, which tells us that

$$0 = -e^{t^2/2}\int_{-\infty}^{\infty} e^{-x^2/2}e^{-ixt}\,dx + \sqrt{2\pi} \iff \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2/2}e^{-ixt}\,dx = e^{-t^2/2}$$

Problem 6

We suppose that $f(z)$ is continuous on $\{z : |z| \le R\}$ and analytic on $\{z : |z| < R\}$. We want to show that $\oint_{|z|=R} f(z)\,dz = 0$. As the hint suggests, consider $f_r(z) := f(rz)$ for $r \in (0,1)$. Because $r < 1$, $f_r(z)$ is analytic on a domain containing the closed R-disk, and so by Cauchy's theorem, $\oint_{|z|=R} f_r(z)\,dz = 0$ for any $r \in (0,1)$. But for $r > 1 - \delta$, we have that $|z - rz| < \delta$, and because f is continuous on the closed R-disk, it is uniformly continuous on the closed R-disk, and so for every ϵ, there is δ s.t.

$$r > 1 - \delta \implies |f(z) - f(rz)| < \epsilon$$

Thus, picking an $r > 1 - \delta$ where δ is as above, we have

$$\left| \oint_{|z|=R} f(z)\,dz \right| = \left| \oint_{|z|=R} f(z)\,dz - \oint_{|z|=R} f_r(z)\,dz \right| = \left| \oint_{|z|=R} f(z) - f_r(z)\,dz \right|$$
$$\leq \oint_{|z|=R} |f(z) - f_r(z)|\,dz \leq \oint_{|z|=R} \epsilon\,dz = 0$$

336 Midterm

Daniel Rui - 4/30/20

Problem 1

Let T be the FLT that maps z_1, z_2, z_3 to w_1, w_2, w_3 resp. (all z's distinct, and w's distinct), and let $T_1(z) = \frac{(z-z_2)(z_1-z_3)}{(z-z_3)(z_1-z_2)}$ and let $T_2(w) = \frac{(w-w_2)(w_1-w_3)}{(w-w_3)(w_1-w_2)}$. We want to prove that $w = T(z)$ if and only if $\frac{(z-z_2)(z_1-z_3)}{(z-z_3)(z_1-z_2)} = \frac{(w-w_2)(w_1-w_3)}{(w-w_3)(w_1-w_2)}$, or in other words that $w = T(z) \iff T_1(z) = T_2(w)$.

($\implies$) because we are given that $w = T(z)$, we want to show that $T_1(z) = T_2(T(z))$ for all $z \in \mathbb{C} \cup \{\infty\}$. Well, we know that $T_1(z_1), T_1(z_2), T_1(z_3)$ are 1, 0, and ∞ resp, and similarly $T_2(w_1), T_2(w_2), T_2(w_3)$ are 1, 0, and ∞. T maps z_1, z_2, z_3 to w_1, w_2, w_3, so we see that $T_1(z)$ and $T_2(T(z))$ agree for $z = z_1, z_2, z_3$. But FLT's are uniquely determined by 3 points, so it must be that $T_1(z)$ and $T_2(T(z))$ are exactly the same FLT.

($\impliedby$) we are given that $T_1(z) = T_2(w)$. Because FLT's have inverses that are also FLT's, we see that $T_2^{-1}(T_1(z)) = w$. Because T_2^{-1} and T_1 are FLT's, their composition is as well, so let's call that composition $T'(z) = T_2^{-1}(T_1(z))$. Well, we know that $T_1(z_1), T_1(z_2), T_1(z_3)$ are 1, 0, and ∞ resp, and $T_2^{-1}(1), T_2^{-1}(0), T_2^{-1}(\infty)$ are w_1, w_2, and w_3. Thus, we get that $T'(z_1), T'(z_2), T'(z_3)$ are w_1, w_2, w_3 resp., and again because FLT's are uniquely determined by 3 points, it must be that $T'(z) = T(z)$ and so $w = T(z)$.

Problem 2

We can represent the FLT $T(z) = \frac{1}{2\cos(\frac{\pi}{n}) - z}$ can be represented as following matrix:

$$\begin{bmatrix} 0 & 1 \\ -1 & 2\cos(\frac{\pi}{n}) \end{bmatrix}$$

it has eigenvalues $e^{i\pi/n}$ and $e^{-i\pi/n}$, as can be verified by multiplying through the matrices in the following diagonalization:

$$\begin{bmatrix} 0 & 1 \\ -1 & 2\cos(\frac{\pi}{n}) \end{bmatrix} = \begin{bmatrix} e^{-i\pi/n} & e^{i\pi/n} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{i\pi/n} & 0 \\ 0 & e^{-i\pi/n} \end{bmatrix} \frac{1}{e^{-\pi/n} - e^{i\pi/n}} \begin{bmatrix} 1 & -e^{i\pi/n} \\ -1 & e^{-i\pi/n} \end{bmatrix}$$

The matrix corresponding to T iterated n times is exactly the above matrix to the nth power (due to the fact that FLT composition corresponds to matrix multiplication; see Gamelin pg. 64). Because the two matrices on the left and right are inverses of each other (call them P and P^{-1}), we just need to raise the diagonal matrix to the nth power, which yields

$$\begin{bmatrix} 0 & 1 \\ -1 & 2\cos(\frac{\pi}{n}) \end{bmatrix}^n = \begin{bmatrix} e^{-i\pi/n} & e^{i\pi/n} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \frac{1}{e^{-\pi/n} - e^{i\pi/n}} \begin{bmatrix} 1 & -e^{i\pi/n} \\ -1 & e^{-i\pi/n} \end{bmatrix}$$

which is just $P(-I)P^{-1} = -I$. The FLT corresponding to this matrix is

$$\frac{-1z+0}{0z-1} = z$$

and thus T iterated n times is exactly the identity FLT.

Problem 3

Let $w_0 \in \mathbb{C}, w_0^n = 1$ but $w_0 \neq 1$. First note that

$$w + w^2 + \ldots + w^n = w\frac{w^n - 1}{w - 1}$$

for $w \neq 1$ (using the elephant-teacup formula). Both sides are analytic on $\mathbb{C} \setminus \{1\}$ (sums of powers of w, and quotients where the denominator is not 0), and so it both sides are complex differentiable at any point in $\mathbb{C} \setminus \{1\}$. Taking the derivative of each side, we see that

$$1 + 2w + \ldots + nw^{n-1} = \frac{w^n - 1}{w - 1} + w\left(\frac{(w-1)(nw^{n-1}) - (w^n - 1)}{(w-1)^2}\right)$$

on $\mathbb{C} \setminus \{1\}$. Plugging in w_0 (which is in $\mathbb{C} \setminus \{1\}$), we get that

$$1 + 2w_0 + \ldots nw_0^{n-1} = 0 + \frac{nw_0^n}{(w_0 - 1)} = \frac{n}{w_0 - 1}$$

Problem 4

We are given that $\sum_1^\infty z_k$ converges, and that $\sum_1^\infty z_k^2$ converges, where $z_k = x_k + iy_k$ have the additional specification that all the x_k are ≥ 0. Well, $\sum_1^\infty z_k$ converges $\iff$ $\sum_1^\infty x_k$ and $\sum_1^\infty y_k$ converge, and similarly $\sum_1^\infty z_k^2$ converges $\iff$ $\sum_1^\infty (x_k^2 - y_k)^2$ and $\sum_1^\infty 2x_k y_k$ converge.

The convergence of $\sum_1^\infty x_k$ along with the fact that $x_k \geq 0$ means that for large enough k, $x_k < 1$ (because $x_k \to 0$ by convergence of the sum) and so $x_k^2 \leq x_k$. This implies that $\sum_1^\infty x_k^2$ converges (by direct comparison). Now the fact that both $\sum_1^\infty x_k^2$ and $\sum_1^\infty (x_k^2 - y_k^2)$ converges means that $\sum_1^\infty y_k^2$ must converge as well (otherwise $\sum_1^\infty (x_k^2 - y_k^2)$ would diverge).

So we have that both $\sum_1^\infty x_k^2$ and $\sum_1^\infty y_k^2$ converge, and so their sum converges, but their sum is $\sum_1^\infty x_k^2 + y_k^2 = \sum_1^\infty |z_k|^2$, so $\sum_1^\infty |z_k|^2$ converges.

Problem 5

We have $f = u + iv$ analytic on an open connected set Ω, and that there are real numbers a, b, c where $a^2 + b^2 \neq 0$ s.t. $au + bv = c$ for all $z \in \Omega$. Define $g(z) = (a - ib)f(z) = (au + bv) - ibu + iav$ (where $a - ib \neq 0$ because if so then $a^2 + b^2 = 0$). Because a, b are real and u, v are real-valued, we have that $\operatorname{Re} g(z) = au + bv = 0$ and so the real part of $g(z)$ is constant, and so the x and y partials of $\operatorname{Re} g(z)$ are 0. Because $g(z)$ is just a constant times an analytic function, it is analytic too, and so

in particular it satisfies the C-R equations, which tell us that the x and y partials of $\operatorname{Im} g(z)$ are 0 as well. On an open connected set Ω, this means that $\operatorname{Im} g(z)$ is constant, and so $g(z)$ is constant. Dividing by $(a - bi)$ (because remember it is not 0), we get that $f(z)$ must be constant on Ω too.

Problem 6

We have an analytic function $f = u + iv$ on $\Omega = \{\operatorname{Re} z > 0\}$, and we have that $u_x + v_y = 0 \iff u_x = -v_y$. Note that one of the C-R equations is that $u_x = v_y$. Combining these two, we see that $u_x = v_y = 0$ on Ω. This means that for every y, $u(x, y)$ takes on the same value for all $x > 0$, and so there must be some function $a(y)$ s.t. $u(x, y) = a(y)$ for all $(x, y) \in \Omega$. Similarly, $v(x, y) = b(x)$ for some function b, for all $(x, y) \in \Omega$.

Analytic functions have u and v that are harmonic, and so $u_{xx} + u_{yy} = 0$, but $u_x = 0$ on Ω so this gives that $u_{yy} = a'' = 0$ on Ω. Similarly for v, we now have that $v_{xx} = b'' = 0$ on Ω. As a and b are both single variable real valued functions defined on a connected set (which are differentiable because u and v are), two applications of the mean value theorem show that $a'' = 0 \implies a(y) = a_1 y + a_2$ and $b'' = 0 \implies b(x) = b_1 x + b_2$ for real numbers a_1, a_2, b_1, b_2. Finally, the other C-R equation gives that $u_y = -v_x$, i.e. $a' = -b'$ and so $a_1 = -b_1$.

Thus, $f(z) = u(x, y) + iv(x, y) = a(y) + ib(x) = a_1 y + a_2 + ib_1 x + ib_2 = -ia_1(iy + x) + a_2 + ib_2 = -ia_i z + (a_2 + ib_2)$. Thus $f(z) = icz + d$ where $c = -a_i$ is real and $d = a_2 + ib_2$ is complex.

Problem 7

The function $f(z) = e^{-1/z^4}$ for $z \neq 0$ and 0 at $z = 0$ is definitely analytic for $z \neq 0$, because there it's just a composition of analytic functions (the denominator of $-\frac{1}{z^4}$ is not zero when $z \neq 0$).

At $z = 0$, the x partial of f (we hold $y = 0$) is

$$\lim_{x \to 0} \frac{e^{-1/x^4} - 0}{x}$$

which exists and equals 0, which we can see by noting that the above limit is the same as $\lim_{x' \to \infty} \frac{e^{-(x')^4}}{1/x'} = \lim_{x' \to \infty} x' e^{-(x')^4}$ which goes to 0 by l'Hopital's, or any other method you so desire. The y partial of f is exactly the same expression because now that we hold $x = 0$, $f(z)$ becomes $e^{-1/(iy)^4} = e^{-1/y^4}$, the same expression as above. Because all the x, y partials are 0 (the real parts were shown above, and the imaginary part of f is 0 along the axes), the C-R equations do in fact hold.

However, f is definitely NOT analytic at $z = 0$ because it's not even continuous: if we approach along the diagonal, i.e. approach along $re^{i\pi/4}$ as $r \to 0$, we get that $f(re^{i\pi/4}) = e^{-1/(r^4 e^{i\pi})} = e^{1/r^4}$ which goes to ∞, which is definitely not 0, which $f(0) = 0$.

Problem 8

From the fundamental theorem of algebra, $z^n - 1$ has n roots (counting multiplicity), and we know that $z^n - 1$ equals 0 if $z = e^{2ki\pi/n}$ for $k \in \{1, \ldots, n\}$. These are n distinct complex numbers that are roots of $z^n - 1$, and so it must be that $z^n - 1$ is exactly $\prod_{k=1}^{n}(z - e^{2ki\pi/n})$ (the leading coefficient is 1, and the degree on both sides is n). Now for any $z \neq 1$, we have that

$$1 + z + \ldots + z^{n-1} = \frac{z^n - 1}{z - 1} = \prod_{k=1}^{n-1}(z - e^{2ki\pi/n})$$

Because $1 + z + \ldots + z^{n-1}$ is analytic, and so is the product on the RHS, and we know they agree everywhere except $z = 1$, they must also agree at 1 due to continuity. Thus, plugging in $z = 1$, we have

$$n = \prod_{k=1}^{n-1}(1 - e^{2ki\pi/n})$$

We can put modulus bars on both sides to get that

$$n = \prod_{k=1}^{n-1}|1 - e^{2ki\pi/n}|$$

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§III.1 - Problem 6

We have $F(w) = \int_\gamma \frac{P\,dx}{z-w} + \int_\gamma \frac{Q\,dy}{z-w}$ for $z \in \gamma$ and $w \in \mathbb{C} \setminus \gamma$. Then we have

$$\begin{aligned}\frac{F(w)-F(w_0)}{w-w_0} &= \frac{1}{w-w_0}\left(\int_\gamma \frac{P}{z-w} - \frac{P}{z-w_0}\,dx + \int_\gamma \frac{Q}{z-w} - \frac{Q}{z-w_0}\,dy\right)\\ &= \frac{1}{w-w_0}\left(\int_\gamma \frac{P\cdot(w-w_0)}{(z-w)(z-w_0)}\,dx + \int_\gamma \frac{Q\cdot(w-w_0)}{(z-w)(z-w_0)}\,dy\right)\\ &= \int_\gamma \frac{P}{(z-w)(z-w_0)}\,dx + \int_\gamma \frac{Q}{(z-w)(z-w_0)}\,dy\end{aligned}$$

which goes to $\int_\gamma \frac{P}{(z-w)^2}\,dx + \int_\gamma \frac{Q}{(z-w)^2}\,dy$ as $w \to w_0$ because the integrands converge uniformly (and limits can be pulled inside integrals when we have uniform convergence); we only look at P here because Q will follow similarly: $\sup_{z\in\gamma,|w-w_0|\le r} |\frac{P(x)}{(z-w)(z-w_0)} - \frac{P(x)}{(z-w_0)^2}| = \sup_{z\in\gamma,|w-w_0|\le r} |\frac{P(x)}{(z-w_0)}|\frac{(w-w_0)}{(z-w)(z-w_0)}| = (w-w_0)\sup_{z\in\gamma,|w-w_0|\le r} |\frac{P(x)}{(z-w_0)(z-w)(z-w_0)}|$ but sup of a continuous function over a compact set (we choose r small enough that the denominator is never 0, which we can do because $\mathbb{C}\setminus\gamma$ is open) is finite, so we can bound the above by just $(w-w_0)\cdot C$, which goes to 0 uniformly as $w \to w_0$. (Compare this to problem II.2.6 from Homework 2). Thus, $F(\omega)$ is analytic on $\mathbb{C} \setminus \gamma$ and

$$F'(\omega) = \int_\gamma \frac{P}{(z-w)^2}\,dx + \int_\gamma \frac{Q}{(z-w)^2}\,dy$$

§III.3 - Problem 2

We want to show that a complex-valued function $h(z)$ on a star-shaped domain D is harmonic iff $h(z) = f(z) + \overline{g(z)}$ for analytic f and g on D.

($\implies$) By definition, $h = u + iv$ is harmonic if u and v are harmonic on D. The theorem at the end of the first page of the section tells us that any harmonic function on D has a harmonic conjugate on D; and so let a and b be the harmonic conjugates of u and v resp., so that $\alpha = u + ia$ and $\beta = v + ib$ are analytic on D. Obviously, we can recover h by $h = \frac{1}{2}(\alpha + \bar{\alpha} + i(\beta + \bar{\beta}))$. Thus we can set $f = \frac{1}{2}(\alpha + i\beta)$ and $\bar{g} = \frac{1}{2}(\bar{\alpha} + i\bar{\beta}) \implies g = \frac{1}{2}(\alpha - i\beta)$ where f and g are just compositions of analytic functions, and so clearly analytic.

($\impliedby$) Let $f = a + bi$ and $g = c + di$ (which are both analytic on D so a, b, c, d are all harmonic), and so $h = f + \bar{g} = (a + c) + i(b - d)$. By linearity of differentiation, a and c being harmonic imply that $(a + c)$ is harmonic, and similarly $(b - d)$ is too. Thus, h is harmonic because its real and imaginary parts are both harmonic.

Problem 4

Let $u(z)$ be harmonic on $\{a < |z| < b\}$. We want to show that there is a constant C s.t. $u(z) - C\log|z|$ has a harmonic conjugate on the annulus, and furthermore that

$$C = \frac{r}{2\pi}\int_0^{2\pi} \frac{\partial u}{\partial r}(re^{i\theta})\,d\theta$$

where r is some fixed number in (a,b). The lemma at the very beginning of this section says that if u is harmonic, then the differential $-\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy$ is closed (i.e. $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, which implies by Green's theorem that $\int_{\partial U} -\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy = 0$ for any U s.t. $\bar{U} \subseteq D$. This means that for any r_1, r_2 s.t. $a < r_1 < r_2 < b$, that

$$\int_{|z|=r_2} -\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy - \int_{|z|=r_1} -\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy = 0$$

and so along any circle with radius in (a,b), the line integral of $-\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy$ along that circle will be some constant c. Let $w = u - \frac{c}{2\pi}\log|z| = u - \frac{c}{2\pi}\log(\sqrt{x^2+y^2})$, which is harmonic because u is harmonic, and $\log(\sqrt{x^2+y^2})$ has $\partial_x = \frac{x}{x^2+y^2} \implies \partial_{xx} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2}$ and $\partial_y = \frac{y}{x^2+y^2} \implies \partial_{yy} = \frac{x^2+y^2-2y^2}{(x^2+y^2)^2}$ and so $\partial_{xx} + \partial_{yy} = 0$. We do this so that

$$\begin{aligned}
\int_{|z|=r} -\frac{\partial w}{\partial y}\,dx + \frac{\partial w}{\partial x}\,dy &= \int_{|z|=r} -\frac{\partial u}{\partial y} + \frac{c}{2\pi}\frac{y}{x^2+y^2}\,dx + \frac{\partial u}{\partial x} - \frac{c}{2\pi}\frac{x}{x^2+y^2}\,dy \\
&= \int_{|z|=r} -\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy + \frac{c}{2\pi}\int_{|z|=r} \frac{y}{x^2+y^2}\,dx - \frac{x}{x^2+y^2}\,dy \\
&= c + \frac{c}{2\pi}\int_0^{2\pi} \frac{r\sin\theta}{r^2}(-r\sin\theta)\,d\theta - \frac{r\cos\theta}{r^2}(r\cos\theta)\,d\theta = c - \frac{c}{2\pi}2\pi = 0
\end{aligned}$$

Now, we can define $v(z) = \int_C -\frac{\partial w}{\partial y}\,dx + \frac{\partial w}{\partial x}\,dy$ where C is the curve going from a point, say $r+0i$ counterclockwise along the circle of radius r, then on radially inward and outward to z. This is well defined, because any counterclockwise path can be transformed continuously to C, and furthermore because the path integral around the circle is 0, any clockwise paths ALSO give the same value. Therefore, this differential form is independent of path. This is the key part of the proof. Finally, from the last page of section III.2, independent of path $\iff$ exact with $dv = \frac{\partial v}{\partial x}\,dx + \frac{\partial v}{\partial y}\,dy = -\frac{\partial w}{\partial y}\,dx + \frac{\partial w}{\partial x}\,dy$, and so the C-R equations are satisfied for $w + iv$, which means that v is the harmonic conjugate of w.

To get the alternative form of $C = \frac{c}{2\pi}$, we use Problem II.3.8 from Homework 3, where we got the polar form C-R equations, the first of which says that $\frac{\partial w}{\partial r} = \frac{\partial u}{\partial r} - \frac{c}{2\pi}\frac{1}{r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$, and so

$$\begin{aligned}
\frac{r}{2\pi}\int_0^{2\pi} \frac{\partial u}{\partial r}(re^{i\theta})\,d\theta &= \frac{r}{2\pi}\int_0^{2\pi} \frac{1}{r}\frac{\partial v}{\partial \theta}(re^{i\theta}) + \frac{c}{2\pi}\frac{1}{r}\,d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} -\frac{\partial w}{\partial y}\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial x}\frac{\partial y}{\partial \theta}\,d\theta + \frac{1}{2\pi}\int_0^{2\pi} \frac{c}{2\pi}\,d\theta = 0 + \frac{c}{2\pi} = C
\end{aligned}$$

§III.4 - Problem 1

Let $f(z)$ be continuous on D. We want to prove that if $f(z)$ has the mean value property for circles, then it has the MVP w.r.t. disks; that is for $z_0 \in D$, and $D_0 \subseteq D$ centered at z_0 with area A, then $f(z_0) = \frac{1}{A}\iint_{D_0} f(z)\,dx\,dy$. First fix $z_0 \in D$. By the definition of MVP for circles, there is $\epsilon > 0$ s.t. for any $r \in (0, \epsilon)$,

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\,d\theta$$

(and of course $\{|z - z_0| < \epsilon\} \subseteq D$). Then fix any $r \in (0, \epsilon)$, and consider D_0 with that radius (obviously with area πr^2). Then,

$$\frac{1}{\pi r^2}\iint_{D_0} f(z)\,dx\,dy = \frac{1}{\pi r^2}\int_0^r \int_0^{2\pi} f(z_0 + re^{i\theta}) r\,d\theta\,dr = \frac{1}{\pi r^2}\int_0^r 2\pi f(z_0) r\,d\theta\,dr = f(z_0)$$

Problem 3

A function $f(t)$ on an interval (a, b) has the mean value property if $f(\frac{s+t}{2}) = \frac{1}{2}(f(s) + f(t))$ for any $s, t \in (a, b)$.
Any affine function $f(t) = At + B$ obviously has the mean value property, because

$$A\frac{s+t}{2} + B = A\frac{s}{2} + \frac{B}{2} + A\frac{t}{2} + \frac{B}{2} = \frac{As + B + At + B}{2}$$

Now we show that if a function is continuous and satisfies this MVP, then it is affine. Well, at a and b, define $A, B \in \mathbb{R}$ s.t. the function is $Aa + B$ and $Ab + B$ resp. The MVP tells us that $f(\frac{a+b}{2}) = \frac{f(a)+f(b)}{2} = A(\frac{a+b}{2}) + B$. Now it is pretty obvious that we can continue like this, proving by induction that

$$f\left(a + \frac{(b-a)}{2^k}\right) = A\left(a + \frac{(b-a)}{2^k}\right) + B$$

for every k. The induction is extremely straightforward, so I will omit it here. Thus, we proved that at every $a + \frac{(b-a)}{2^k}$, $f(x)$ and $Ax + B$ agree. These points (let's call them "midpoint-type points") are dense in (a, b) (one could prove this by showing that points of the form $\frac{1}{2^k}$ are dense in $(0, 1)$, which is a very easy proof, and then transforming to the (a, b) case). Now suppose for contradiction that there is some $x_0 \in (a, b)$ s.t. $f(x_0) \neq Ax_0 + B$. Let us say w.l.o.g. that $f(x_0) > Ax_0 + B$ where $A \geq 0$ (the other cases are very similar), and let $\epsilon = f(x_0) - (Ax_0 + B)$. But for every $x < x_0$, we can find x_1 s.t. $f(x_1) \leq Ax_0 + B$ because for every $x < x_0$, we can find a midpoint-type point x_1 in (x, x_0) and $f(x_1) = Ax_1 + B \leq Ax_0 + B < f(x_0) - \frac{\epsilon}{2}$. Thus, there is no $\delta > 0$ s.t. $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \frac{\epsilon}{2}$, contradiction the assumption that f was continuous. This of course means that $f(x) = Ax + B$ on (a, b).

§III.5 - Problem 5

Let $f(z)$ be a bounded analytic function on the right half-plane. Suppose f extends continuously to the imaginary axis and satisfies $|f(iy)| \leq M$ for all $y \in \mathbb{R}$. As the hint suggests, we consider the function $\frac{f(z)}{(z+1)^\epsilon}$. We know $f(z)$ is bounded, so say $|f(z)| \leq B$ on the right half plane. Fix some

small $\epsilon > 0$, and take R to be some constant large enough s.t. $\frac{B}{R^\epsilon} \leq M$. Now as is obvious from the Pythagorean theorem, for any z in the right half-plane, $|z+1| > |z|$, and so on the boundary of any half-circle with radius $r > R$, we have that

$$\left|\frac{f(z)}{(z+1)^\epsilon}\right| = \frac{|f(z)|}{|z+1|^\epsilon} \leq \frac{B}{|z|^\epsilon} \leq \frac{B}{R^\epsilon} \leq M$$

But $|f(z)| \leq M$ on the imaginary axis, so the axis together with the large half-circle bound a region D. The maximum principle tells us that $|f(z)| \leq M$ on $\partial D \implies |f(z)| \leq M$ for all $z \in D$. Because D can be any half-disk with $r > R$, every point in the right-half plane can be in one of these D, and so $\left|\frac{f(z)}{(z+1)^\epsilon}\right| \leq M$ for all z in the right half-plane. Finally, because $(z+1)^\epsilon$ approaches 1, it is impossible that $|f(z)| > M$ for some z because then it would mean that for some small enough ϵ, $\left|\frac{f(z)}{(z+1)^\epsilon}\right| > M$ as well, which contradicts what we just proved.

Problem 7

Let $f(z)$ be a bounded analytic function on the open unit disk D, and suppose it extends continuously to ∂D except finitely many points $z_1, \ldots, z_n$. At all the points at which it is extended continuously, $|f(z)| \leq M$. Now consider the function

$$g_\epsilon(z) = f(z) \prod_{k=1}^{n} (z - z_k)^\epsilon$$

For every $\delta > 0$, we can find ϵ small enough s.t. $|g_\epsilon(z)| \leq M + \delta$ for all $z \in \partial D$ (because as $\epsilon \to 0$, the $(z - z_i)^\epsilon$ terms approach 1; and at the $z_1, \ldots, z_n$ $g_\epsilon(z)$ is 0). Furthermore this function is analytic on D as well, so by the maximum principle $|g_\epsilon(z)| \leq M + \delta$ for all $z \in D$. Similar to the above problem, it can not be that $|f(z)| > M + \delta$ for some point $z' \in D$ because then there would be some small enough $\epsilon > 0$ s.t. $|g_\epsilon(z')| > M + \delta$ as well due to the convergence of the $(z - z_i)^\epsilon$ terms to 1. Thus, we showed that for every $\delta > 0$, $|g_\epsilon(z)| \leq< M + \delta$ for all $\epsilon > 0$ sufficiently small, which implies that $|f(z)| \leq M + \delta$ on D. Because δ was arbitrary, it must be that $|f(z)| \leq M$ on D.

Problem 8

We consider a couple transformations to go from any horizontal strip to the unit disk: first, we shift and scale the horizontal strip to become the strip $S := \{z : \operatorname{Im} z \in [-1, 1]\}$. Then, we consider the transformation $z \mapsto e^{\pi z/2}$ which maps S to the right half-plane (because $e^{\pi z/2} = e^{\pi x/2} e^{i\pi y/2}$ and $y \in [-1, 1]$, and $e^{\pi x/2}$ non-negative and surjective onto $(0, \infty)$. There's a little minor issue at the origin, but it will not matter much. This map also maps the boundary lines to the imaginary axis (not including the origin).

From section II.7 about fractional linear transformations, one example said that the FLT $\frac{z-i}{z+i}$ mapped the upper-half plane to the unit disk. We take inspiration from this and consider the FLT $z \mapsto \frac{z-1}{z+1}$ which should map the right half-plane to the unit disk, with the imaginary axis being mapped to the

boundary of the disk. The problem at the origin I mentioned above is translated to a problem at -1.

All of these transformations are analytic (the right-half-plane does not contain -1 so the denominator for that is never 0), so we've kind of squished the horizontal strip into the unit circle. But notice that we've only changed the input, not the output so every output reached on the strip is reached in the circle and vice versa; thus if we prove that $|f(z')| \leq M$ on the disk (where z' is the transformed version of z) then it must be that $|f(z)| \leq M$ on the strip. Well, we have everything continued nicely except at the one point -1, and everywhere on the boundary except that 1 point (as I've said before) has $|f(e^{i\theta})| \leq M$, and so by Problem 7 above, $|f(z')| \leq M$.

336 Homework 3

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§II.3 - Problem 3

The very last theorem in the book tells us that if a function is analytic and real-valued on a domain D, then that function is constant. Well, here we are given that f and $\bar{f}$ are analytic on D, which means that $\frac{f+\bar{f}}{2} = \operatorname{Re} f$ and $\frac{f-\bar{f}}{2i} = \operatorname{Im} f$ are both analytic on D as well (just sums and scalar multiplication of analytic functions). $\operatorname{Re} f$ and $\operatorname{Im} f$ are therefore real-valued and analytic on D, hence constant on D, which of course means that $f = \operatorname{Re} f + i \operatorname{Im} f$ is also constant on D.

Problem 4

As the hint says, we can write $\bar{f} = |f|^2/f$ but only if $f \neq 0$ anywhere in D. If this is the case, then $|f|^2/f$ is analytic ($|f|$ is a constant so analytic, and f is never 0). As per Problem 3, we have f and $\bar{f}$ both analytic, so f is constant.

If however f does equal 0 somewhere, then obviously $|f|$ equals 0 there as well, but because $|f|$ is constant on D, $|f|$ is 0 everywhere on D, which forces that $f = 0$ everywhere on D as well (thereby being constant).

Problem 5

Analyticity gives us the C-R equations and equation (3.1), which tells us that $f'(z) = [\partial_x u](x,y) + i[\partial_x v](x,y)$, and so $|f'| = \sqrt{(\partial_x u)^2 + (\partial_x v)^2}$. On the other hand, $|\nabla u| = \sqrt{(\partial_x u)^2 + (\partial_y u)^2}$, and $|\nabla v| = \sqrt{(\partial_x v)^2 + (\partial_y v)^2}$. But the Cauchy-Riemann equations give that $\partial_y u = -\partial_x v$ and $\partial_y v = \partial_x u$ so substituting in the second term for each of $|\nabla u|$ and $|\nabla v|$ give us the same expression as $|f'|$.

Problem 6

Analyticity gives us the C-R equations. The rotation matrix for rotation 90° degrees counterclockwise is $[[0,-1],[1,0]]$. When we multiply ∇u by this, we get $\langle -\partial_y u, \partial_x u\rangle$ which by the C-R equations is exactly $\langle \partial_x v, \partial_y v\rangle = \nabla v$. In English, this says that ∇v is ∇u rotated 90° degrees counterclockwise everywhere on D.

Problem 8

We have $f = u(x,y) + iv(x,y)$ where $x = r\cos\theta, y = r\sin\theta$. The chain rule says that

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \partial_x u \cos\theta + \partial_y u \sin\theta \quad \text{and} \quad \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = \partial_x u(-r\sin\theta) + \partial_y u(r\cos\theta)$$

and similarly

$$\partial_r v = \partial_x v \cos\theta + \partial_y v \sin\theta \quad \text{and} \quad \partial_\theta v = \partial_x v(-r\sin\theta) + \partial_y v(r\cos\theta)$$

The C-R equations ($\partial_x v = -\partial_y u$ and $\partial_y v = \partial_x u$) can be easily substituted into the above to get that

$$\partial_r u = \frac{1}{r}\partial_\theta v \quad \text{and} \quad \partial_\theta u = -r\partial_r v$$

For an example, if we take the functions $u(re^{i\theta}) = r^m \cos(m\theta)$ and $v(re^{i\theta}) = r^m \sin(m\theta)$, then (for $m \neq 0$)

$$\partial_r u = mr^{m-1}\cos(m\theta) \quad \text{and} \quad \partial_\theta u = -mr^m \sin(m\theta)$$

and

$$\partial_r v = mr^{m-1}\sin(m\theta) \quad \text{and} \quad \partial_\theta v = mr^m \cos(m\theta)$$

which at a cursory glance can be seen to satisfy the polar C-R equations we've just derived. For $m = 0$, all derivatives are 0 and so it's even more obvious.

§II.4 - Problem 5

We are given that $g(z)$ is an analytic branch of $\arccos(z) = -i\log(z \pm \sqrt{z^2-1})$ defined on a domain D (i.e. we fix either the plus or the minus, and choose a branch of $\log z$ and $\sqrt{z}$ to work with). Henceforth when I write $\pm$ I mean whatever sign we fixed above. We have (no matter what branch of $\log z$ we are on):

$$g'(z) = -i\frac{1 \pm z/\sqrt{z^2-1}}{z \pm \sqrt{z^2-1}} = -i\frac{(\sqrt{z^2-1} \pm z)/\sqrt{z^2-1}}{z \pm \sqrt{z^2-1}} = \frac{-i}{\sqrt{z^2-1}}$$

The derivatives may be different on different branches (by choosing a different branch of the square root, i.e. the negative of the principal branch).

Problem 7

We have f, a bounded one-to-one analytic function defined on a bounded $D \subset \mathbb{C}^2$. Now as page 51 of Gamelin says, the Jacobian of this function is $|f'(z)|^2$ and so we can change the variables of integration to get that the area of $f(D)$ is

$$\int_{f(D)} 1\, du\, dv = \int_D |f'(x+iy)|^2\, dx\, dy$$

Problem 9

For $f(z) = z^2$ on D the open unit disk, and converting to polar coordinates for easy integration (remembering that $dx\,dy = r\,dr\,d\theta$ from 335):

$$\int_D |f'(z)|^2\,dx\,dy = \int_0^1 \int_0^{2\pi} |2re^{i\theta}|^2 r\,dr\,d\theta = \int_0^1 \int_0^{2\pi} = 4r^3\,dr\,d\theta = 2\pi$$

Interpreting this in terms of area, this is the "area" of $f(D)$ in that although $f(D) = D$, it covers it twice (f is not one to one), and so we count the "area" twice.

§II.5 - Problem 2

We have that $f = u + iv$ is analytic (on some domain) $\iff$ v is the harmonic conjugate of u (this is the definition of harmonic conjugate). We want to show that $-u$ is the harmonic conjugate of v, i.e. WTS that $v - iu = v + i(-u)$ is analytic: the x partial of the real function is in fact equal to the y partial of the imaginary function ($\partial_x v = \partial_y(-u) = -\partial_y u$, borrowed from the second C-R equation of the original f), and the x partial of the imaginary function is in fact equal to the negative y partial of the real function($\partial_x(-u) = -\partial_y v \iff \partial_x u = \partial_y v$, borrowed from the first C-R equation of the original f). Thus $v - iu$ satisfies both C-R equations and is thus analytic.

Problem 4

We are given that $h(z) = u(x,y) + iv(x,y)$ and $zh(z) = U(x,y) + iV(x,y) = xu(x,y) - yv(x,y) + i(yu(x,y) + xv(x,y))$ are harmonic functions. Thus, the 2nd partials of $h(z)$ show that

$$\partial_x U = u + x\partial_x u - y\partial_x v \implies \partial_{xx} U = \partial_x u + \partial_x u + x\partial_{xx} u - y\partial_{xx} v$$

and

$$\partial_y U = x\partial_y u - v - y\partial_y v \implies \partial_{yy} U = x\partial_{yy} u - \partial_y v - \partial_y v - y\partial_{yy} v$$

which sum to 0:

$$2\partial_x u + x\partial_{xx} u - y\partial_{xx} v + x\partial_{yy} u - 2\partial_y v - y\partial_{yy} v = 2\partial_x u - 2\partial_y v = 0 \iff \partial_x u = \partial_y v$$

and again for V:

$$\partial_x V = y\partial_x u + v + x\partial_x v \implies \partial_{xx} V = y\partial_{xx} u + \partial_x v + \partial_x v + x\partial_{xx} v$$

and

$$\partial_y V = u + y\partial_y u + x\partial_y v \implies \partial_{yy} V = \partial_y u + \partial_y u + y\partial_{yy} u + x\partial_{yy} v$$

which sum to 0:

$$y\partial_{xx} u + 2\partial_x v + x\partial_{xx} v + 2\partial_y u + y\partial_{yy} u + x\partial_{yy} v = 2\partial_x v + 2\partial_y u = 0 \iff \partial_x v = -\partial_y u$$

Thus both C-R equations are satisfied and $h(z)$ is analytic.

§II.7 - Problem 8

Letting $a' = \frac{a}{\sqrt{ad-bc}}, b' = \frac{b}{\sqrt{ad-bc}}, c' = \frac{c}{\sqrt{ad-bc}}$, and $d' = \frac{d}{\sqrt{ad-bc}}$ we have that

$$\frac{a'z+b'}{c'z+d'} = \frac{\frac{az+b}{\sqrt{ad-bc}}}{\frac{cz+d}{\sqrt{ad-bc}}} = \frac{az+b}{cz+d}$$

all with the benefit of having

$$a'd' - b'c' = \frac{ad}{ad-bc} - \frac{bc}{ad-bc} = 1$$

This is not unique because we could have used $(-\sqrt{ad-bc})$ instead of $\sqrt{ad-bc}$.

Problem 9

Suppose that $f(z) = \frac{az+b}{cz+d}$ sends the extended real line to the extended real line (we don't know yet if it's surjective). Well, first we can make a a real number (because we can always multiply all coefficients by a constant and keep the FLT the same). We now have a couple equations:

$$f(0) = \frac{b}{d}, f(\infty) = \frac{a}{c}, \quad \text{and} \quad f(1) = \frac{a+b}{c+d}$$

We know that all FLTs are one-to-one mappings from $\mathbb{C} \cup \{\infty\}$ to itself (because in the definition of FLT we have $ad - bc \neq 0$). Now assuming that $c \neq 0$ and $d \neq 0$ and $c + d \neq 0$, we have that $\frac{b}{d} = r_1 \in \mathbb{R} \iff b = r_1 d$, and similarly $a = r_2 c$ and $a + b = r_3(c + d)$. First, we fixed a as a real number, so c is also a real number. Thus this last equation can be written:

$$a - r_3 c = r_3 d - r_1 d \implies d = \frac{a - r_3 c}{r_3 - r_1}$$

where $r_1 \neq r_3$ because if it did, then $\frac{a+b}{c+d} = \frac{b}{d} \implies da + bd = bc + bd \implies ad - bc = 0$ which is forbidden. Thus d is real and then b is real.

Only one of the three terms can be infinite (because FLTs are one-to-one), so first case: $d = 0$, and $c \neq 0$ and $c + d \neq 0$. Again, because we fixed a to be real, c is real. d is obviously real, so now the third equation becomes $a + b = r_3 c \implies b = r_3 c - a$ and so b is real too.

Second case: $c = 0$, $d \neq 0$ and $c + d \neq 0$. We still have $b = r_1 d$, and $a + b = r_3 d$. a is fixed to be real, and $c = 0$ is real. Substituting the first equation into the third we get $a + r_1 d = r_3 d \implies d = \frac{a}{r_3 - r_1}$ where again $r_3 - r_1 \neq 0$. Thus d is real and so b is real too.

Final case: $c + d = 0$, $c \neq 0, d \neq 0$. a is still real implying that c is real, and $b = r_1 d$. But $d = -c$ so d is real, and therefore b is real too.

Conversely, if a, b, c, d are all real, then it's very easy to see that the FLT takes $\mathbb{R} \cup \{\infty\}$ to $\mathbb{R} \cup \{\infty\}$ (we can split up by cases; if $c = 0$ then we have just a line, so $f(\mathbb{R}) = \mathbb{R}$ and we can set $f(\infty) = \infty$; if $c \neq 0$, then we can set $f(-\frac{d}{c}) = \infty$ and $f(\infty) = \frac{a}{c}$ and everything else accordingly).

Problem 10

From Problem 9, we know that if a FLT maps the real line to the real line, then we can represent it using real a, b, c, d. From problem 8, we know that we can divide by $\sqrt{ad - bc}$ (or the negative) to get an FLT with $ad - bc = 1$. Well, $\sqrt{ad - bc}$ is either real or pure imaginary, so when we divide we get our new FLT has coefficients either real or pure imaginary.

But how do we know that this there isn't a representation where we can multiply a, b, c, d by a non-pure imaginary number to get another equivalent FLT with $a'd' - b'c' = 1$? If we could, then we would get that $a'd' - b'c' = z^2ad - z^2bc = z^2(ad - bc) = 1$. But a, b, c, d are real, so the only way for this to happen is if z^2 is real, so $z = \pm 1$ or $\pm i$. So we find that the only multiple you can multiply by to keep the "determinant" equal to 1 is pure real or pure imaginary, so we are done.

336 Homework 2

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§I.8 - Problem 7

We proceed as Gamelin did with $\sin(w)$. Let w be a value of $\arccos(z)$; i.e.

$$\cos(w) = \frac{e^{iw} + e^{-iw}}{2} = z \iff (e^{iw})^2 - 2ze^{iw} + 1 = 0$$

which by the quadratic formula gives that this happens if and only if

$$e^{iw} = \frac{2z \pm \sqrt{4z^2 - 4}}{2} = z \pm \sqrt{z^2 - 1}$$

and this happens if and only if w is a value attained by

$$\frac{1}{i}\log(z \pm \sqrt{z^2 - 1}) = -i\log(z \pm \sqrt{z^2 - 1})$$

These if and only if statements mean that

$$\arccos(z) = -i\log(z \pm \sqrt{z^2 - 1})$$

(an equality of sets).

§II.1 - Problem 6

We define for $\alpha \in \mathbb{C}$ and $n \in \mathbb{Z}_{\geq 0}$:

$$\binom{\alpha}{0} = 1 \quad \text{and} \quad \binom{\alpha}{n} = \frac{\alpha \cdots (\alpha - n + 1)}{n!}, n \geq 1$$

(a) We know that the sequence of $\binom{\alpha}{n}$ is bounded iff the sequence $|\binom{\alpha}{n}|$ is bounded, where

$$\left|\binom{\alpha}{n}\right| = \prod_{k=1}^{n} \frac{|\alpha - k + 1|}{k} = \prod_{k=1}^{n} \frac{|\operatorname{Re}\alpha - k + 1 + i\operatorname{Im}\alpha|}{k}$$

Of course, $|\operatorname{Re}\alpha - k + 1 + i\operatorname{Im}\alpha| \geq |\operatorname{Re}\alpha - k + 1|$, and so if $\operatorname{Re}\alpha < -1$, we have that $|\operatorname{Re}\alpha - k + 1 + i\operatorname{Im}\alpha| \geq |\operatorname{Re}\alpha - k + 1| = k + c$ where $c = (-\operatorname{Re}\alpha) - 1 > 0$. Then we have that

$$\log\left(\left|\binom{\alpha}{n}\right|\right) \geq \log\left(\prod_{k=1}^{n} \frac{k+c}{k}\right) = \sum_{k=1}^{n} \log\left(1 + \frac{c}{k}\right) = \sum_{k=1}^{n} \left|\log\left(1 + \frac{c}{k}\right)\right|$$

But we know from 335, problem §6.3.5 that $\sum a_n$ converges absolutely if and only if $\sum \ln(1+a_n)$ is absolutely convergent. Using this fact, we see that the above sum is divergent and so $\operatorname{Re}\alpha < -1 \implies |\binom{\alpha}{n}| \to \infty$ as $n \to \infty$.

In the case that $\operatorname{Re}\alpha = -1$, our product becomes

$$\left|\binom{\alpha}{n}\right| = \prod_{k=1}^{n} \frac{|-k + i\operatorname{Im}\alpha|}{k} = \prod_{k=1}^{n} \sqrt{1 + \frac{(\operatorname{Im}\alpha)^2}{k^2}}$$

or in other words

$$\log\left(\left|\binom{\alpha}{n}\right|\right) = \sum_{k=1}^{n} \frac{1}{2}\log\left(1 + \frac{(\operatorname{Im}\alpha)^2}{k^2}\right) = \frac{1}{2}\sum_{k=1}^{n}\left|\log\left(1 + \frac{(\operatorname{Im}\alpha)^2}{k^2}\right)\right|$$

which converges by the fact referenced above from 335. Thus $\operatorname{Re}\alpha = -1 \implies |\binom{\alpha}{n}|$ remains bounded as $n \to \infty$.

In the case that $\operatorname{Re}\alpha > -1$, our product becomes

$$\left|\binom{\alpha}{n}\right| = \prod_{k=1}^{n} \frac{\sqrt{(\operatorname{Re}\alpha + 1 - k)^2 + (\operatorname{Im}\alpha)^2}}{k} = \prod_{k=1}^{n} \sqrt{1 - \frac{2(\operatorname{Re}\alpha + 1)}{k} + \frac{(\operatorname{Re}\alpha + 1)^2}{k^2} + \frac{(\operatorname{Im}\alpha)^2}{k^2}}$$

and so

$$\log\left(\left|\binom{\alpha}{n}\right|\right) = \frac{1}{2}\sum_{k=1}^{n}\log\left(1 - \frac{2(\operatorname{Re}\alpha + 1)}{k} + \frac{(\operatorname{Re}\alpha + 1)^2}{k^2} + \frac{(\operatorname{Im}\alpha)^2}{k^2}\right)$$

I claim that for k large enough, the term inside the log becomes < 1 (and so we start going down instead of up) — let's denote $a = \operatorname{Re}\alpha + 1 > 0$ and $b = \operatorname{Im}\alpha$ for convenience:

$$-\frac{2a}{k} + \frac{a^2}{k^2} + \frac{b^2}{k^2} < 0 \iff -2ak + a^2 + b^2 < 0 \iff k > \frac{a^2 + b^2}{2a}$$

This means that for large enough n, $\log(|\binom{\alpha}{n}|)$ is decreasing, which means that $|\binom{\alpha}{n}|$ is decreasing, and it's bounded below by 0, so it must converge (and therefore it's also bounded).

(b) We've established that it is not bounded for $\operatorname{Re}\alpha < -1$, and for $\operatorname{Re}\alpha = -1$, we saw that $|\binom{\alpha}{n}|$ was a product of things > 1, so it can't possibly approach 0. Now for $\operatorname{Re}\alpha > -1$; we use that $\log(1 + x) \le x$, and so looking at the expression for $\log(|\binom{\alpha}{n}|)$, we see that this must go to $-\infty$, which means that $|\binom{\alpha}{n}|$ must go to 0, which means that $\binom{\alpha}{n}$ goes to 0.

(c) Already established in (b), we have $\binom{\alpha}{n+1} = \frac{\alpha - n}{n+1}\binom{\alpha}{n}$, and for $\alpha \neq 0, 1, 2, \ldots$ $\binom{\alpha}{n}$ is not zero so we can divide by it on both sides to get

$$\frac{\binom{\alpha}{n+1}}{\binom{\alpha}{n}} = \frac{\alpha - n}{n+1}$$

which approaches -1 obviously from limit properties.

(d) As established from (c) (for all $\alpha \neq 0, 1, 2\ldots$ because the region I will soon mention does not contain these points),

$$\left|\frac{\binom{\alpha}{n+1}}{\binom{\alpha}{n}}\right| = \frac{|\alpha - n|}{n+1} = \frac{|n - \alpha|}{|n - (-1)|}$$

and for all $\alpha \in \{z \in \mathbb{C} : \operatorname{Re} z \leq -1\} \setminus \{-1 + 0i\}$, the distance from n (a non-negative integer) to α will strictly exceed the distance from n to -1 and so $|\binom{\alpha}{n+1}| > |\binom{\alpha}{n}|$ on the aforementioned region.

(e) Piggybacking off of (d), on the region $\operatorname{Re}(\alpha) > -1$, $\alpha \neq 0, 1, 2, \ldots$ (so we can use the formula from (c)), we again have

$$\left| \frac{\binom{\alpha}{n+1}}{\binom{\alpha}{n}} \right| = \frac{|\alpha - n|}{n+1} = \frac{\sqrt{(\operatorname{Re}\alpha - n)^2 + (\operatorname{Im}\alpha)^2}}{n+1}$$

Observe that

$$\begin{aligned} \frac{\sqrt{(\operatorname{Re}\alpha - n)^2 + (\operatorname{Im}\alpha)^2}}{n+1} < 1 &\iff \sqrt{n^2 - 2n\operatorname{Re}\alpha + (\operatorname{Re}\alpha)^2 + (\operatorname{Im}\alpha)^2} < n+1 \\ &\iff n^2 - 2n\operatorname{Re}\alpha + |a|^2 < n^2 + 2n + 1 \\ &\iff |\alpha|^2 - 1 < 2n(\operatorname{Re}\alpha + 1) \\ &\iff \frac{|\alpha|^2 - 1}{2(\operatorname{Re}\alpha + 1)} < n \end{aligned}$$

so for n large enough, $|\binom{\alpha}{n+1}| < |\binom{\alpha}{n}|$.

Problem 14

Let $h(t)$ be continuous complex valued function on $[0, 1]$, and consider

$$H(z) = \int_0^1 \frac{h(t)}{t - z}\, dt$$

This is defined on $\mathbb{C} \setminus [0, 1]$ because then $t - z$ will not be 0. A quotient of continuous functions with non-zero denominator is continuous, and so $f(t, z) = \frac{h(t)}{t-z}$ is continuous. Thus for any $z_0 \in \mathbb{C} \setminus [0, 1]$ and $\epsilon > 0$, there is $\delta > 0$ s.t. $|z - z_0| < \delta \implies |f(t, z) - f(t, z_0)| < \epsilon$. This implies that

$$|z - z_0| < \delta \implies |H(z) - H(z_0)| \leq \int_0^1 |f(t, z) - f(t, z_0)\, dt < \epsilon$$

and so $H(z)$ is continuous at every point of $\mathbb{C} \setminus [0, 1]$.

Problem 19

We have a non-constant polynomial, $p(z)$, say $a_n z^n + \ldots + a_0$ for $n \geq 1$ and $a_n \neq 0$. Now $|\frac{p(z)}{z^n}| = |a_n + a_{n-1}/z + \ldots + a_0/z^n|$ approaches $|a_n| > 0$ as $|z| \to \infty$, so $|p(z)|$ must go to ∞ as $|z| \to \infty$. Thus for every z with $|z|$ greater than some R, $|p(z)|$ must be $> |a_0|$, where we choose a_0 because we know $|p(z)|$ attains $|a_0|$ in $\{z \in \mathbb{C} : |z| \leq R\}$ (because continuous functions attain their minimum on compact sets), and so the local minimum m of $|p(z)|$ on that set will be the global minimum of $|p(z)|$. Suppose that the polynomial does not have any zeroes, and so $|m| > 0$. Let's say that $|p|$ attains the minimum global at z_0. Then, define $q(z) = \frac{1}{|m|} p(z + z_0) = 1 + b_k z^k + \ldots + b_n z^n$ so that $|q|$ has a

minimum value of 1 (globally) at $z = 0$, and $b_k \neq 0$. Now because

$$|b_k| \geq 2(|b_{k+1}z^1| + \ldots + |b_n z^{n-k}|)$$

for z with sufficiently small modulus, say $\epsilon > 0$, (0 on the RHS if $n = k$), multiplying by $|z_k|$ on both sides yields

$$|z| \leq \epsilon \implies |b_k z^k| \geq 2(|b_{k+1}z^{k+1}| + \ldots + |b_n z^n|)$$

Choose δ s.t. $0 < \delta < \min\{1, |b_k|\epsilon^k\}$. There must exist z_1 s.t. $b_k z_1^k = -\delta$. Notice that $|z_1^k| = \delta/|b_k| < \epsilon^k$, and so we can use the above inequality to see that

$$\delta \geq 2(|b_{k+1}z_1^{k+1}| + \ldots + |b_n z_1^n|)$$

Putting everything together, we have

$$|q(z_1)| = |1 - \delta + b_{k+1}z_1^{k+1} + \ldots + b_n z_1^n| \leq 1 - \delta + \frac{\delta}{2} = 1 - \frac{\delta}{2} < 1$$

which contradicts that the minimum value of $|q|$ was 1. Thus, p must have at least 1 root.

§II.2 - Problem 4

Define $f(z) = az^2 + bz\bar{z} + c\bar{z}^2$. We differentiate f by hand to get

$$\frac{f(z+h) - f(z)}{h} = 2az + b\overline{z} + ah + b\overline{h} + c\frac{\overline{h}^2}{h} + (bz + 2c\overline{z})\frac{\overline{h}}{h}$$

As $h \to 0$, $\overline{h}$ also $\to 0$. Furthermore, $|\overline{h}^2/h| = |\overline{h}|^2/|h| = |h|$ also $\to 0$, which is $\iff \overline{h}^2/h \to 0$. However, $\overline{h}/h$ does not have a limit as $h \to 0$ (approaching from the right ($\mathbb{R}^+$) the limit is 1, but approaching from above ($i\mathbb{R}^+$) the limit is -1). Thus, $\frac{f(z+h)-f(z)}{h}$ has a limit if and only if $(bz + 2c\overline{z})$ is 0. If $b = c = 0$ we get that the solution set to $bz + 2c\overline{z} = 0$ is all of $\mathbb{C}$, and so $b = c = 0 \implies f$ is everywhere analytic. Otherwise, the solution set is just some line (null interior), which means that f is nowhere analytic.

Problem 6

$$\begin{aligned}\frac{H(z) - H(z_0)}{z - z_0} &= \frac{1}{z - z_0}\left(\int_0^1 \frac{h(t)}{t - z}\,dt - \int_0^1 \frac{h(t)}{t - z_0}\,dt\right) \\ &= \frac{1}{z - z_0}\left(\int_0^1 \frac{h(t)(z - z_0)}{(t - z)(t - z_0)}\,dt\right) = \int_0^1 \frac{h(t)}{(t - z)(t - z_0)}\,dt\end{aligned}$$

which goes to $\int_0^1 \frac{h(t)}{(t-z_0)^2}$ as $z \to z_0$ because the integrands converge uniformly: $\sup_{0\leq t\leq 1} |\frac{h(t)}{(t-z)(t-z_0)} - \frac{h(t)}{(t-z_0)^2}| = \sup_{0\leq t\leq 1} |\frac{h(t)}{(t-z_0)}||\frac{(z-z_0)}{(t-z)(t-z_0)}| = (z - z_0)\sup_{0\leq t\leq 1} |\frac{h(t)}{(t-z_0)(t-z)(t-z_0)}|$ but $(z - z_0)$ goes to 0 and the sup of a continuous function over a compact set is finite.

Thus we see that $H'(z)$ exists everywhere in $\mathbb{C} \setminus [0, 1]$. $H'(z)$ is continuous (on the same domain) with much the same argument as Problem 14 from above, and so by definition $H(z)$ is analytic on $\mathbb{C} \setminus [0, 1]$.

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§I.1 - Problem 6

We know that $|z| = 1 \implies |\bar{z}| = 1$ and $z\bar{z} = |z| = 1$. Observe that

$$|z-a| = \frac{|z-a|}{|z|} = \left|\frac{z-a}{z}\right| = \left|1 - \frac{a}{z}\right| = |1 - a\bar{z}| = |\overline{1-a\bar{z}}| = |1-\bar{a}z|$$

and so if $1 - \bar{a}z \neq 0$, we can divide to get $\frac{|z-a|}{|1-\bar{a}z|} = 1$.

Problem 7

We want to show that for $\rho > 0$ and $\neq 1$, $|z-z_0| = \rho|z-z_1|$ is a circle. Let $z = x+yi$, and $z_0 = a_0+ib_0$ and $z_1 = a_1 + ib_1$. Then the equation is

$$\begin{aligned}
&|(x-a_0)+i(y-b_0)| = \rho|(x-a_1)+i(y-b_1)| \iff (x-a_0)^2+(y-b_0)^2 = \rho^2\Big((x-a_1)^2+(y-b_1)^2\Big) \\
\iff& x^2 - 2a_0x + a_0^2 + y^2 - 2b_0y + b_0^2 = \rho^2\Big(x^2 - 2a_1x + a_1^2 + y^2 - 2b_1y + b_1^2\Big) \\
\iff& (1-\rho^2)x^2 - 2(a_0 - \rho^2a_1)x + (1-\rho^2)y^2 - 2(b_0-\rho^2b_1)y = \rho^2a_1^2 - a_0^2 + \rho^2b_1^2 - b_0^2 \\
\iff& x^2 - 2\frac{(a_0-\rho^2a_1)}{1-\rho^2}x + y^2 - 2\frac{(a_b-\rho^2b_1)}{1-\rho^2}y = \frac{\rho^2a_1^2 - a_0^2 + \rho^2b_1^2 - b_0^2}{1-\rho^2} \\
\iff& \left(x - \frac{(a_0-\rho^2a_1)}{1-\rho^2}\right)^2 + \left(y - \frac{(b_0-\rho^2b_1)}{1-\rho^2}\right)^2 = \frac{\rho^2a_1^2 - a_0^2 + \rho^2b_1^2 - b_0^2}{1-\rho^2} + \frac{(a_0-\rho^2a_1)^2 + (b_0-\rho^2b_1)^2}{(1-\rho^2)^2} \\
\iff& \left(x - \frac{(a_0-\rho^2a_1)}{1-\rho^2}\right)^2 + \left(y - \frac{(b_0-\rho^2b_1)}{1-\rho^2}\right)^2 = \frac{\rho^2((a_1-a_0)^2 + (b_1-b_0)^2)}{(1-\rho^2)^2}
\end{aligned}$$

This is the graph of a circle centered at $(\frac{a_0-\rho^2a_1}{1-\rho^2}, \frac{b_0-\rho^2b_1}{1-\rho^2})$ with radius $\frac{\rho}{1-\rho^2}\sqrt{(a_1-a_0)^2+(b_1-b_0)^2}$. When $\rho = 1$, the fourth step from above simplifies to

$$-2(a_0-a_1)x - 2(b_0-b_1)y = a_1^2 - a_0^2 + b_1^2 - b_0^2 \iff y = \frac{(a_0-a_1)}{(b_1-b_0)}x + \frac{a_1^2-a_0^2+b_1^2-b_0^2}{2(b_1-b_0)}$$

which is a line. I will sketch the case where $z_0 = 0, z_1 = 1$, and $\rho = 1/2$ or 2 here:

§I.2 - Problem 6

(a) We have the nth roots of unity $\omega_0, \ldots, \omega_{n-1}$, i.e. $\omega_k = e^{2\pi ik/n}$, which satisfy $\omega_k^n = e^{2\pi ik} = 1$. Thus, all $\omega_0, \ldots, \omega_{n-1}$ are roots of the equation $z^n - 1 = 0$. The fundamental theorem of algebra says that an n degree polynomial has exactly n roots (counting multiplicity), and we've found exactly n solutions, and so $z^n - 1$ must be $\prod_{k=0}^{n-1}(z-\omega_k)$.

(b) Expanding out $\prod_{k=0}^{n-1}(z-\omega_k)$, we see that the coefficient for the z^{n-1} term is $-(\omega_0+\ldots+\omega_{n-1})$. For $n \geq 2$, the coefficient for the z^{n-1} term is 0, and so $\omega_0+\ldots+\omega_{n-1}=0$. For $n=1$, there is only one root, which sums to 1 not 0.

(c) Expanding out $\prod_{k=0}^{n-1}(z-\omega_k)$, we see that the coefficient for the z^0 term is $(-1)^n\omega_0\cdots\omega_{n-1}$. For $n \geq 1$, the coefficient of the z^0 is -1, and so $-1=(-1)^n\omega_0\cdots\omega_{n-1} \implies \omega_0\cdots\omega_{n-1}=(-1)^{n-1}$

(d) Using the formula for finite geometric series, if $1 \leq k \leq n-1$, we have

$$\sum_{j=0}^{n-1}\omega_j^k=\sum_{j=0}^{n-1}e^{2\pi kij/n}=\frac{1-e^{2\pi ik(n-1+1)/n}}{1-e^{2\pi ik/n}}=\frac{0}{1-e^{2\pi ik/n}}$$

The denominator is not 0 because $1 \leq k \leq n-1$, so this is 0. For $k=n$, $\omega_j^k=1$, so the sum will equal n.

Problem 7

From the triangle inequality, $|z^n|=|z^n+1-1| \leq |z^n+1|+|-1| \implies |z^n+1| \geq |z^n|-1$. We also know that if $|z|=R>1$, $|z^n|=|z|^n=R^n$. Putting everything together, we have

$$\left|\frac{z^m}{z^n-1}\right|=\frac{|z^m|}{|z^n+1|} \leq \frac{R^m}{|z^n|-1}=\frac{R^m}{R^n-1}$$

Equality holds when $|z^n+1|=|z^n|-1 \iff |-1|+|z^n+1|=|z^n|$. We know from literally the first day of 334 that the triangle inequality is an equality if and only if one vector is a non-negative multiple of the other; here that means that $z^n+1=-c$ for some $c \in \mathbb{R}_{\geq 0}$. If we think of $z=Re^{i\theta}$, then $z^n=R^ne^{in\theta}=-c-1$. R is positive, so the only way to make it negative is to set $e^{in\theta}=-1$. Thus, $n\theta$ must be of the form $\pi+2k\pi$ for $k \in \mathbb{Z}$, and so $z^n=R^ne^{i(\pi+2k\pi)}$, i.e. $z=Re^{i(\pi+2k\pi)/n}$ for any $k \in \mathbb{Z}$ make the inequality an equality. A sketch follows below of what the set where equality holds looks like for $n=5$:

§I.7 - Problem 4

By definition $(zw)^\alpha=\exp(\alpha(\log|zw|+i\operatorname{Arg}(zw)+2k\pi i))$ for any $k \in \mathbb{Z}$. This can be broken into $\exp(\alpha(\log|z|+\log|w|+i(\operatorname{Arg}(z)+\operatorname{Arg}(w))+2k\pi i))=\exp(\alpha(\log|z|+i\operatorname{Arg}(z)+2k_1\pi i))\exp(\alpha(\log|w|+i\operatorname{Arg}(w)+2k_2\pi i)$ for any $k_1,k_2 \in \mathbb{Z}, k_1+k_2=k$, which are values attained by $z^\alpha w^\alpha$. k,k_1,k_2 are all arbitrary; i.e. every value attained by $(zw)^\alpha$ can be attained by choosing k_1 and k_2 when doing $z^\alpha w^\alpha$, and every value attained by $z^\alpha w^\alpha$ can be attained by $(zw)^\alpha$ by choosing $k=k_1+k_2$.

Problem 5

We want to compute $(i^i)^i$. We know that $i^i=\exp(i(\log|i|+i\operatorname{Arg}(i)+2k\pi i))=\exp(i(0+\pi i/2+2k\pi i))=\exp(-\pi/2-2k\pi)$ for $k \in \mathbb{Z}$. Thus, $(i^i)^i=\exp(i(\log|i^i|+i\operatorname{Arg}(i^i)+2k'\pi i))=\exp(i((-\pi/2-$

$2k\pi) + i \cdot 0 + 2k'\pi i)) = e^{-2k'\pi} \exp(i(-\pi/2 - 2k\pi))$ for $k, k' \in \mathbb{Z}$. We can further simplify to get $(i^i)^i = e^{-2k'\pi} e^{-i\pi/2} e^{-2k\pi i} = -ie^{-2k'\pi}$. If we instead consider $i^{i \cdot i} = i^{-1} = -i$, we see that this is one of the values of $(i^i)^i$; i.e. take $k' = 0$.