524 Homework 10

Daniel Rui - 12/9/20

Problem 1 - 2.7.64

We are asked to investigate which real values of a and b allow $f(x) = |x|^a \big|\log|x|\big|^b$ to be integrable over $A = \{x \in \mathbb{R}^n : |x| < 1/2\}$ or $B = \{x \in \mathbb{R}^n : |x| > 2\}$.

Corollary 2.51 in Folland says that if f is a non-negative measurable function on $\mathbb{R}^n$ s.t. $f(x) = g(|x|)$ for some function g on $(0,\infty)$, then $\int_{\mathbb{R}^n} f(x)\,dx = \sigma(S^{n-1})\int_0^\infty g(r)r^{n-1}\,dr$, where σ is defined in Theorem 2.49 of Folland and evaluated in Prop 2.54 as $\sigma(S^{n-1}) = \frac{2\pi^{n/2}}{\Gamma(n/2)} < \infty$.

First, let us establish that $f \in L^+(\mathbb{R})$. It is clearly ≥ 0, and it is measurable because it is continuous everywhere it is defined (so everywhere except maybe $\{-1,0,1\}$), meaning we can split up f into 4 chunks f_1, f_2, f_3, f_4 defined (and continuous) on $(-\infty,-1),(-1,0),(0,1),(1,\infty)$ respectively, so the inverse image via f of (a,∞) for all $a > 0$ will just be union of the inverse images of (a,∞) via f_i for $i \in [4]$ (all open sets because the f_i are continuous), union maybe $\{-1\}$, $\{0\}$, and/or $\{1\}$. Because all these (open and singleton) set that we are union-ing are Borel sets, we see that $f^{-1}((a,\infty))$ is a measurable set. Thus, because (a,∞) generate $\mathscr{B}_\mathbb{R}$, f is measurable.

We can now apply Corollary 2.51: define $g : (0,\infty) \to \mathbb{R}$ as $g(x) = x^a|\log x|^b$. Then,

$$\int_A f(x)\,dx = \int f(x)1_A(x)\,dx = \int_0^\infty g(r)r^{n-1}1_{\{x\in(0,\infty):x<\frac{1}{2}\}}\,dr = \int_0^{1/2} r^{a+n-1}|\log(r)|^b\,dr$$

and similarly $\int_B f(x)\,dx = \int_2^\infty r^{a+n-1}|\log(r)|^b\,dr$. So the problem boils down to which values of $a, b \in \mathbb{R}$ allow these two easier-to-understand integrals to be finite. We tackle the B integral first; note that with the variable change $t = s^{-1}$, we get that

$$\int_0^{1/2} r^{a+n-1}|\log(r)|^b\,dr = -\int_\infty^2 t^{1-n-a}|\log(t)|^b(-\tfrac{1}{t^2})\,dt = \int_2^\infty t^{-(a+n+1)}|\log(t)|^b\,dt,$$

so once we know for which $a, b \in \mathbb{R}$ the B integral is finite, it becomes trivial to find the values for which the A integral is finite. Moreover because the B integral takes place on $(2,\infty)$ on which log is positive, we can get rid of the absolute values to see that the B integral is $\int_2^\infty r^{a+n-1}(\log(r))^b\,dr$. Doing a variable change $u = \log r$, we see that this integral is equal to $\int_{\log 2}^\infty e^{u(a+n)}u^b\,du$. If $a > -n$, then the integral will have $e^{\epsilon u}$ for some $\epsilon > 0$, and exponential growth dominates any u^b, so the integral will not be finite. If $a < -n$, the exponent of the e^u term will be negative, and exponential decay dominates any u^b, so the integral will be finite.

Problem 2

Suppose (X, d) is a compact metric space, and $f : X \to \mathbb{R}$ is some function s.t. for all $t \in \mathbb{R}$, $f^{-1}([t, \infty))$ is closed. We want to show that there is some $x_0 \in X$ s.t. $f(x_0) = \sup_{x \in X} f(x) < \infty$ (i.e. f attains its supremum, and the supremum is finite).

We first show that the supremum is finite (i.e. the function is bounded). Let's define the sequence of closed sets $F_n = f^{-1}([n, \infty))$ (for all $n \in \mathbb{N}$); notice that this is a decreasing sequence, i.e. for any $I \subseteq \mathbb{N}$, $\bigcap_{i \in I} F_i = F_{\max I}$. We now suppose that f is not bounded. This would imply that all the F_n are non-empty. From page 3 of the Lecture of 10/5, we know that for a compact metric space (X, d), every collection of closed sets with the finite intersection property (i.e. any finite subcollection has non-empty intersection) has non-empty intersection. Our sequence $\{F_n\}_{n \in \mathbb{N}}$ indeed satisfies the finite intersection property (from the above assertions that all the F_n are non-empty, and that for any $I \subseteq \mathbb{N}$, $\bigcap_{i \in I} F_i = F_{\max I}$), so it must now be that $\bigcap_{n=1}^{\infty} F_n \neq \varnothing$. But this is impossible, because for any $x \in X$, $f(x) \in \mathbb{R}$ (i.e. it is a finite number), and so x would not be in any F_n for $n > f(x)$. This is a contradiction; thus it must be that f is bounded.

Now let $M := \sup_{x \in X} f(x)$, which we've just proven is finite. Suppose that there is no $x_0 \in X$ s.t. $f(x_0) = M$ (i.e. for all $x_0 \in X$, $f(x_0) < M$, because M is the unattained supremum). Then, the function g defined as $g(x) = \frac{1}{M - f(x)}$ is a function $X \to \mathbb{R}$ (the denominator is never 0), or more specifically $X \to (0, \infty)$. We now examine the sets $g^{-1}([t, \infty))$. If $t \leq 0$, then $g^{-1}([t, \infty)) = X$. If $t > 0$, then we have that

$$\begin{aligned} g^{-1}([t, \infty)) &= \{x \in X : \tfrac{1}{M - f(x)} \geq t\} = \{x \in X : 1 \geq tM - tf(x)\} \\ &= \{x \in X : f(x) \geq \tfrac{tM - 1}{t}\} = f^{-1}([\tfrac{tM-1}{t}, \infty)), \end{aligned}$$

which we know to be closed. That is to say, we know that for any $t \in \mathbb{R}$, $g^{-1}([t, \infty))$ is closed. But we proved in the above paragraph that such functions must be bounded! So there must be some $N \in \mathbb{N}$ s.t. for all $x \in X$, $g(x) = \frac{1}{M - f(x)} < N$, i.e. that for all $x \in X$, $f(x) < M - \frac{1}{n}$. But this contradicts the definition of supremum, and so our initial supposition was wrong — there must in fact be some $x_0 \in X$ s.t. $f(x_0) = M$.

Problem 3

We want to show that if a measurable $E \subseteq [0, 1]$ satisfies $m(E \cap I) \geq \alpha m(I)$ for some $\alpha > 0$ and all intervals I in $[0, 1]$, then E has measure 1.

I now copy a theorem + corollary we covered in class on 12/2: if $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ (in particular we can take $f = 1_E$ for some measurable $E \subseteq \mathbb{R}^n$), then it is true that $(*) \lim_{m(B) \to 0, x \in B, B \text{ ball}} \frac{1}{m(B)} \int_B f(y)\, dy = f(x)$ for m-a.e. x. In fact, it is true that $(**) \lim_{m(B) \to 0, x \in B, B \text{ ball}} \frac{1}{m(B)} \int_B |f(y) - f(x)|\, dy = 0$ for m-a.e. x. Thus in particular as a corollary, for $f = 1_E$ where $E \subseteq \mathbb{R}^n$ is measurable, defining D_E to be the density points of E, i.e. the points x s.t. $(*) \lim_{m(B) \to 0, x \in B, B \text{ ball}} \frac{m(B \cap E)}{m(B)} = 1$, and N_E

("non-density points") to be the set of points x s.t. the limit is 0, $1_E =_{\text{a.e.}} 1_{D_E}$ and $1_{E^\complement} =_{\text{a.e.}} 1_{N_E}$.

Now to the problem — we proceed by contradiction: suppose that measurable $E \subseteq [0,1]$ satisfies $m(E \cap I) \geq \alpha m(I)$ for some α and for all intervals I in $[0,1]$, and that E has measure < 1. Then, $[0,1] \backslash E$ has measure > 0, implying that $m((0,1)\backslash E) = m((0,1)\backslash E) + 0 + 0 = m(((0,1)\backslash E) \cup \{0\} \cup \{1\}) = m([0,1] \setminus E) > 0$ too, and because (from the above paragraph) $1_{E^\complement} =_{\text{a.e.}} 1_{N_E}$, it can not be that $1_{E^\complement}(x) \neq 1_{N_E}(x)$ for all $x \in (0,1) \setminus E$. Thus we have some $x_0 \in N_E$ also in $(0,1) \setminus E$, i.e. we have that $\lim_{m(B)\to 0, x_0 \in B, B \text{ ball}} \frac{m(B \cap E)}{m(B)} = 0$. By the definition of limit, there is some $\delta > 0$ s.t. $m(B) < \delta, x_0 \in B$, B a ball implies that $m(B \cap E) < \alpha m(B)$. Since $x_0 \in (0,1)$ and $(0,1)$ is open, all small enough balls containing x_0 are contained in $(0,1)$, and so the previous sentence tells us that there is some small enough ball $B \subseteq (0,1) \subset [0,1]$ such that $m(B \cap E) < \alpha m(B)$. But balls in $\mathbb{R}$ are just intervals, and so we've found a contradiction to the initial claim that E satisfied $m(E \cap I) \geq \alpha m(I)$ for all intervals I in $[0,1]$. We conclude that the initial supposition was wrong, i.e. E must actually have measure 1.

Problem 4

Let μ be a doubling measure on $\mathbb{R}^n$ (i.e. there exists a constant $C \geq 1$ s.t. for every $x \in \mathbb{R}^n$ and $r > 0$, $\mu(B(x,2r)) \leq C\mu(B(x,r))$; this implicitly assumes that all $B(x,r)$ are μ-measurable). Assume that $\mu(B(x_0,1)) < \infty$ for some $x_0 \in X$. Note that the implicit assumption that all $B(x,r)$ are μ-measurable makes μ a Borel measure on $\mathbb{R}^n$.

(a) We want to show that if K is compact, then $\mu(K) < \infty$. Note that the "doubling" condition and the condition $\mu(B(x_0,1)) < \infty$ implies that for any $n \in \mathbb{N}_0 := \{0,1,\ldots\}$, $\mu(B(x_0,2^n)) \leq C^n \mu(B(x_0,1)) < \infty$. Also observe that $\{B(x_0,2^n)\}_{n=0}^{\infty}$ is an open cover of $\mathbb{R}^n$ (and hence also K). Thus, by the definition of compactness, there is some N s.t. $K \subseteq \bigcup_{n=0}^{N} B(x_0,2^n) = B(x_0,2^N)$. By monotonicity of measures, $\mu(K) \leq \mu(B(x_0,2^N)) < \infty$.

(b) We want to show that if $E \in \mathscr{B}_{\mathbb{R}^n}$ with $\mu(E) < \infty$, then given any $\epsilon > 0$ we can find some compact K, $K \subseteq E$, s.t. $\mu(E \setminus K) < \epsilon$. We use the following theorem from the Lecture of 10/16: if μ is a Borel measure on (X,d) a metric space, and $\mu(B(x,r)) < \infty$ for all $x \in X$ and $r > 0$, then for any $E \in \mathscr{B}_X$, for all $\epsilon > 0$, there is an open set O and closed set F s.t. $F \subseteq E \subseteq O$ and $\mu(O \setminus E), \mu(E \setminus F) < \epsilon$ (a Borel measure satisfying this property is known as a Borel regular measure).

Let us fix $\epsilon > 0$. Note that for any $x \in X$, $r > 0$, $\overline{B(x,r)} \subseteq \mathbb{R}^n$ is compact and hence has finite measure (from part (a)), implying that $\mu(B(x,r)) \leq \mu(\overline{B(x,r)}) < \infty$, so the above theorem gives that for any $E \in \mathscr{B}_{\mathbb{R}^n}$, there is some closed $F \subseteq E$ s.t. $\mu(E \setminus F) = \mu(E) - \mu(F) < \frac{\epsilon}{2}$ (we can split up the measure like that because $E, F, E \setminus F$ are all in $\mathscr{B}_{\mathbb{R}^n}$, and measure is additive for disjoint measurable sets). For each $n \in \mathbb{N}$, define $K_n = F \cap \overline{B(x,n)}$. Note that all the K_n are compact (intersection of closed sets are closed, and $\overline{B(x,n)}$ is bounded), and increasing to F. Then, by continuity of measure from below (see the Lecture of 10/9), we know that $\mu(K_n) \to \mu(F)$ as $n \to \infty$. We now use the assumption that $\mu(E) < \infty$: because $F \subseteq E$,

$\mu(F) < \infty$ too, so by the definition of convergence to a finite number, there is some N s.t. $n \geq N \implies |\mu(F) - \mu(K_n)| = \mu(F) - \mu(K_n) < \frac{\epsilon}{2}$. Finally, we can put the two inequalities together to yield $\mu(E \setminus K_N) = \mu(E) - \mu(K_N) = \mu(E) - \mu(F) + \mu(F) - \mu(K_N) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. We've indeed found, for our arbitrary fixed $\epsilon > 0$, $K_N \subseteq E$ compact satisfying $\mu(E \setminus K_N) < \epsilon$, so we are done.

Problem 5

We want to show that for any open $U \subseteq \mathbb{R}^d$ and $\delta > 0$, there exists a countable collection $\mathcal{G}$ of disjoint closed balls in U with diameter $\leq \delta$ s.t. $m(U \setminus \bigcup_{B \in \mathcal{G}} B) = 0$.

For reference, I state the Vitali covering lemma and the 1st corollary we covered in class on 11/23. Vitali's covering lemma says that letting $\mathcal{F}$ be any collection of non-degenerate closed balls in $\mathbb{R}^n$ s.t. the sup of their diameters is finite, there exists a countable family $\mathcal{G}$ of disjoint balls in $\mathcal{F}$ s.t. $\bigcup_{B \in \mathcal{F}} B \subseteq \bigcup_{B \in \mathcal{G}} \hat{B}$ (where $\hat{B}$ is the closed ball with same center as B but 5 times the radius), and moreover for every $B \in \mathcal{F}$ there is some $S \in \mathcal{G}$ s.t. $B \cap S \neq \varnothing$ and $B \subseteq \hat{S}$.
The 1st corollary we proved was that assuming $\mathcal{F}$ is a fine cover of A by non-degenerate closed balls (i.e. for all $x \in A$ and $\delta > 0$, there is some $B \in \mathcal{F}$ containing x with diameter $\leq \delta$) with sup of their diameters finite, there exists a countable family $\mathcal{G}$ of disjoint balls in $\mathcal{F}$ (namely the $\mathcal{G}$ from Vitali's covering lemma) s.t. for any finite subset $\{B_1, \ldots, B_m\} \subseteq \mathcal{F}_k$, we have $A \setminus \bigcup_{k=1}^m B_k \subseteq \bigcup_{B \in \mathcal{G} \setminus \{B_1, \ldots, B_m\}} \hat{B}$.

We shall proceed via expanding upon the proof the 2nd corollary: we start with some fixed open $U \subseteq \mathbb{R}^n$ and $\delta > 0$. Let us further assume FOR NOW that U has finite measure. Consider $\mathcal{F}_0 = \{\overline{B(x,r)} : x \in U, r \in (0, \frac{\delta}{2}), \overline{B(x,r)} \subseteq U\}$. It is clear that $\mathcal{F}_0$ is a fine cover of U by non-degenerate closed balls (around any $x \in U$, there is some $\delta_x > 0$ s.t. $r \in (0, \delta_x) \implies \overline{B(x,r)} \subseteq U$ by the openness of U), and all balls in $\mathcal{F}_0$ have diameter $\leq \delta$. By the 1st corollary (stated above), there is some countable disjoint subcollection $\mathcal{G} = \{B_i\}_{i=1}^\infty$ of $\mathcal{F}_0$ s.t. for any $m \in \mathbb{N}$, $E \setminus \bigcup_{k=1}^m B_k \subseteq \bigcup_{B \in \mathcal{G} \setminus \{B_1, \ldots, B_m\}} \hat{B}$. Now notice (exactly as in the proof of the 2nd corollary we did in class):

$$\sum_{i=1}^\infty m(\hat{B}_i) = \sum_{i=1}^\infty 5^n m(B_i) = 5^n \sum_{i=1}^\infty m(B_i) = 5^n m\left(\bigcup_{i=1}^\infty B_i\right) \leq 5^n m(U) < \infty$$

Thus for every $n \in \mathbb{N}$, there is some m_n s.t. $\sum_{i=m_n+1}^\infty m(\hat{B}_i) < \frac{1}{n}$, implying that for all $n \in \mathbb{N}$,

$$m(A_n) := m\left(E \setminus \bigcup_{i=1}^{m_n} B_i\right) \leq m\left(\bigcup_{i=m_n+1}^\infty \hat{B}_i\right) \leq \sum_{i=m_n+1}^\infty m(\hat{B}_i) < \frac{1}{n}.$$

Note that the A_n form a decreasing sequence of sets, going to $\bigcap_{n=1}^\infty A_n \supseteq E \setminus \bigcup_{i=1}^\infty B_i$ (because $\bigcap_{n=1}^\infty A_n$ is the set of all points $x \in E$ s.t. for all $n \in \mathbb{N}$, x is not in $\bigcup_{i=1}^{m_n} B_i$, something that is implied if $x \in E \setminus \bigcup_{i=1}^\infty B_i$). Thus by monotonicity, and upper continuity of measure (see the Lecture of 10/9)

which we can apply because $m(A_1) < 1 < \infty$, we have that

$$0 \leq m\left(E \setminus \bigcup_{i=1}^{\infty} B_i\right) \leq m\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} m(A_n) \leq \lim_{n\to\infty} \frac{1}{n} = 0.$$

To recap, we've shown that for open sets $U \subseteq \mathbb{R}^n$ with finite measure, there is indeed a countable collection $\mathscr{G}$ of disjoint closed balls in U with diameter $\leq \delta$ s.t. $m(U \setminus \bigcup_{B\in\mathscr{G}} B) = 0$ (where the $\{B_i\}_{i=1}^{\infty}$ we found are in fact disjoint, from the 1st corollary; and are in U and have diameter $\leq \delta$, by definition of $\mathscr{F}_0$).

Now suppose we have open $U \subseteq \mathbb{R}^d$ with infinite measure. Then, define the "annuli" $A_n = \{x \in \mathbb{R}^d : n - 1 < |x| < n\}$ (for all $n \in \mathbb{N}$). These A_n are all open, so $U_n := U \cap A_n$ will all be open as well (as well as disjoint from each other, and bounded $\implies$ have finite measure). The above then gives that for each U_n, there is some countable collection $\mathscr{G}_n$ of disjoint closed balls in U_n with diameter $\leq \delta$ s.t. $m(U_n \setminus \bigcup_{B\in\mathscr{G}_n} B) = 0$, where because all the U_n are disjoint and all balls in $\mathscr{G}_n$ are $\subseteq U_n$, $\mathscr{G} := \bigcup_{n=1}^{\infty} \mathscr{G}_n$ is a countable collection of disjoint closed balls in U with diameters $\leq \delta$. Disjointness of the U_n and the fact that for all $B_n \in \mathscr{G}_n$, $B_n \subseteq U_n$ also give that

$$\bigcup_{n=1}^{\infty}\left(U_n \setminus \bigcup_{B\in\mathscr{G}_n} B\right) = \left(\bigcup_{n=1}^{\infty} U_n\right) \setminus \left(\bigcup_{n=1}^{\infty}\left(\bigcup_{B\in\mathscr{G}_n} B\right)\right) = \left(\bigcup_{n=1}^{\infty} U_n\right) \setminus \left(\bigcup_{B\in\mathscr{G}} B\right).$$

Also define the "shells" $S_n = \{x \in \mathbb{R}^d : |x| = n\}$ for all $n \in \mathbb{N}$ (as well as $S_0 = \{\mathbf{0}\}$), and note that all the S_n have Lebesgue measure 0 because $S_n \subseteq B(x, n(1+\epsilon)^{1/d}) \setminus B(x, n(1-\epsilon)^{1/d})$ for any $\epsilon > 0$, so we have that for each $n \in \mathbb{N}$, $m(S_n) \leq m(B(x, n(1+\epsilon)^{1/d})) - m(B(x, n(1-\epsilon)^{1/d})) = n^d(1+\epsilon) - n^d(1-\epsilon) = 2n^d\epsilon$, which goes to 0 as $\epsilon > 0$ (because n, d are fixed). Obviously S_0 has measure 0 too (it's just one point, so in particular it's countable, and countable sets have measure 0). Note that $\bigcup_{n=0}^{\infty} S_n \cup \bigcup_{n=1}^{\infty} A_n = \mathbb{R}^n$. Thus, we have:

$$\begin{aligned}
0 &= \sum_{n=0}^{\infty} m(S_n) + \sum_{n=1}^{\infty} m\left(U_n \setminus \bigcup_{B\in\mathscr{G}_n} B\right) \geq m\left(U \cap \bigcup_{n=0}^{\infty} S_n\right) + m\left(\bigcup_{n=1}^{\infty}\left(U_n \setminus \bigcup_{B\in\mathscr{G}_n} B\right)\right) \\
&\geq m\left(\left(U \cap \bigcup_{n=0}^{\infty} S_n\right) \cup \bigcup_{n=1}^{\infty}\left(U_n \setminus \bigcup_{B\in\mathscr{G}_n} B\right)\right) = m\left(\left(U \cap \bigcup_{n=0}^{\infty} S_n\right) \cup \left[\left(\bigcup_{n=1}^{\infty} U_n\right) \setminus \left(\bigcup_{B\in\mathscr{G}} B\right)\right]\right) \\
&= m\left(\left[\left(U \cap \bigcup_{n=0}^{\infty} S_n\right) \cup \bigcup_{n=1}^{\infty}(U \cap A_n)\right] \setminus \left(\bigcup_{B\in\mathscr{G}} B\right)\right) = m\left(\left[\left(U \cap \bigcup_{n=0}^{\infty} S_n\right) \cup \left(U \cap \bigcup_{n=1}^{\infty} A_n\right)\right] \setminus \left(\bigcup_{B\in\mathscr{G}} B\right)\right) \\
&= m\left(\left[U \cap \left(\bigcup_{n=0}^{\infty} S_n \cup \bigcup_{n=1}^{\infty} A_n\right)\right] \setminus \left(\bigcup_{B\in\mathscr{G}} B\right)\right) = m\left(U \cap \mathbb{R}^n \setminus \left(\bigcup_{B\in\mathscr{G}} B\right)\right) = m\left(U \setminus \left(\bigcup_{B\in\mathscr{G}} B\right)\right) \geq 0
\end{aligned}$$

where the first equality on the third line follows because $\left(\bigcup_{B\in\mathscr{G}} B\right) \subseteq \left(\bigcup_{n=1}^{\infty} U_n\right)$ and hence are disjoint from all the S_n. Thus, $\mathscr{G}$ is our desired countable collection of disjoint closed balls in U with diameter $\leq \delta$ s.t. $m(U \setminus \bigcup_{B\in\mathscr{G}} B) = 0$, and we are done.

524 Homework 8

Daniel Rui - 11/25/20

Problem 1 - 2.5.46

Let $X = Y = [0,1]$, $\mathscr{M} = \mathscr{N} = \mathscr{B}_{[0,1]}$, μ be Lebesgue measure and ν be counting measure, and define $D = \{(x,x) : x \in [0,1]\}$. We now calculate the three integrals (1) $\iint 1_D \, d\mu \, d\nu$, (2) $\iint 1_D \, d\nu \, d\mu$, and (3) $\int 1_D \, d[\mu \times \nu]$.

(3) is by definition $[\mu \times \nu](D)$, which is in turned defined using Theorem 1.14 as $\inf\{\sum_{k=1}^{\infty} \mu(A_k)\nu(B_k) : D \subseteq \bigcup_{k=1}^{\infty}(A_k \times B_k)\}$. Now suppose we are given any arbitrary covering $\{A_k \times B_k\}_{k=1}^{\infty}$ of D, where each $A_k \in \mathscr{M}, B_k \in \mathscr{N}$. Because for every $x \in [0,1]$, $(x,x) \in D$, so it must be that there must be some $k \in \mathbb{N}$ s.t. $(x,x) \in A_k \times B_k \iff x \in A_k$ and $x \in B_k \iff x \in A_k \cap B_k$. That is to say, $[0,1] \subseteq \bigcup_{k=1}^{\infty}(A_k \cap B_k)$.

But by monotonicity and subadditivity of Lebesgue measure m, it must be that $1 = m([0,1]) \leq \sum_{k=1}^{\infty} m(A_k \cap B_k)$. This tells us that there must be some $K \in \mathbb{N}$ s.t. $m(A_K \cap B_K) > 0$, and because $A_K \cap B_K \subseteq A_K, B_K$, we must also have that $m(A_K), m(B_K) \geq m(A_K \cap B_K) > 0$. In particular, this implies that B_K must be uncountable (because countable sets have 0 Lebesgue measure), and so the counting measure applied to B_K, i.e. $\nu(B_K)$, must be infinite. But then (using the fact that $\mu(A_K) = m(A_K) > 0$), $[\mu \times \nu](A_K \times B_K) = \mu(A_K)\nu(B_K) = \infty$. It goes without saying that $\sum_{k=1}^{\infty} \mu(A_k)\nu(B_k) = \infty$ too. Because the cover $\{A_k \times B_k\}_{k=1}^{\infty}$ was arbitrary, we see that $[\mu \times \nu](D) = \inf\{\sum_{k=1}^{\infty} \mu(A_k)\nu(B_k) : D \subseteq \bigcup_{k=1}^{\infty}(A_k \times B_k)\}$ must be ∞.

(1): we consider $\iint 1_D \, d\mu \, d\nu = \int \left(\int (1_D)^y(x) \, d\mu(x) \right) d\nu(y)$. But from the definition of 1_D, we see that $(1_D)^y(x) = 1_{\{y\}}(x)$. Then, $\int (1_D)^y(x) \, d\mu(x) = \mu(\{y\}) = m(\{y\}) = 0$ (because Lebesgue measure of a point is 0), and so $\iint 1_D \, d\mu \, d\nu = \int 0 \, d\nu(y) = 0$.

(2): we consider $\iint 1_D \, d\nu \, d\mu = \int \left(\int (1_D)_x(y) \, d\nu(y) \right) d\mu(x)$. But from the definition of 1_D, we see that $(1_D)_x(y) = 1_{\{x\}}(y)$. Then, $\int (1_D)_x(y) \, d\nu(y) = \nu(\{x\}) = 1$ (because counting measure of a point is 1), and so $\iint 1_D \, d\nu \, d\mu = \int 1 \, d\mu(x) = 1 \cdot \mu(X) = \mu([0,1]) = m([0,1]) = 1$.

Thus, in summary, we have that the three integrals all have different values: (1) $\iint 1_D \, d\mu \, d\nu = 0$, (2) $\iint 1_D \, d\nu \, d\mu = 1$, and (3) $\int 1_D \, d[\mu \times \nu] = \infty$.

Problem 2 - 2.5.48

Let $X = Y = \mathbb{N}$, $\mathscr{M} = \mathscr{N} = \mathscr{P}(\mathbb{N})$, and μ and ν both be counting measure. Define $f(m,n) = 1$ if $m = n$ and $f(m,n) = -1$ if $m = n+1$, and $f(m,n) = 0$ otherwise. We now examine three integrals (1) $\int |f| \, d[\mu \times \nu]$, (2) $\iint f \, d\mu \, d\nu$, and (3) $\iint f \, d\nu \, d\mu$.

(1) Because $|f| = 1_D$ where $D = \{(m,n) \in \mathbb{N} \times \mathbb{N} : m = n \text{ or } m = n+1\}$, we have by definition that $\int |f| \, d[\mu \times \nu] = [\mu \times \nu](D)$. By construction of $[\mu \times \nu]$ (see first page of Section 2.5 in Folland), we have that for rectangles $(m,n) = \{m\} \times \{n\}$, $[\mu \times \nu](\{m\} \times \{n\}) = \mu(\{m\})\nu(\{n\})$. Because both μ and ν are counting measure, we get that $[\mu \times \nu]((m,n)) = 1 \cdot 1 = 1$. Because $[\mu \times \nu]$ is a measure by construction, and because D can be written as the countable disjoint union $\big(\bigcup_{n\in\mathbb{N}}(n,n)\big) \cup \big(\bigcup_{n\in\mathbb{N}}(n+1,n)\big)$, countable additivity yields

$$[\mu \times \nu](D) = \Big(\sum_{n=1}^{\infty} [\mu \times \nu]((n,n))\Big) + \Big(\sum_{n=1}^{\infty} [\mu \times \nu]((n+1,n))\Big) = \Big(\sum_{n=1}^{\infty} 1\Big) + \Big(\sum_{n=1}^{\infty} 1\Big) = \infty$$

(2): we consider $\iint f \, d\mu \, d\nu = \int \big(\int (f)^n(m) \, d\mu(m)\big) \, d\nu(n)$. But from the definition of f, we see that $(f)^n(m)$ is $1_{\{n\}}(m) - 1_{\{n+1\}}(m)$ for all $m \in \mathbb{N}$. Then, for all $m \in \mathbb{N}$, $\int (f)^n(m) \, d\mu(m) = \mu(\{n\}) - \mu(\{n+1\}) = 1 - 1 = 0$ (where we used that for simple functions, we can interchange finite sums and integrals — Prop 2.13(b)). Thus, $\iint f \, d\mu \, d\nu = \int 0 \, d\nu(n) = 0$.

(3): we consider $\iint f \, d\nu \, d\mu = \int \big(\int (f)_m(n) \, d\nu(n)\big) \, d\mu(m)$. But from the definition of f, we see that $(f)_m(n)$ is $1_{\{m\}}(n)$ for $m = 1$, and $1_{\{m\}}(n) - 1_{\{m-1\}}(n)$ for $m \geq 2$. Then, for $m = 1$, $\int (f)_m(n) \, d\nu(n) = \nu(\{m\}) = 1$ (because counting measure of a point is 1), but for $m \geq 2$, $\int (f)_m(n) \, d\nu(n) = \nu(\{m\}) - \nu(\{m-1\}) = 1 - 1 = 0$ (where we used that for simple functions, we can interchange finite sums and integrals — Prop 2.13(b)). That is to say, $\int (f)_m(n) \, d\nu(n) = 1_{\{1\}}(m)$. Thus, $\iint f \, d\nu \, d\mu = \int 1_{\{1\}}(m) \, d\mu(m) = \mu(\{1\}) = 1$.

Thus, in summary, because (1) $\int |f| \, d[\mu \times \nu] = \infty$, we can not apply Fubini, and indeed the conclusions of Fubini do not hold here as (2) $\iint f \, d\mu \, d\nu = 0$ while (3) $\iint f \, d\nu \, d\mu = 1$.

Problem 3

Let f be a measurable finite-valued function on [0,1] and suppose that $h(x,y) = |f(x) - f(y)|$ is integrable on $[0,1] \times [0,1]$, i.e. $h \in L^1([0,1] \times [0,1])$. We want to show that f is integrable on $[0,1]$.

Since $h \in L^1([0,1] \times [0,1])$, we can use Fubini, which tells us that for almost all $y \in [0,1]$, $h^y(x) = h(x,y) = |f(x) - f(y)|$ is in $L^1([0,1])$. Let y_0 be one of these y, so that we have that $h^{y_0}(x) = h(x,y_0) = |f(x) - f(y_0)| \in L^1([0,1])$. But note that $|f(x)| = |f(x) - f(y_0) + f(y_0)| \leq |f(x) - f(y_0)| + |f(y_0)|$ (where $|f(y_0)|$ is a constant function and so in $L^+([0,1])$), so using that the integral is monotone for L^+ functions (end of Prop 2.13), and sum-integral interchange for L^+ functions (Theorem 2.15), we see that

$$\int_{[0,1]} |f(x)| \, dx \leq \int_{[0,1]} |f(x) - f(y_0)| + |f(y_0)| \, dx = \int_{[0,1]} |f(x) - f(y_0)| \, dx + \int_{[0,1]} |f(y_0)| \, dx.$$

But $|f(x) - f(y_0)| \in L^1([0,1]) \implies \int_{[0,1]} |f(x) - f(y_0)| \, dx$ is finite, and also $\int_{[0,1]} |f(y_0)| \, dx = m([0,1])|f(y_0)| = |f(y_0)|$ is finite (because we supposed that f is finite-valued), so we get that $\int_{[0,1]} |f(x)| \, dx$ is finite. That is, we've shown that f is integrable on $[0,1]$, as desired.

Problem 4

Let f be integrable on $\mathbb{R}^d$, and for each $\alpha > 0$ define $E_\alpha = \{x \in \mathbb{R}^d : |f(x)| > \alpha\}$. We want to show that $\int_{\mathbb{R}^d} |f(x)|\, dx = \int_0^\infty m(E_\alpha)\, d\alpha$.

Consider $h : [0,\infty) \times \mathbb{R}^d \to [0,1]$ defined by $h(\alpha, x) = 1_{E_\alpha}(x)$. Note that h is only 0 or 1, i.e. it is of the form $h = 1_E$ for some set $E \subseteq [0,\infty) \times \mathbb{R}^d$, where the set E is easily seen to be $E := \{(\alpha, x) \in [0,\infty) \times \mathbb{R}^d : |f(x)| > \alpha\}$. We can write E as $g^{-1}((0,\infty])$, where g is defined as $g(\alpha, x) = |f(x)| - \alpha$.

Because f is measurable (on $\mathbb{R}^d$), $|f|$ is measurable (on $\mathbb{R}^d$) too (it is the sum of f^+ and f^- which are measurable because f is measurable). Also, the function $\alpha \mapsto -\alpha$ (on $[0,\infty)$) is a continuous function and hence measurable. Thus, extending these measurable functions to measurable functions on $[0,\infty) \times \mathbb{R}^d$ using Corollary 3.7 on pg 85 of Stein and Shakarchi, their sum, i.e. $g(\alpha, x)$, must be measurable on $[0,\infty) \times \mathbb{R}^d$ as well. Thus, $E = g^{-1}((0,\infty])$ is measurable, and so h is a measurable function on $[0,\infty) \times \mathbb{R}^d$.

Thus we can apply Tonelli on $h(\alpha, x) = 1_E$ to get that

$$\int_{\mathbb{R}^d} \left(\int_0^\infty (1_E)^x \, d\alpha \right) dx = \int_{\mathbb{R}^d} \int_0^\infty 1_E \, d\alpha \, dx = \int_0^\infty \int_{\mathbb{R}^d} 1_E \, dx \, d\alpha = \int_0^\infty \left(\int_{\mathbb{R}^d} (1_E)_\alpha \, dx \right) d\alpha.$$

Finally observe that

$$(1_E)_\alpha = 1_{E_\alpha} \implies \int_{\mathbb{R}^d} (1_E)_\alpha \, dx = m(E_\alpha)$$

and

$$(1_E)^x = 1_{\{\alpha \in [0,\infty) : |f(x)| > \alpha\}} = 1_{[0, |f(x)|)} \implies \int_0^\infty (1_E)^x \, d\alpha = m([0, |f(x)|)) = |f(x)|$$

so that

$$\int_{\mathbb{R}^d} |f(x)|\, dx = \int_{\mathbb{R}^d} \left(\int_0^\infty (1_E)^x \, d\alpha \right) dx = \int_0^\infty \left(\int_{\mathbb{R}^d} (1_E)_\alpha \, dx \right) d\alpha = \int_0^\infty m(E_\alpha)\, d\alpha,$$

as desired.

Problem 5

Let f and g both be measurable functions on $\mathbb{R}^d$.

(a) We show that $f(x-y)g(y)$ is measurable on $\mathbb{R}^{2d}$: Proposition 3.9 of Stein and Shakarchi says that if f is measurable on $\mathbb{R}^d$, $f(x-y)$ is measurable on $\mathbb{R}^{2d}$. Corollary 3.7 of Stein and Shakarchi then says that $\tilde{g} : \mathbb{R}^{2d} \to \mathbb{R}$ defined as $\tilde{g}(x,y) = g(y)$ is measurable on $\mathbb{R}^{2d}$. We know that a product of measurable functions is measurable, so indeed $f(x-y)g(y)$ is measurable on $\mathbb{R}^{2d}$.

(b) If $f, g \in L^1(\mathbb{R}^d)$, then we want to show that $f(x-y)g(y)$ is integrable on $\mathbb{R}^{2d}$. We know from (a) that $f(x-y)g(y) = L^+(\mathbb{R}^{2d}) = L^+(\mathbb{R}^d \times \mathbb{R}^d)$. That means we can use Tonelli, which tells us

that

$$\int_{\mathbb{R}^{2d}} |f(x-y)g(y)|\, d(x,y) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(x-y)g(y)|\, dx\, dy.$$

For every fixed $y \in \mathbb{R}^d$, $\int_{\mathbb{R}^d} |f(x-y)g(y)|\, dx = \int_{\mathbb{R}^d} |f(x-y)| \cdot |g(y)|\, dx$. We can pull out $|g(y)|$ since it is a constant (we've fixed the y) so $\int_{\mathbb{R}^d} |f(x-y)g(y)|\, dx = |g(y)| \int_{\mathbb{R}^d} |f(x-y)|\, dx$. Furthermore, because $|f(x-y)|$ is just a shifted copy of $|f(x)|$ y units to the right, and Lebesgue measure is translation invariant (i.e. we are using Theorem 2.42 in Folland), we have that $\int_{\mathbb{R}^d} |f(x-y)|\, dx = \int_{\mathbb{R}^d} |f(x)|\, dx$. Putting these new insights into the big equation above, we see that

$$\int_{\mathbb{R}^{2d}} |f(x-y)g(y)|\, d(x,y) = \int_{\mathbb{R}^d} |g(y)| \int_{\mathbb{R}^d} |f(x)|\, dx\, dy$$

Noting that $\int_{\mathbb{R}^d} |f(x)|\, dx$ is just a number, we can pull it out of the integral to see that

$$\int_{\mathbb{R}^{2d}} |f(x-y)g(y)|\, d(x,y) = \int_{\mathbb{R}^d} |f(x)|\, dx \int_{\mathbb{R}^d} |g(y)|\, dy.$$

As we assumed that f and g are both integrable, the two quantities on the right are both finite, and so $\int_{\mathbb{R}^{2d}} |f(x-y)g(y)|\, d(x,y)$ is finite; hence $f(x-y)g(y)$ is integrable on $\mathbb{R}^{2d}$, as desired.

(c) Piggybacking off of (b) (so we also assume $f, g \in L^1(\mathbb{R}^d)$), we define

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x-y)g(y)\, dy.$$

We want to show that $(f*g)$ is well defined for almost every x, i.e. that $f(x-y)g(y)$ is integrable (w.r.t. y) on $\mathbb{R}^d$ for almost every x. Because in part (b), we showed that $f(x-y)g(y) \in L^1(\mathbb{R}^{2d})$, we can use Fubini, which tells us that for almost every $x \in \mathbb{R}^d$, the x-section of $f(x-y)g(y)$ is in $L^1(\mathbb{R}^d)$; i.e. for almost every $x \in \mathbb{R}^d$, $\int_{\mathbb{R}^d} f(x-y)g(y)\, dy$ exists and is finite, which is exactly what we wanted.

(d) We want to show that $(f*g)$ is integrable whenever f and g are integrable, and that $||f*g||_{L^1(\mathbb{R}^d)} \leq ||f||_{L^1(\mathbb{R}^d)}||g||_{L^1(\mathbb{R}^d)}$. That is, we assume f, g are integrable, and show that $||f * g||_{L^1(\mathbb{R}^d)} \leq ||f||_{L^1(\mathbb{R}^d)}||g||_{L^1(\mathbb{R}^d)}$, implying that $(f * g)$ is integrable too. As above in part (b), we know from (a) that $f(x-y)g(y) = L^+(\mathbb{R}^{2d}) = L^+(\mathbb{R}^d \times \mathbb{R}^d)$, so we can use Tonelli, which tells us that

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(x-y)g(y)|\, dy\, dx = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(x-y)g(y)|\, dx\, dy.$$

In the proof of part (b), we already established that

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(x-y)g(y)|\, dx\, dy = \int_{\mathbb{R}^d} |f(x)|\, dx \int_{\mathbb{R}^d} |g(y)|\, dy = ||f||_{L^1(\mathbb{R}^d)}||g||_{L^1(\mathbb{R}^d)}.$$

Now let us examine the LHS: we know that if f, g are integrable, part (c) yields that for almost every $x \in \mathbb{R}^d$, the x-section of $f(x-y)g(y) \in L^1(\mathbb{R}^d)$, so we can use Proposition 2.22 in Folland,

which tells us that for such x,

$$\left|\int_{\mathbb{R}^d} f(x-y)g(y)\,dy\right| \leq \int_{\mathbb{R}^d} |f(x-y)g(y)|\,dy$$

Note that the RHS is a measurable function of x (given by the application of Tonelli from paragraph 1 of this part, part (d)). We can also get that the LHS is a measurable function of x, because we can use Fubini on $f(x-y)g(y) \in L^1(\mathbb{R}^{2d})$ to get that $\int_{\mathbb{R}^d} f(x-y)g(y)\,dy$ is a measurable (and integrable) function of x — and the absolute value does not change it being measurable, of course. Although the above inequality holds only for almost all x, it doesn't impact anything, because integration does not care what happens on a null set (Prop 2.23(b) in Folland). Thus, we can use monotonicity of the integral for L^+ functions (end of Prop 2.13 in Folland) to get that

$$\int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d} f(x-y)g(y)\,dy\right| dx \leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(x-y)g(y)|\,dy\,dx.$$

By definition, $(f*g)(x) = \int_{\mathbb{R}^d} f(x-y)g(y)\,dy$, and so combining all the information together, we get that

$$\begin{aligned}
||f*g||_{L^1(\mathbb{R}^d)} &:= \int_{\mathbb{R}^d} |(f*g)(x)|\,dx = \int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d} f(x-y)g(y)\,dy\right| dx \\
&\leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(x-y)g(y)|\,dy\,dx \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(x-y)g(y)|\,dx\,dy = ||f||_{L^1(\mathbb{R}^d)}||g||_{L^1(\mathbb{R}^d)},
\end{aligned}$$

as desired.

524 Homework 7

Daniel Rui - 11/18/20

Problem 1

Suppose F is a closed set in $\mathbb{R}$ whose complement has finite measure, and let $\delta(x)$ denote the distance from x to F, i.e. $\delta(x) = d(x, F) = \inf\{|x - y| : y \in F\}$. Consider

$$I(x) = \int_{\mathbb{R}} \frac{\delta(y)}{|x - y|^2}\, dy.$$

(a) We will show that δ is continuous, by showing that it satisfies that Lipschitz condition $|\delta(x) - \delta(y)| \leq |x - y|$ (with this condition, continuity is trivial because we can just take $\delta = \epsilon$ in the $\epsilon - \delta$ definition of continuity — sorry about the double use of "δ"). We will show this, i.e. that for all $x, y \in \mathbb{R}$, $|d(x, F) - d(y, F)| \leq d(x, y)$, by showing that for all $x, y \in \mathbb{R}$, $d(x, F) - d(y, F) \leq d(x, y) \iff d(x, F) \leq d(x, y) + d(y, F)$. This shows the desired claim because one can swap the x and y to get both $d(x, F) - d(y, F) \leq d(x, y)$ and $d(y, F) - d(x, F) \leq d(y, x) = d(x, y)$, recovering the absolute value.

Fix arbitrary $x, y \in \mathbb{R}$. Note that for all $z \in F$, the triangle inequality gives that $d(x, z) \leq d(x, y) + d(y, z)$. Taking the infimum over all $z \in F$ of both sides, it must then be that $\inf_{z \in F}\{d(x, z)\} \leq \inf_{z \in F}\{d(x, y) + d(y, z)\}$. But $d(x, y)$ is just a constant here, so $\inf_{z \in F}\{d(x, y) + d(y, z)\} = d(x, y) + \inf_{z \in F}\{d(y, z)\}$ (see Lemma below). Thus, we have by definition of $d(x, F)$ and $d(y, F)$ that

$$d(x, F) = \inf_{z \in F}\{d(x, z)\} \leq \inf_{z \in F}\{d(x, y) + d(y, z)\} = d(x, y) + \inf_{z \in F}\{d(y, z)\} = d(x, y) + d(y, F).$$

And as stated previously, this proves that $|d(x, F) - d(y, F)| \leq d(x, y) \iff |\delta(x) - \delta(y)| \leq |x - y|$.

Lemma: in general for a set $S \subseteq \mathbb{R}$, $\inf\{c + x : x \in S\} = c + \inf S$. We prove this by showing that (1) all $c + x$ for $x \in S$ are $\geq c + \inf S$, and that (2) for every $\epsilon > 0$, there is some $x \in S$ s.t. $c + x < c + \inf S + \epsilon$. (1) is true because for all $x \in S$, $x \geq \inf S$, and so $c + x \geq c + \inf S$. (2) is true because by definition of $\inf S$, there is some $x \in S$ with $x < \inf S + \epsilon$, and so $c + x < c + \inf S + \epsilon$. BTW, $S = \{d(y, z) : z \in F\}$ in our case above.

(b) We want to show that $I(x) = \infty$ for each $x \notin F$. Fix arbitrary $x \notin F$. First, note that because $F^{\complement}$ is open, there is some $r > 0$ s.t. $B(x, r) \subseteq F^{\complement}$. Thus, $\delta(x)$ must be $\geq r > 0$. By continuity of δ, there must be some $\alpha > 0$ s.t. for all $y \in B(x, \alpha)$, $\delta(y) \geq \frac{r}{2} > 0$. Then, consider

$$J_n := \int_{[x+1/n, x+\alpha/2]} \frac{\delta(y)}{|x - y|^2}\, dy$$

for all n such that $n \geq N$ where $\frac{1}{N} < \alpha/2$. First, let us establish that $f(y) = \frac{\delta(y)}{|x-y|^2} \in L^+(\mathbb{R})$.

It is clearly ≥ 0, and we will show that it's measurable by defining a sequence of measurable functions f_n such that f is their sup (which gives that f is measurable by Proposition 2.7). Define f_n as follows:

$$f_n(y) = \delta(y) \cdot \begin{cases} \min\{n, |y-x|^{-2}\} & \text{for } y \neq x \\ n & \text{for } y = x \end{cases}.$$

Observe that each f_n is continuous: since $|y-x|^{-2}$ is continuous on $\mathbb{R} \setminus \{x\}$ (a quotient of continuous functions where the denominator is non-zero on the domain in question), for small enough neighborhoods around each $y \neq x$, f_n will be the minimum of 2 continuous functions and hence continuous at y; and for all y in a neighborhood of x small enough such that it's contained in $(x-1/\sqrt{n}, x+1/\sqrt{n})$, $f_n(y) = n\delta(y)$, and so f_n is continuous at x too. We know that continuous $\implies$ measurable (inverse image of $(-\infty, a)$, an open set, is open by continuity, and open sets are measurable), and it's clear that the f_n increase to f, and so we do in fact have that f is the sup of a sequence of measurable functions and hence measurable itself.

By the end of the proof of Proposition 2.13, the integral is monotone for L^+ functions, and so we have that

$$J_n := \int_{[x+1/n, x+\alpha/2]} \frac{\delta(y)}{|x-y|^2}\, dy \geq \int_{[x+1/n, x+\alpha/2]} \frac{r/2}{(y-x)^2}\, dy,$$

where the function $\frac{r/2}{(y-x)^2}$ is in L^+, because it is bounded ($\leq \frac{r/2}{((x+1/n)-x)^2} = n^2 r/2$), continuous and real valued on $[x+1/n, x+\alpha/2]$, and hence by Theorem 2.28, it is Lebesgue measurable (hence in L^+) and integrable on $[x+1/n, x+\alpha/2]$. Furthermore, Theorem 2.28 also tells us that $\int_{[x+1/n, x+\alpha/2]} \frac{r/2}{(y-x)^2}\, dy$ is equal to the Riemann integral $\frac{r}{2} \int_{x+1/n}^{x+\alpha/2} \frac{1}{(y-x)^2}\, dy$, which we know from elementary calculus equals $(-\frac{1}{y-x})|_{y=x+1/n}^{y=x+\alpha/2} = (n - \frac{2}{\alpha})$. Thus, putting these assertions together, we now have that $J_n \geq n - \frac{2}{\alpha}$ for all $n \geq N$. Finally, we have by the MCT that

$$\int_{(x, x+\alpha/2]} \frac{\delta(y)}{|x-y|^2}\, dy = \int \lim_{n\to\infty} \frac{\delta(y)}{|x-y|^2} \cdot 1_{[x+1/n, x+\alpha/2]} = \lim_{n\to\infty} J_n \geq \lim_{n\to\infty} (n - \alpha/2) = \infty.$$

Again using monotonicity of the integral for L^+ functions, we have that $I(x) \geq \int_{(x, x+\alpha/2]} \frac{\delta(y)}{|x-y|^2}\, dy$ and hence $I(x) = \infty$.

(c) We want to show that $I(x) < \infty$ for almost every $x \in F$. We show this by showing that $\int_F I(x)\, dx$ is finite, which implies that $I(x) \cdot 1_F$ is almost everywhere finite by Proposition 2.20, which we proved in Homework 5 (where $I(x) \cdot 1_F$ is a product of L^+ functions and hence in L^+, from Exercise 2.1.2 which we did in Homework 4). $I(x) \cdot 1_F$ being almost everywhere finite is of course equivalent to the desired claim that $I(x) < \infty$ for almost every $x \in F$.

We now want to show that $\int_F I(x)\, dx = \int_F \int_{\mathbb{R}} \frac{\delta(y)}{|x-y|^2}\, dy\, dx$ is finite. First, let us establish that $f(x,y) = \frac{\delta(y)}{|x-y|^2} \in L^+(\mathbb{R} \times \mathbb{R})$. It is clearly ≥ 0, and we will show that it's measurable by defining a sequence of measurable functions f_n such that f is their sup (which gives that f is measurable

by Proposition 2.7). Define f_n as follows:

$$f_n(x,y) = \delta(y) \cdot \begin{cases} \min\{n, |y-x|^{-2}\} & \text{for } y \neq x \\ n & \text{for } y = x \end{cases}.$$

Observe that each f_n is continuous: since $|y-x|^{-2}$ is continuous on $S := \{(x,y) \in \mathbb{R}\times\mathbb{R} : y \neq x\}$ (a quotient of continuous functions where the denominator is non-zero on the domain in question), for small enough neighborhoods around each $(x,y) \notin S$, f_n will be the minimum of 2 continuous functions and hence continuous at (x,y); and on a neighborhood of any (x,x) small enough such that it's contained in $\{(x,y) \in \mathbb{R}\times\mathbb{R} : |x-y| < 1/\sqrt{n}\}$, $f_n(x,y) = n\delta(y)$, and so f_n is continuous at all such (x,x) too. We know that continuous $\implies$ measurable (inverse image of $(-\infty, a)$, an open set, is open by continuity, and open sets are measurable), and it's clear that the f_n increase to f, and so we do in fact have that f is the sup of a sequence of measurable functions and hence measurable itself.

Now that we know $f \in L^+(\mathbb{R}\times\mathbb{R})$, we can use Tonelli's theorem, which gives us that

$$\int_F \int_{\mathbb{R}} \frac{\delta(y)}{|x-y|^2}\, dy\, dx = \int_{\mathbb{R}} \int_F \frac{\delta(y)}{|x-y|^2}\, dx\, dy.$$

So now let us examine $\int_F \frac{\delta(y)}{|x-y|^2}\, dx = \delta(y) \int_F \frac{1}{|x-y|^2}\, dx$. If $y \in F$, $\delta(y) = 0$, and so this integral will be 0. This means that we only need to concern ourselves with $y \notin F$, so let us pick y arbitrary in $F^{\complement}$. Now notice that $\Delta_y = (y-\delta(y), y+\delta(y))$ is contained completely within $F^{\complement}$; this is because if x is in this neighborhood around y, then $|x-y| < \delta(y)$, and because $\delta(y) = d(y,F)$, x can not possibly be in F. Thus, again by monotonicity of the integral for L^+ functions, we have that

$$\delta(y) \int_F \frac{1}{|x-y|^2}\, dx \leq \delta(y) \int_{\mathbb{R}\setminus\Delta_y} \frac{1}{|x-y|^2}\, dx.$$

If we define $g_n(x) = \frac{1}{|x-y|^2} \cdot 1_{(-n, y-\delta(y))\cup(y+\delta(y), n)}$ for $n \geq N$ s.t. $N \geq |y-\delta(y)|, |y+\delta(y)|$, we can use Theorem 2.28 to compute $\int_{\mathbb{R}} g_n(x)\, dx$ via our knowledge of elementary calculus:

$$\begin{aligned} \int_{\mathbb{R}} g_n(x)\, dx &= \left(-\frac{1}{x-y}\right)\Big|_{x=-n}^{x=y-\delta(y)} + \left(-\frac{1}{x-y}\right)\Big|_{x=y+\delta(y)}^{x=n} \\ &= \frac{1}{\delta(y)} + \frac{1}{-n-y} - \frac{1}{n-y} + \frac{1}{\delta(y)} = \frac{2}{\delta(y)} - \left(\frac{1}{n+y} + \frac{1}{n-y}\right). \end{aligned}$$

Because the g_n converge upward pointwise to $\frac{1}{|x-y|^2} \cdot 1_{\mathbb{R}\setminus\Delta_y}$, the MCT tells us that $\int_{\mathbb{R}\setminus\Delta_y} \frac{1}{|x-y|^2}\, dx = \lim_{n\to\infty} \left[\frac{2}{\delta(y)} - \left(\frac{1}{n+y} + \frac{1}{n-y}\right)\right] = \frac{2}{\delta(y)}$, and so putting together all our assertions, we have that

$$\delta(y) \int_F \frac{1}{|x-y|^2}\, dx \leq \delta(y) \int_{\mathbb{R}\setminus\Delta_y} \frac{1}{|x-y|^2}\, dx = \delta(y) \cdot \frac{2}{\delta(y)} = 2$$

and so

$$\int_F I(x)\,dx = \int_F \int_{\mathbb{R}} \frac{\delta(y)}{|x-y|^2}\,dy\,dx = \int_{\mathbb{R}} \int_F \frac{\delta(y)}{|x-y|^2}\,dx\,dy$$
$$= \int_{F^{\complement}} \int_F \frac{\delta(y)}{|x-y|^2}\,dx\,dy \leq \int_{F^{\complement}} 2\,dy = 2\mu(F^{\complement}).$$

Recall our assumption at the very very beginning that F was a set whose complement has finite measure, and so we get that $I(x) < \infty$, as desired.

Problem 2 - 2.4.34

Suppose $|f_n| \leq g \in L^1$ and $f_n \to f$ in measure. NOTE: in this proof, we will use that fact that if a sequence of functions converges in measure to f, then a subsequence of those functions converges a.e. to f. This almost comes from Theorem 2.30 as written in Folland; we just need the extra step of showing that converging in measure $\implies$ Cauchy in measure. We do this here: fix $\eta > 0$ arbitrary, and also fix arbitrary $\epsilon > 0$. Convergence in measure says that there is some $N \in \mathbb{N}$ s.t. $n \geq N \implies \mu(\{x \in X : |f_n(x) - f(x)| \geq \frac{\epsilon}{2}\}) < \frac{\eta}{2}$. Note also that $|f_n(x) - f_m(x)| \geq \epsilon \implies |f_n(x) - f(x)| + |f(x) - f_m(x)| \geq \epsilon \implies |f_n(x) - f(x)| \geq \frac{\epsilon}{2}$ or $|f_m(x) - f(x)| \geq \frac{\epsilon}{2}$, and so $\{x \in X : |f_n(x) - f_m(x)| \geq \epsilon\} \subseteq \{x \in X : |f_n(x) - f(x)| \geq \frac{\epsilon}{2}\} \cup \{x \in X : |f_m(x) - f(x)| \geq \frac{\epsilon}{2}\}$, and so if $n, m \geq N$, the union bound gives that

$$\mu(\{x \in X : |f_n(x) - f_m(x)| \geq \epsilon\}) \leq \mu(\{x \in X : |f_n(x) - f(x)| \geq \tfrac{\epsilon}{2}\})$$
$$+ \mu(\{x \in X : |f_m(x) - f(x)| \geq \tfrac{\epsilon}{2}\}) < \frac{\eta}{2} + \frac{\eta}{2} = \eta$$

By the definition of convergence, for every $\epsilon > 0$, $\mu(\{x \in X : |f_n(x) - f_m(x)| \geq \epsilon\}) \to 0$ as $m, n \to \infty$ and so we do in fact have that the f_n are Cauchy in measure.

(a) We want to show that $\int f\,d\mu = \lim_{n\to\infty} \int f_n\,d\mu$. Suppose not. Then, for some $\epsilon_0 > 0$, there are infinitely many indices, say $\{n_k\}_{k=1}^{\infty} \subseteq \mathbb{N}$ such that $|\int f\,d\mu - \int f_{n_k}\,d\mu| \geq \epsilon_0$. Call this claim (*). Since we know that $f_n \to f$ in measure, i.e. for every $\epsilon > 0$, $a_{\epsilon,n} := \mu(\{x \in X : |f_n(x) - f(x)| \geq \epsilon\}) \to 0$ as $n \to \infty$, it must be that $f_{n_k} \to f$ in measure as well, since the subsequence a_{ϵ,n_k} of a convergent sequence $a_{\epsilon,n}$ converging to 0 must converge to 0 as well.

Theorem 2.30 in Folland (see NOTE above) tells us that there is a further subsequence $\{n_{k_i}\}_{i=1}^{\infty}$ such that $f_{n_{k_i}} \to f$ μ-a.e., and the given $|f_n| \leq g \in L^1$ clearly implies that $|f_{n_{k_i}}| \leq g$ for all $i \in \mathbb{N}$. Thus, by the DCT, we have that $f \in L^1$, and $\int f\,d\mu = \lim_{i\to\infty} \int f_{n_{k_i}}\,d\mu$, i.e. in particular for the ϵ_0 above, there is some I_0 s.t. $i \geq I_0 \implies |\int f\,d\mu - \int f_{n_{k_i}}\,d\mu| < \epsilon_0$. But this contradicts (*), because each $n_{k_i} \in \{n_k\}_{k=1}^{\infty}$, and (*) says that for every $n_k \in \{n_k\}_{k=1}^{\infty}$, $|\int f\,d\mu - \int f_{n_k}\,d\mu| \geq \epsilon_0$. Thus, it must be that $\int f\,d\mu = \lim_{n\to\infty} \int f_n\,d\mu$.

(b) We want to show that $f_n \to f$ in L^1, i.e. that $\int |f_n - f|\,d\mu \to 0$ as $n \to \infty$. By checking that $|f_n - f| \leq g + |f| \in L^1$ and $|f_n - f| \to 0$ in measure, we can use part (a) to conclude that $0 = \int 0\,d\mu = \lim_{n\to\infty} \int |f_n - f|\,d\mu$.

First, $|f_n - f| \leq |f_n| + |f|$ by the triangle inequality, and we already know that $|f_n| \leq g \in L^1$ for all $n \in \mathbb{N}$. We know that $f \in L^1$ from the proof of part (a) (recall it is a consequence of the

DCT), and a sum of L^1 functions is in L^1, and so we do in fact have that $|f_n - f| \leq g + |f| \in L^1$. Lastly, $|f_n - f| \to 0$ in measure because $|f_n(x) - f(x)| \geq \epsilon \iff \big||f_n(x) - f(x)| - 0\big| \geq \epsilon$, and so for every $\epsilon > 0$, $\mu(\{x \in X : \big||f_n - f|(x) - 0\big| \geq \epsilon\}) = \mu(\{x \in X : |f_n(x) - f(x)| \geq \epsilon\}) \to 0$. Thus, the conditions are checked, and the conclusion $\int |f_n - f| \, d\mu \to 0$ holds.

Problem 3 - 2.4.42

Let μ be counting measure on $\mathbb{N}$. We want to show that $f_n \to f$ in measure iff $f_n \to f$ uniformly.

($\implies$) Recall that the definition of convergence in measure is that for every $\epsilon > 0$, $\mu(\{x \in X : |f_n(x) - f(x)| \geq \epsilon\}) \to 0$ as $n \to \infty$. In this particular case, we are assuming that for every $\epsilon > 0$, $a_n := \mu(\{x \in \mathbb{N} : |f_n(x) - f(x)| \geq \epsilon\}) \to 0$ as $n \to \infty$. First, we fix an $\epsilon \in (0, 1)$. Because μ outputs in $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, the definition of convergence to 0 implies that for some $N \in \mathbb{N}$, $n \geq N \implies |a_n| < \epsilon \implies a_n = 0$. That is, we've just proved that for all $n \geq N$, $a_n = \mu(\{x \in \mathbb{N} : |f_n(x) - f(x)| \geq \epsilon\}) = 0$. Because counting measure on $\mathbb{N}$ being 0 means that the set is the null set, this must be that for $n \geq N$, there are NO $x \in \mathbb{N}$ s.t. $|f_n(x) - f(x)| \geq \epsilon$. That is to say, $n \geq N \implies |f_n(x) - f(x)| < \epsilon$ for ALL $x \in \mathbb{N}$. Because ϵ was arbitrary in $(0, 1)$, this means that $f_n \to f$ uniformly.

($\impliedby$) $f_n \to f$ uniformly means that for every $\epsilon > 0$, there is $N \in \mathbb{N}$ s.t. for all $x \in \mathbb{N}$, $n \geq N \implies |f_n(x) - f(x)| < \epsilon$. Thus, the set $S_n = \{x \in \mathbb{N} : |f_n(x) - f(x)| \geq \epsilon\}$ is empty for all $n \geq N$, and so $\mu(S_n) = 0$ for all $n \geq N$. This of course means that $\mu(S_n) \to 0$ as $n \to \infty$, and because $\epsilon > 0$ was arbitrary, we have convergence in measure of $f_n \to f$.

524 Homework 6

Daniel Rui - 11/11/20

Problem 1 - 2.3.20

Suppose that we have $f_n, g_n, f, g \in L^1(\mu)$, and $f_n \to f$ and $g_n \to g$ (both μ-a.e.), and $|f_n| \leq g_n$ μ-a.e. and $\int g_n \, d\mu \to \int g \, d\mu$. We want to show that $\int f_n \, d\mu \to \int f \, d\mu$. First observe that $|f_n| \leq g_n$ a.e. $\iff -g_n \leq f_n \leq g_n \implies 0 \leq f_n + g_n$ a.e. and $g_n - f_n \geq 0$ a.e. (the sum/difference is defined a.e., because indeterminate forms $\pm\infty + \mp\infty$ only occur on a null set, since L^1 functions are only infinite on a null set). Thus, by Fatou (again, as Folland says, we can use Fatou because we can modify the functions in question on a null set without affecting the integrals, by Proposition 2.16), we have that

$$\int \liminf_{n\to\infty}(f_n + g_n)\, d\mu \leq \liminf_{n\to\infty}\left(\int (f_n + g_n)\, d\mu\right)$$

and the same for $(g_n - f_n)$. Because $f_n \to f$ and $g_n \to g$ a.e., $\lim_{n\to\infty}(f_n + g_n)\, d\mu = f + g$, and so the LHS of the above inequality becomes $\int f + g \, d\mu$. Now by Proposition 2.21, because $f_n, f, g_n, g \in L^1(\mu)$, $\int f + g \, d\mu = \int f \, d\mu + \int g \, d\mu$, and $\int f_n + g_n \, d\mu = \int f_n \, d\mu + \int g_n \, d\mu$. With all these reductions, the above inequality now says that

$$\int f \, d\mu + \int g \, d\mu \leq \liminf_{n\to\infty}\left(\int f_n \, d\mu + \int g_n \, d\mu\right)$$

Similarly, the inequality for $(g_n - f_n)$ yields

$$-\int f \, d\mu + \int g \, d\mu \leq \liminf_{n\to\infty}\left(-\int f_n \, d\mu + \int g_n \, d\mu\right).$$

Now, consider the following fact: if we have a two sequences of real numbers $\{a_n\}_1^\infty$ and $\{b_n\}_1^\infty$, where $b_n \to b$, then $\liminf(a_n + b_n) = (\liminf a_n) + b$. This is true, because for every $\epsilon > 0$, there is some N s.t. $n \geq N \implies b_n < b + \epsilon$, and so we get that $\liminf(a_n + b_n) \leq \liminf(a_n + b + \epsilon)$ (because for $n \geq N$, the LHS sequence is $<$ the RHS sequence). But $b + \epsilon$ is not dependent on n, so $\liminf(a_n + b_n) \leq (\liminf a_n) + b + \epsilon$, which implies $\liminf(a_n + b_n) \leq (\liminf a_n) + b$ because $\epsilon > 0$ was arbitrary. The $\geq$ direction is proven in the same way, but using that $n \geq N \implies b_n > b - \epsilon$ instead. Thus, the two big inequalities above become

$$\int f \, d\mu + \int g \, d\mu \leq \liminf_{n\to\infty}\left(\int f_n \, d\mu\right) + \int g \, d\mu \quad \text{and} \quad -\int f \, d\mu + \int g \, d\mu \leq \liminf_{n\to\infty}\left(-\int f_n \, d\mu\right) + \int g \, d\mu$$

Recall that $g \in L^1(\mu)$, so $\int g \, d\mu < \infty$, so we can subtract it from both sides. Also, notice that $\liminf(-a_n) = -\limsup(a_n)$, and so we have that

$$\int f \, d\mu \leq \liminf_{n\to\infty}\left(\int f_n \, d\mu\right) \quad \text{and} \quad -\int f \, d\mu \leq -\limsup_{n\to\infty}\left(\int f_n \, d\mu\right) \iff \int f \, d\mu \geq \limsup_{n\to\infty}\left(\int f_n \, d\mu\right),$$

or in other words $\limsup_{n\to\infty}\left(\int f_n \, d\mu\right) \leq \int f \, d\mu \leq \liminf_{n\to\infty}\left(\int f_n \, d\mu\right) \implies \int f \, d\mu = \lim_{n\to\infty} \int f_n \, d\mu$.

Problem 2 - 2.3.26

Suppose $f \in L^1(m)$, and define $F(x) = \int_{-\infty}^{x} f\,dm$ (where m is Lebesgue measure on $\mathbb{R}$). Note that F is everywhere finite (because $f \in L^1$), and also that we don't need to specify exactly what set on which we perform such an integral, because whether or not we include the endpoints of the interval does not matter to the value of the integral, since integrals are the same up to adjustments on a null set. We want to show that F is continuous on $\mathbb{R}$. Recall the proposition on page 4 of the lecture of 10/30: on a complete measure space $(X, \mathscr{M}, \mu)$, if $f \in L^1(\mu)$, then for all $\epsilon > 0$, there is $\delta > 0$ s.t. $m(E) < \delta \implies \int_E |f|\,d\mu < \epsilon$. Recall also from Proposition 2.22 that $|\int f| \leq \int |f|$, and so a weaker (but sufficient) version of the proposition from class is that $m(E) < \delta \implies |\int_E f\,d\mu| < \epsilon$.

Using this proposition, we see that if $|x - x_0| < \delta$, then letting E be the (let's say open) interval with endpoints x and x_0 (which we can also write as (y_1, y_2), where $y_1 = \min\{x, x_0\}$ and $y_2 = \max\{x, x_0\}$), then $\mu(E) < \delta$, and so $|\int_E f\,dm| < \epsilon$. But $\int_E f\,dm = \int_{y_1}^{y_2} f\,dm = \int_{-\infty}^{y_2} f\,dm - \int_{-\infty}^{y_1} f\,dm = |F(x) - F(x_0)|$ (where we could split up the integral like that because all the integrals are finite). In other words, we've shown that $|x - x_0| < \delta \implies |F(x) - F(x_0)| < \epsilon$, and because x_0 was arbitrary and δ did not depend on x_0, F is in fact uniformly continuous on $\mathbb{R}$ (and in particular just regular continuous on $\mathbb{R}$).

I will give a brief sketch of the proof of the proposition from class, just to be thorough: suppose first that f is bounded (i.e. $|f| \leq M$). Then, $\int_E |f|\,d\mu \leq \int_E M\,d\mu \leq M\mu(E) < \epsilon$ for $\mu(E) < \frac{\epsilon}{M}$. Now let $g = |f|$, so $g \geq 0$. Define $F_n = \{x \in X : f(x) < n\}$, so $F_n \subseteq F_{n+1}$. Then define $g_n = g \cdot 1_{F_n}$, so $0 \leq g_1 \leq g_2 \leq \ldots \leq g$, where $g_n \to g$ a.e. (because the set on which $g = \infty$ is a null set). Thus, the monotone convergence theorem tells us that $\int g_n\,d\mu \to \int g\,d\mu$. Now fixing any $\epsilon > 0$, we know there is $N \in \mathbb{N}$ s.t. $n \geq N \implies \int f\,d\mu - \int f_n\,d\mu < \frac{\epsilon}{2}$, and so $\int_E f\,dmu = \int_E (f - f_N)\,d\mu + \int E f_N\,d\mu < \frac{\epsilon}{2} + N\mu(E)$, which is $< \epsilon$ whenever $\mu(E) < \delta := \frac{\epsilon}{2N}$.

Problem 3 - 2.4.36

Suppose we have E_n (measurable) s.t. $\mu(E_n) < \infty$ for all $n \in \mathbb{N}$, and $1_{E_n} \to f$ in L^1 (where we require $\mu(E_n) < \infty$ to ensure 1_{E_n} is in L^1). We want to show that f is a.e. equal to the characteristic function of some measurable set. Corollary 2.32 says that because $1_{E_n} \to f$ in L^1, there is a subsequence $\{n_k\}_{k=1}^{\infty} \subseteq \{n\}_{n=1}^{\infty}$ s.t. $1_{E_{n_k}} \to f$ a.e. (as $k \to \infty$). In particular, we know that $1_{E_{n_k}}(x)$ converges for almost all $x \in X$. This means that at such an x, all $1_{E_{n_k}}(x)$ are 1 for $k \geq K$ (for some $K \in \mathbb{N}$), or all are 0 for $k \geq K$ (there are no other options; characteristic functions only take the values 1 and 0). That is to say, if x is a point at which $1_{E_{n_k}}(x)$ converges, then $x \in E := \bigcup_{K=1}^{\infty} \bigcap_{k=K}^{\infty} E_{n_k} \implies$ convergence to 1; otherwise, it converges to 0. Thus, $1_{E_{n_k}}$ converges a.e. to 1_E, where E is a countable union and intersection of measurable sets and hence measurable, and because $1_{E_{n_k}}$ also converges a.e. to f, f must be equal to 1_E almost everywhere.

Problem 4 - 2.4.44

Let $f : [a, b] \to \mathbb{C}$ be Lebesgue measurable, and $\epsilon > 0$. We want to show that there is a compact set $K \subseteq [a, b]$ s.t. $\mu([a, b] \setminus K) < \epsilon$, and $f|_K$ is continuous. First, let us refer to Theorem 2.26 which tells us that for any $\epsilon > 0$, there is a continuous function g (with compact support) s.t. $\int |f - g| \, d\mu < \epsilon$. Thus, we can construct a sequence of continuous functions f_n s.t. $\int |f - f_n| \, d\mu < \frac{1}{n}$. That is to say, we now have a sequence of continuous functions f_n that converge to f in L^1.

Corollary 2.32 now says that there must be a subsequence $\{n_k\}_{k=1}^{\infty} \subseteq \{n\}_{n=1}^{\infty}$ s.t. $f_{n_k} \to f$ a.e. (as $k \to \infty$). Let us now fix an arbitrary $\epsilon > 0$. Theorem 2.33 (Egoroff) tells us that because $\mu([a, b]) < \infty$ and $\{f_{n_k}\}_{k=1}^{\infty}$ is a sequence of measurable (complex) functions that converge a.e. to f, there exists some (measurable) $E \subseteq [a, b]$ s.t. $\mu([a, b] \setminus E) < \frac{\epsilon}{2}$ and $f_n \to f$ uniformly on E. The sets $[a, b]$ and E are both measurable (and $E \subseteq [a, b]$), so $\mu([a, b] \setminus E) = \mu([a, b]) - \mu(E)$.

To get to a compact set K, let us use Theorem 1.18, which tells us that there is some K compact s.t. $K \subseteq E$, and $\mu(K) > \mu(E) - \frac{\epsilon}{2} \iff \mu(E) - \mu(K) < \frac{\epsilon}{2}$. Adding this to the previous inequality ($\mu([a, b]) - \mu(E) < \frac{\epsilon}{2}$), we have that $\mu([a, b]) - \mu(K) = \mu([a, b] \setminus K) < \epsilon$. Moreover, because $K \subseteq E$, and $f_n \to f$ uniformly on E, $f_n \to f$ uniformly on K as well. Finally, because all the f_n are continuous, f (restricted to K) must be continuous as well, because a uniform limit of continuous functions is continuous. Thus, for an arbitrary $\epsilon > 0$, we have indeed found a compact $K \subseteq [a, b]$ s.t. $\mu([a, b] \setminus K) < \epsilon$, and $f|_K$ is continuous.

Problem 5

Let $L^1(\mathbb{R}^d)$ be the set of Lebesgue integrable functions on $\mathbb{R}^d$, m be the Lebesgue measure in $\mathbb{R}^d$, and $||f||_1 := \int |f| \, dm$. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in $L^1(\mathbb{R}^d)$. Suppose there exists $f \in L^1(\mathbb{R}^d)$ s.t. $f_n \to f$ m-a.e. and $||f_n||_1 \to ||f||_1$.

(a) We want to show that for all Lebesgue measurable sets in $E \subseteq \mathbb{R}^d$, $||f_n \cdot 1_E||_1 \to ||f \cdot 1_E||_1$. Fix a Lebesgue measurable set $E \subseteq \mathbb{R}^d$. We use Problem 1, which says that for "$f_n, g_n, f, g \in L^1(\mu)$, and $f_n \to f$ and $g_n \to g$ (both μ-a.e.), and $|f_n| \leq g_n$ μ-a.e. and $\int g_n \, d\mu \to \int g \, d\mu$, it is true that $\int f_n \, d\mu \to \int f \, d\mu$".

Here, our "f_n" will be $|f_n \cdot 1_E|$, our "f" will be $|f \cdot 1_E|$, our "g_n" will be $|f_n|$, and "g" will be $|f|$. Now the conditions: we are given that $f_n \to f$ a.e., implying that $|f_n| \to |f|$ a.e. and $|f_n \cdot 1_E| \to |f \cdot 1_E|$ a.e.; clearly $|f_n \cdot 1_E| \leq |f_n|$ a.e.; and we also are given $\int |f_n| \, dm \to \int |f| \, dm$. Moreover, it is clear that all the functions here are L^1. The conditions are satisfied, so we conclude that $\int |f_n \cdot 1_E| \, dm \to \int |f \cdot 1_E| \, dm$, or in other words $||f_n \cdot 1_E||_1 \to ||f \cdot 1_E||_1$.

(b) We want to show that $||f_n - f||_1 \to 0$. We again use Problem 1: our "f_n" will be $|f_n - f|$, our "f" will be 0, our "g_n" will be $|f_n| + |f|$, and "g" will be $2|f|$. Now the conditions: we are given that $f_n \to f$ a.e., implying that $|f_n - f| \to 0$ a.e. and $|f_n| + |f| \to 2|f|$ a.e.; the triangle inequality gives $|f_n - f| \leq |f_n| + |f|$ a.e.; and we also are given $\int |f_n| \, dm \to \int |f| \, dm$ (all finite) which implies that $\int |f_n| + |f| \, dm = \int |f_n \, dm + \int |f| \, dm \to \int 2|f| \, dm$ (all finite) — note the usage of Prop

2.21, i.e the linearity of the integral for L^1 functions. Moreover, it is clear that all the functions here are L^1. The conditions are satisfied, so we conclude that $\int |f_n - f|\, dm \to \int 0\, dm = 0$, i.e. $||f_n - f||_1 \to 0$.

(c) We want to show that for every continuous function with compact support $\varphi \in C_c(\mathbb{R}^d)$, $\int f_n\varphi\, dm \to \int f\varphi\, dm$. Let $\text{supp}(\varphi)$ denote the support of φ, We know that continuous functions on compact sets are bounded, and so we know that on $\text{supp}(\varphi)$, $|\varphi| \leq M$ for some $M \in \mathbb{N}$; but outside $\text{supp}(\varphi)$, $\varphi = 0$, so $|\varphi| \leq M$ everywhere.

We again use Problem 1: our "f_n" will be $f_n\varphi$, our "f" will be $f\varphi$, our "g_n" will be $|f_n|M$, and "g" will be $|f|M$. Now the conditions: we are given that $f_n \to f$ a.e., implying that $f_n\varphi \to f\varphi$ a.e. and $|f_n|M \to |f|M$ a.e.; from the above paragraph it is clear that $|f_n\varphi| \leq |f_n|M$ a.e.; and we also are given $\int |f_n|\, dm \to \int |f|\, dm$ (all finite) which implies that $\int M|f_n|\, dm = M\int |f_n|\, dm \to M\int |f|\, dm = \int M|f|\, dm$ (all finite). Moreover, it is clear that all the functions here are L^1. The conditions are satisfied, so we conclude that $\int f_n\varphi\, dm \to \int f\varphi\, dm$.

524 Homework 5

Daniel Rui - 11/4/20

Problem 1 - 2.2.12

We want to show that if $f \in L^+$ (the space of all measurable functions from $X \to [0, \infty]$), and $\int f \, d\mu < \infty$, then $\{x \in X : f(x) = \infty\}$ is a null set and $\{x \in X : f(x) > 0\}$ is σ-finite.

First, $E := f^{-1}(\{\infty\})$ is a null set: suppose not. Then $\mu(E) > 0$. Recall the definition of the integral of a function $f \in L^+$: $\int f \, d\mu = \sup\{\int \phi \, d\mu : 0 \leq \phi \leq f, \phi \text{ simple}\}$. Well, consider the simple function $\phi_n := n \cdot 1_E$. Obviously, $\int \phi_n \, d\mu = n\mu(E)$, and because for each $n \in \mathbb{N}$, ϕ_n is simple, non-negative, and $\leq f$ (for each $n \in \mathbb{N}$, $n \leq \infty$), and so $\int f \, d\mu \geq n\mu(E)$ for all $n \in \mathbb{N}$, which gets arbitrarily large because $\mu(E) > 0$. This forces $\int f \, d\mu$ to be ∞; contradiction.

For each $n \in \mathbb{N}$, consider the set $E_n := \{x \in X : f(x) > \frac{1}{n}\} = f^{-1}((\frac{1}{n}, \infty])$ (which is measurable because $(\frac{1}{n}, \infty] \in \mathscr{B}_{\overline{\mathbb{R}}}$). It must be that for all $n \in \mathbb{N}$, $\mu(E_n) < \infty$; if not, then for some $N \in \mathbb{N}$, $\mu(E_N) = \infty$, and the simple function $\varphi_N = \frac{1}{N} \cdot 1_{E_N}$ (which satisfies $0 \leq \varphi_N \leq f$) has integral $= \frac{1}{N}\mu(E_N) = \infty$, which would force $\int f \, d\mu = \infty$, a contradiction. To summarize, we have shown that each E_n is measurable and has finite measure. Furthermore, note that $E_0 := \{x \in X : f(x) > 0\} = \bigcup_{n=1}^{\infty} E_n$, and so indeed E_0 is σ-finite.

Problem 2 - 2.2.14

If $f \in L^+$, let $\lambda(E) = \int_E f \, d\mu$ for $E \in \mathscr{M}$. We want to show that λ is a measure on $\mathscr{M}$, and that for any $g \in L^+$, $\int g \, d\lambda = \int fg \, d\mu$.

First, λ is a measure on $\mathscr{M}$ — check the two properties (measure of null set is 0, and countably additive):

- $\lambda(\varnothing) = \int_\varnothing f \, d\mu = \int f \cdot 1_\varnothing \, d\mu = \int 0 \, d\mu = 0$
- Let $\{A_k\}_{k=1}^{\infty}$ be a sequence of disjoint measurable sets, and let A be their union. Then,

$$\int_A f \, d\mu = \int f \cdot 1_A \, d\mu = \int f \cdot \sum_{k=1}^{\infty} 1_{A_k} \, d\mu = \int \sum_{k=1}^{\infty} f \cdot 1_{A_k} \, d\mu = \sum_{k=1}^{\infty} \int f \cdot 1_{A_k} \, d\mu = \sum_{k=1}^{\infty} \int_{A_k} f \, d\mu$$

  where the second-to-last equality comes from Theorem 2.15, which applies because $\{f \cdot 1_{A_k}\}_{k=1}^{\infty}$ is a sequence of functions in L^+ (measurable because it's the product of measurable functions, and finite integral because $\int f \cdot 1_{A_k} \, d\mu \leq \int f \, d\mu < \infty$). Thus, we've shown that for disjoint measurable sets $\{A_k\}_{k=1}^{\infty}$, $\lambda(\bigcup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} \lambda(A_k)$.

For the second claim: (Step 1) suppose for now that g is an indicator function, say 1_E. Then, we have that

$$\int g \, d\lambda = \int 1_E \, d\lambda = \lambda(E) := \int_E f \, d\mu = \int f \cdot 1_E \, d\mu = \int fg \, d\mu.$$

Theorem 2.15 says that for L^+ functions, sums and integrals can be interchanged (this fact will be used in the second equality and second-to-last equality below). (Step 2) Now suppose that g is a simple function in L^+, i.e. $g = \sum_{i=1}^N c_i \cdot 1_{E_i}$ (where in particular all the $c_i \geq 0$; I write this explicitly so we can be sure that part (a) of Prop 2.13 can be used). Then we have that

$$\begin{aligned}\int g\, d\lambda &= \int \sum_{i=1}^N c_i \cdot 1_{E_i}\, d\lambda = \sum_{i=1}^N \int c_i \cdot 1_{E_i}\, d\lambda = \sum_{i=1}^N c_i \cdot \int 1_{E_i}\, d\lambda = \sum_{i=1}^N c_i \lambda(E_i) \\ &= \sum_{i=1}^N c_i \int_{E_i} f\, d\mu = \sum_{i=1}^N \int c_i \cdot f \cdot 1_{E_i}\, d\mu = \int f \cdot \sum_{i=1}^N c_i 1_{E_i}\, d\mu = \int fg\, d\mu\end{aligned}$$

Now suppose g is some arbitrary function in L^+. Let $\phi_1, \phi_2, \ldots$ be the sequence of simple functions from Theorem 2.10 (so $0 \leq \phi_1 \leq \phi_2 \leq \ldots \leq g$, and $\phi_n \to f$ pointwise — i.e the conditions for the MCT). The monotone convergence theorem then gives that

$$\int g\, d\lambda = \int \lim_{n\to\infty} \phi_n\, d\lambda = \lim_{n\to\infty} \int \phi_n\, d\lambda = \lim_{n\to\infty} \int f\phi_n\, d\mu = \int \lim_{n\to\infty} (f\phi_n)\, d\mu = \int fg\, d\mu$$

where the third equality was from $\int \phi_n\, d\lambda = \int f\phi_n\, d\mu$ (Step 2), and where we used the MCT on the second and second-to-last equality. Note that the MCT usage on the second-to-last equality was justified because $f \geq 0$ gives that $0 \leq \phi_1 \leq \phi_2 \leq \ldots \leq g \implies 0 \leq f\phi_1 \leq f\phi_2 \leq \ldots \leq fg$, and $\phi_n \to g \implies f\phi_n \to fg$. Thus, the second claim is proven.

Problem 3 - 2.2.15

Suppose we are given $\{f_n\} \subset L^+$ where the f_n decrease pointwise to f, and $\int f_1\, d\mu < \infty$. We want to show that $\int f\, d\mu = \lim_{n\to\infty} \int f_n\, d\mu$. Define $g_n = f_1 - f_n$. Because the f_n decrease pointwise, i.e. $f_1 \geq f_2 \geq \ldots \geq f$, we have that $f_1 - f_1 \leq f_1 - f_2 \leq \ldots \leq f_1 - f$, or written using the g_n: $0 = g_1 \leq g_2 \leq \ldots \leq f_1 - f$. Moreover, because the f_n decrease pointwise TO f, the g_n increase pointwise to $f_1 - f$. That is to say, the g_n are measurable (difference of measurable functions), non-negative and increasing pointwise to $f_1 - f$. Thus, we can use the MCT which tells us that

$$\int f_1 - f\, d\mu = \lim_{n\to\infty} \int f_1 - f_n\, d\mu$$

Theorem 2.15 tells us that because $f_1 - f$ and all the $f_1 - f_n$, and f and all the f_n are all in L^+, we have that

$$\int (f_1 - f) + f\, d\mu = \int f_1 - f\, d\mu + \int f\, d\mu \quad \text{and} \quad \int (f_1 - f_n) + f_n\, d\mu = \int f_1 - f_n\, d\mu + \int f_n\, d\mu$$

But because $\int f_1\, d\mu < \infty$, and f and all the f_n are $\leq f_1$ (because the f_n decrease pointwise to f), we have that $\int f\, d\mu$ and all the $\int f_n\, d\mu < \infty$, and so we can subtract them from both sides to get:

$$\int f_1\, d\mu - \int f\, d\mu = \int f_1 - f\, d\mu \quad \text{and} \quad \int f_1\, d\mu - \int f_n\, d\mu = \int f_1 - f_n\, d\mu$$

Thus the equality given by the MCT becomes $\int f_1\,d\mu - \int f\,d\mu = \lim_{n\to\infty}\big(\int f_1\,d\mu - \int f_n\,d\mu\big)$, and because $\int f_1\,d\mu$ does not depend on n, we can pull it out to get

$$\int f_1\,d\mu - \int f\,d\mu = \int f_1\,d\mu - \lim_{n\to\infty}\int f_n\,d\mu \implies -\int f\,d\mu = -\lim_{n\to\infty}\int f_n\,d\mu$$
$$\implies \int f\,d\mu = \lim_{n\to\infty}\int f_n\,d\mu$$

Problem 4 - 2.2.16

We want to show that if $f \in L^+$, and $\int f\,d\mu < \infty$, for every $\epsilon > 0$ there exists some $E \in \mathscr{M}$ s.t. $\mu(E) < \infty$ but $\int_E f\,d\mu > (\int f\,d\mu) - \epsilon$. Consider the set $E_n := \{x \in X : f(x) > \frac{1}{n}\} = f^{-1}((\frac{1}{n}, \infty])$. As we showed in paragraph 3 of Problem 1, the E_n are measurable, and the conditions that $f \in L^+$ and has finite integral implies that for all $n \in \mathbb{N}$, $\mu(E_n) < \infty$. Now consider $\phi_n := f \cdot 1_{E_n}$. Because the E_n are increasing, i.e. $E_1 \subseteq E_2 \subseteq \ldots$, the ϕ_n are increasing, and obviously they are non-negative (f is ≥ 0, and $1_{E_n} \geq 0$). Finally, the ϕ_n increase to f pointwise, because if $f(x) = 0$, then all the $\phi_n(x)$ will be 0 because x is not in any of the E_n; and if $f(x) > 0$, then x must be in all the E_n for n large enough, and so $\phi_n(x)$ will be exactly $f(x)$ for all n large enough. Thus, the we can apply the MCT, which gives us that

$$\int f\,d\mu = \lim_{n\to\infty}\int \phi_n\,d\mu$$

(where all the integrals are finite because we were given $\int f\,d\mu < \infty$). Thus by the definition of limit, for $N \in \mathbb{N}$ large enough,

$$\int f\,d\mu - \int_{E_N} f\,d\mu = \left|\int f\,d\mu - \int \phi_N\,d\mu\right| < \epsilon \implies \int_{E_N} f\,d\mu > \left(\int f\,d\mu\right) - \epsilon.$$

Problem 5 - 2.3.19

Suppose $\{f_n\} \subseteq L^1(\mu)$ (the set of equivalence classes of a.e.-defined integrable (= measurable and finite integral in absolute value) functions, where the equivalence relation is $f = g$ μ-a.e.), and $f_n \to f$ uniformly.

(a) Assume that $\mu(X) < \infty$. We want to show that then, $f \in L^1(\mu)$ and $\int f_n\,d\mu \to \int f\,d\mu$. By uniform convergence, we have that for any $\epsilon > 0$, there is some $N \in \mathbb{N}$ s.t. $n \geq N \implies |f_n - f| < \epsilon$ (μ-a.e.). That is to say, $n \geq N \implies |f_n| \leq |f_n - f| + |f| < \epsilon + |f|$ μ-a.e., and similarly $n \geq N \implies |f| \leq |f - f_n| + |f_n| < \epsilon + |f_n|$ μ-a.e.. This "latter remark", in conjunction with the fact that the f_n are in $L^1(\mu)$, the fact that the integral is monotone for L^+ functions (as given by the ending remarks of Prop 2.13), and the fact that for L^+ functions we can exchange sums and integrals, yields that

$$\int |f|\,d\mu \leq \int \epsilon + |f_n|\,d\mu = \int \epsilon\,d\mu + \int |f_n|\,d\mu = \epsilon\mu(X) + \int |f_n|\,d\mu < \infty$$

and so $f \in L^1(\mu)$ (measurable because it's the limit of a sequence of measurable functions). Now we can use the "former remark" from above to see that for $n \geq N$, $|f_n| < |f| + \epsilon$ (μ-a.e.), where

we have just shown that $|f| + \epsilon$ is in $L^1(\mu)$ (the $+\epsilon$ contributes $\epsilon\mu(X)$, a finite quantity because $\mu(X) < \infty$). Moreover, $f_n \to f$ uniformly implies that $f_n \to f$ pointwise (μ-a.e.). Thus, by the dominated convergence theorem (keeping in mind that "g" need only dominate the *tail* of the sequence of the "f_n's" — the first terms do not matter), we have that

$$\int f \, d\mu = \lim_{n\to\infty} \int f_n \, d\mu,$$

which is just another way of writing $\int f_n \, d\mu \to \int f \, d\mu$.

(b) If $\mu(X) = \infty$, the conclusions of (a) can fail (we'll be considering functions on $\mathbb{R}$; i.e. μ denotes Lebesgue measure on $\mathbb{R}$):

- f not in $L^1(\mu)$, but $\int f_n \, d\mu \to \int f \, d\mu$: consider $f_n(x) = \frac{1}{x} \cdot 1_{[1,n]}$. These converge uniformly to $f(x) = \frac{1}{x} \cdot 1_{[1,\infty)}$, because we have that for every $n \in \mathbb{N}$, $|f_n - f| \leq \frac{1}{n}$ for all $x \in \mathbb{R}$. Furthermore, for all $n \in \mathbb{N}$, $\int f_n \, d\mu = \ln(n) < \infty$. Also notice that the f_n are non-negative and increasing (and uniform convergence $\implies$ pointwise convergence); thus applying the MCT we see that $\int f_n \, d\mu \to \int f \, d\mu$, and so $\int f \, d\mu$ must be ∞, i.e. f is NOT in $L^1(\mu)$.
- $f \in L^1(\mu)$, but $\int f_n \, d\mu \not\to \int f \, d\mu$: consider $f_n(x) = \frac{1}{n} \cdot 1_{[0,n]}$. We have that these converge uniformly to $f = 0$, because for each $n \in \mathbb{N}$, $|f_n - f| \leq \frac{1}{n}$ μ-a.e.. Obviously, $f = 0$ is in $L^1(\mu)$, and has integral 0. Also, we have that for each $n \in \mathbb{N}$, $\int f_n \, d\mu = \frac{1}{n} \cdot \mu([0, n]) = \frac{1}{n} \cdot n = 1$. Thus, we have that $\int f_n \, d\mu \to 1$, NOT 0, which is the value of $\int f \, d\mu$.

524 Homework 4

Daniel Rui - 10/28/20

Problem 1 - 2.1.2

Let $(X, \mathscr{M})$ be a measurable space, and suppose that $f, g : X \to \overline{\mathbb{R}}$ are measurable. We actually do (b) BEFORE (a).

(b) Fix some $\alpha \in \overline{\mathbb{R}}$ and define $h(x) = \alpha$ if $f(x) = -g(x) = \pm\infty$ and $h(x) = f(x) + g(x)$ otherwise. We want to show that h is $(\mathscr{M}, \mathscr{B}_{\overline{\mathbb{R}}})$-measurable, where $\mathscr{B}_{\overline{\mathbb{R}}}$ is defined as the σ-algebra generated by $[-\infty, a)$ and $(a, \infty]$ (going over all $a \in \mathbb{R}$). First note that $(a, \infty] = \overline{\mathbb{R}} \setminus [-\infty, a] = \overline{\mathbb{R}} \setminus \bigcap_{r\in\mathbb{Q}, r>a}[-\infty, r)$, so in fact $\{[-\infty, a)\}_{a\in\mathbb{R}}$ alone generates $\mathscr{B}_{\overline{\mathbb{R}}}$. Thus, we only need to prove that $h^{-1}([-\infty, a)) \in \mathscr{M}$ for all $a \in \mathbb{R}$. First, if $a \le \alpha$, $h(x) < a \implies h(x) < \alpha \implies h(x) = f(x) + g(x)$, so

$$\begin{aligned} h^{-1}([-\infty, a)) &= \{x \in X : f(x) + g(x) < a\} \\ &= \bigcup_{r\in\mathbb{Q}} \{x \in X : f(x) < r, g(x) < a - r\} \\ &= \bigcup_{r\in\mathbb{Q}} \Big(\{x \in X : f(x) < r\} \cap \{x \in X : g(x) < a - r\}\Big) \\ &= \bigcup_{r\in\mathbb{Q}} \Big(f^{-1}([-\infty, r)) \cap g^{-1}([-\infty, a - r))\Big) \end{aligned}$$

which is in $\mathscr{M}$ because it is the countable union and intersection of sets in $\mathscr{M}$. Note that in the above proof we used the fact that $f(x)+g(x) < a \iff$ there is $r \in \mathbb{Q}$ s.t. $f(x) < r, g(x) < a-r$; ($\Longleftarrow$) is trivial, and for ($\Longrightarrow$), note that if one of $f(x), g(x)$ is $-\infty$, the claim is trivial, and if both are finite, then we have that $f(x) < a - g(x)$ and so by denseness there is some $r \in \mathbb{Q}$ s.t. $f(x) < r < a - g(x)$.

Now for $a > \alpha$. Let us denote $\Gamma = \{x \in X : f(x) = -g(x) = \pm\infty\}$. Then, we can write

$$h^{-1}([-\infty, a)) = \{x \in X : h(x) < a\} = \{x \in X : x \notin \Gamma, f(x) + g(x) < a\} \cup \Gamma.$$

But note that $\{-\infty\} = \bigcap_{n\in\mathbb{N}}[\infty, -n)$ (similarly, $\{\infty\} = \bigcap_{n\in\mathbb{Z}}(n, \infty]$) and so both preimages $f^{-1}(\{\pm\infty\}) \in \mathscr{M}$ (and same with g). Thus, from above we know that $\{x \in X : x \notin \Gamma, f(x) + g(x) < a\} = \bigcup_{r\in\mathbb{Q}}\{x \in X : f(x) < r, g(x) < a - r\} \in \mathscr{M}$, and we can write Γ as

$$\begin{aligned} \Gamma &= \{x \in X : f(x) = -g(x) = \infty\} \cup \{x \in X : f(x) = -g(x) = -\infty\} \\ &= \Big(f^{-1}(\{\infty\}) \cap g^{-1}(\{-\infty\})\Big) \cup \Big(f^{-1}(\{-\infty\}) \cap g^{-1}(\{\infty\})\Big) \end{aligned}$$

and so $\Gamma \in \mathscr{M}$ as well. In other words, we have that $h^{-1}([-\infty, a))$ is a union of sets in $\mathscr{M}$, and so it is in $\mathscr{M}$. Thus for all $a \in \mathbb{R}$, $h^{-1}([-\infty, a)) \in \mathscr{M}$, implying that h is $(\mathscr{M}, \mathscr{B}_{\overline{\mathbb{R}}})$-measurable.

(a) We want to show that fg is $(\mathscr{M}, \mathscr{B}_{\overline{\mathbb{R}}})$-measurable (taking $0 \cdot (\pm\infty) = 0$). As above, it suffices to show that $(fg)^{-1}([-\infty, a)) \in \mathscr{M}$ for all $a \in \mathbb{R}$. First recall that if f is measurable, then cf is too for any $c \in \mathbb{R}$ (the proof from page 7 of Lec 10/19 still holds here even though here $f : X \to \overline{\mathbb{R}}$ instead of $X \to \mathbb{R}$); we will only use the cases $c = -1, \frac{1}{4}$ in this proof. Let us now define $[f + g]$ for functions $X \to \overline{\mathbb{R}}$ to be 0 if $f(x) = -g(x) = \pm\infty$, and $f(x) + g(x)$ otherwise. From (b), we know that such a function is in fact measurable. Observe that if $a \leq 0$, we can write

$$\begin{aligned}
(fg)^{-1}([-\infty, a)) &= \{x \in X : f(x)g(x) < a\} \\
&= \{x \in X : f(x), g(x) \text{ finite}, \, f(x)g(x) < a\} \cup \{x \in X : f(x)g(x) = -\infty\} \\
&= \{x \in X : f(x), g(x) \text{ finite}, \, \tfrac{1}{4}([f(x) + g(x)]^2 - [f(x) - g(x)]^2) < a\} \\
&\quad \cup \{x \in X : f(x) = -\infty, g(x) > 0\} \cup \{x \in X : g(x) = -\infty, f(x) > 0\} \\
&\quad \cup \{x \in X : f(x) = \infty, g(x) < 0\} \cup \{x \in X : g(x) = \infty, f(x) < 0\} \\
&= \Big(\{x \in X : \tfrac{1}{4}([f(x) + g(x)]^2 - [f(x) - g(x)]^2) < a\} \cap f^{-1}((-\infty, \infty)) \cap g^{-1}((-\infty, \infty))\Big) \\
&\quad \cup \Big(f^{-1}(\{-\infty\}) \cap g^{-1}((0, \infty])\Big) \cup \Big(g^{-1}(\{-\infty\}) \cap f^{-1}((0, \infty])\Big) \\
&\quad \cup \Big(f^{-1}(\{\infty\}) \cap g^{-1}([-\infty, 0))\Big) \cup \Big(g^{-1}(\{\infty\}) \cap f^{-1}([-\infty, 0))\Big)
\end{aligned}$$

$\frac{1}{4}([f(x) + g(x)]^2 - [f(x) - g(x)]^2)$ is a sum of measurable functions (using the convention defined above), and so as stated before it is measurable; we know that $f^{-1}(\{\pm\infty\})$ are measurable (same with g), and so $f^{-1}((-\infty, \infty))$ is too (same with g); thus, because the above expression is just a union and intersection of sets from $\mathscr{M}$, $(fg)^{-1}([-\infty, a)) \in \mathscr{M}$ as well. Lastly, the $a > 0$ case is not much different — the convention $0 \cdot \pm\infty = 0$ means we just need to union in the sets $\{x \in X : f(x) = 0\}$ and $\{x \in X : g(x) = 0\}$ (which are measurable because $f^{-1}(\{0\}) = X \setminus (f^{-1}([-\infty, 0)) \cup f^{-1}((0, \infty]))$, and same with g):

$$\begin{aligned}
(fg)^{-1}([-\infty, a)) &= \{x \in X : f(x), g(x) \text{ finite}, \, \tfrac{1}{4}([f(x) + g(x)]^2 - [f(x) - g(x)]^2) < a\} \\
&\quad \cup \{x \in X : f(x) = -\infty, g(x) > 0\} \cup \{x \in X : g(x) = -\infty, f(x) > 0\} \\
&\quad \cup \{x \in X : f(x) = \infty, g(x) < 0\} \cup \{x \in X : g(x) = \infty, f(x) < 0\} \\
&\quad \cup \{x \in X : f(x) = 0\} \cup \{x \in X : g(x) = 0\},
\end{aligned}$$

and so in all cases $a \in \mathbb{R}$, $(fg)^{-1}([-\infty, a)) \in \mathscr{M}$, and so fg is $(\mathscr{M}, \mathscr{B}_{\overline{\mathbb{R}}})$-measurable.

Problem 2 - 2.1.8

We want to show that if $f : \mathbb{R} \to \mathbb{R}$ is monotone, then f is Borel measurable. We only need to tackle the case where f is non-decreasing, because if it were non-increasing instead, we could just take $-f$ (recall that scalar multiplication preserves measurability). Because $\{(-\infty, a]\}_{a \in \mathbb{R}}$ generates $\mathscr{B}_{\mathbb{R}}$, we just need to check that $S_a := f^{-1}((-\infty, a]) = \{x \in \mathbb{R} : f(x) \leq a\} \in \mathscr{B}_{\mathbb{R}}$ for all $a \in \mathbb{R}$. First denote $s_a := \sup(S_a)$; then I claim that if $x < s_a$, then $x \in S_a$, and if $x > s_a$, then $x \notin S_a$: if $x < s_a$, then by the definition of supremum there must be some x_0 s.t. $x < x_0 \leq s_a$ and $x_0 \in S_a \implies f(x_0) \leq a$. Then by non-decreasing, $f(x) \leq f(x_0) \leq a$, and so $x \in S_a$. If $x > s_a$, then obviously $x \notin S_a$, because

everything in a set is $\leq$ the supremum of that set.

This means that S_a must be either $(-\infty, s_a)$, or $(-\infty, s_a]$; but both sets are in $\mathscr{B}_{\mathbb{R}}$, so either way, $f^{-1}((-\infty, a]) \in \mathscr{B}_{\mathbb{R}}$, and so f is Borel measurable.

Problem 3 - 2.1.9

(a) From the last 2 pages of Section 1.5 of Folland, we know that the Cantor function $f : [0,1] \to [0,1]$ is continuous, surjective, and (weakly) increasing (surjective and increasing force $f(0) = 0$ and $f(1) = 1$). Now define $g : [0,1] \to [0,2]$ as $g(x) = f(x) + x$.

Because f and x are both continuous on $[0,1]$, and sums of continuous functions are continuous, g must also be continuous on $[0,1]$. Then we have $g(0) = f(0) + 0 = 0$ and $g(1) = f(1) + 1 = 1 + 1 = 2$, and the intermediate value theorem gives us that g must attain every value $\in [0,2]$; i.e. it is surjective. Finally, because f is weakly increasing and x is strictly increasing, their sum g must also be strictly increasing ($x < y \implies f(x) \leq f(y)$, and so $g(x) = f(x) + x \leq f(y) + x < f(y) + y = g(y)$), and so g must also be injective. Thus, g is a bijection from $[0,1]$ to $[0,2]$.

Moreover, $h = g^{-1} : [0,2] \to [0,1]$ (which is a function because $g : [0,1] \to [0,2]$ is bijective) is continuous. First we prove that h is strictly increasing as well: suppose not; then there is $y_1 < y_2$, $y_1, y_2 \in [0,2]$ s.t. $h(y_1) \geq h(y_2)$. But because $g : [0,1] \to [0,2]$ is strictly increasing, and $h(y_1), h(y_2) \in [0,1]$, we have that $g(h(y_1)) \geq g(h(y_2))$. But g and h are inverses, and so this statement is actually saying that $y_1 \geq y_2$, which is a contradiction. Now to continuity:

Let $y_0 \in (0,2)$. We want to prove that for all $\epsilon > 0$, there is $\delta > 0$ s.t. $|y - y_0| < \delta \implies |h(y) - h(y_0)| < \epsilon \iff h(y_0) - \epsilon < h(y) < h(y_0) + \epsilon$. Fix any $\epsilon > 0$. We know because g is strictly increasing that:

$$g(h(y_0) - \epsilon) < g(h(y_0)) = y_0 < g(h(y_0) + \epsilon).$$

Because these inequalities are strict, we know we can choose $\delta > 0$ small enough such that

$$g(h(y_0) - \epsilon) < y_0 - \delta < y_0 < y_0 + \delta < g(h(y_0) + \epsilon)$$

(for example we could take $\delta = \frac{1}{2}\min\{y_0 - g(h(y_0) - \epsilon), g(h(y_0) + \epsilon) - y_0\}$). Then because h is strictly increasing, we have that

$$y_0 - \delta < y < y_0 + \delta \implies h(g(h(y_0) - \epsilon)) < h(y_0 - \delta) < h(y) < h(y_0 + \delta) < h(g(h(y_0) + \epsilon)).$$

This expression on the right can be simplified to just $h(y_0) - \epsilon < h(y) < h(y_0) + \epsilon$ (because g

and h are inverses). Thus, we have proved that for any $\epsilon > 0$, we can find $\delta > 0$ s.t.

$$y_0 - \delta < y < y_0 + \delta \implies h(y_0) - \epsilon < h(y) < h(y_0) + \epsilon$$

and so h is continuous at all $y_0 \in (0, 2)$. As for the endpoints, they follow in a similar manner: fix an $\epsilon > 0$; then we know that $h(0) < h(0) + \epsilon$ and $h(2) > h(2) - \epsilon$, and so because g is strictly increasing we have that $0 < g(h(0) + \epsilon)$ and $2 > g(h(2) - \epsilon)$, and so we can choose $\delta > 0$ s.t. $0 < 0 + \delta < g(h(0) + \epsilon)$ and $2 > 2 - \delta > g(h(2) - \epsilon)$. Then because h is strictly increasing, we have that

$$0 \leq y < \delta < g(h(0) + \epsilon) \implies h(0) \leq h(y) < h(0) + \epsilon$$

and

$$2 \geq y > 2 - \delta > g(h(2) - \epsilon) \implies h(2) \geq h(y) > h(2) + \epsilon.$$

Rephrased, these assertions become: for $y \in [0, 2]$, $|y - 0| < \delta \implies |h(y) - h(0)| < \epsilon$ and for $y \in [0, 2]$, $|2 - y| < \delta \implies |h(2) - h(y)| < \epsilon$, and because $\epsilon > 0$ above was arbitrary, we have verified that h is continuous at 0 and 2.

(b) Let C be the Cantor set. Because g is bijective (i.e. injective and surjective), $g(C)$ and $g([0, 1] \setminus C)$ are disjoint (from injectivity) and union to the codomain $[0, 2]$ (from surjectvity). From the construction of the Cantor set, we know that $[0, 1] \setminus C$ is just the countable union of disjoint open intervals. Again referring to the last 2 pages of Section 1.5 of Folland, the Cantor function f is constant on each of those open intervals; thus if (a, b) is one of the open intervals in $[0, 1] \setminus C$, and c is the constant value f takes on (a, b), then $g((a, b)) = \{f(x) + x : x \in (a, b)\} = \{c + x : x \in (a, b)\} = (c + a, c + b)$, i.e. a shifted copy of (a, b). Furthermore note that if we have two disjoint open intervals from $[0, 1] \setminus C$, then by injectivity of g, the images of those intervals under g will also be disjoint.

Thus writing $[0, 1] \setminus C$ as $\bigcup_{i=1}^{\infty}(a_i, b_i)$ where all the (a_i, b_i) are disjoint (open intervals $\subseteq [0, 1]$), we see that $g([0, 1] \setminus C) = \bigcup_{i=1}^{\infty} g((a_i, b_i)) =: \bigcup_{i=1}^{\infty}(a'_i, b'_i)$, where all the (a'_i, b'_i) are disjoint (open intervals $\subseteq [0, 2]$), and where each (a'_i, b'_i) is just a shifted copy of (a_i, b_i) (and so in particular, $m((a_i, b_i)) = m((a'_i, b'_i))$ because Lebesgue measure m is translation invariant). Because open sets are measurable, and all the intervals are disjoint, we see that

$$m([0,1]\setminus C) = m\left(\bigcup_{i=1}^{\infty}(a_i, b_i)\right) = \sum_{i=1}^{\infty} m((a_i, b_i)) = \sum_{i=1}^{\infty} m((a'_i, b'_i)) = m\left(\bigcup_{i=1}^{\infty}(a'_i, b'_i)\right) = m(g([0,1]\setminus C))$$

But recall from the last 2 pages of Section 1.5 of Folland that C is measurable and has measure 0, and so $[0, 1] \setminus C$ has measure 1, and so $m(g([0, 1] \setminus C)) = 1$.

Going back to the very first sentence, we know that $g(C)$ and $g([0, 1] \setminus C)$ are disjoint and union to $[0, 2]$; now we know that $g([0, 1] \setminus C)$ is measurable with measure 1, and so $g(C)$ must also be measurable with measure 1 (because measure of disjoint measurable sets is additive, and $[0, 2]$ is measurable with measure 2). So we conclude that $m(g(C)) = 1$.

(c) As Folland gives us, $g(C)$ contains a Lebesgue non-measurable set A. Let $B = g^{-1}(A)$. Then

I claim that B is Lebesgue measurable but not Borel. Note that because $A \subseteq g(C)$, $B = g^{-1}(A) \subseteq g^{-1}(g(C))$ because preimages are monotone. Because g is bijective, $g^{-1}(g(C)) = C$, so we have that $B \subseteq C$, where again C is the Cantor set. The Cantor set has measure 0 and is Borel measurable, and so B is a subset of a Borel measurable null set, and so it is Lebesgue measurable.

However, B can not be Borel measurable, because if it were, then we would have that $A = g(B) = h^{-1}(B)$ is as well: we know that h is continuous, and continuous functions are measurable, implying that the preimage A of the Borel measurable set B is ALSO Borel measurable. This contradicts that A is Lebesgue non-measurable, and so B is NOT Borel measurable.

(d) We are asked to show that there exists a Lebesgue measurable function F and continuous function G on $\mathbb{R}$ s.t. $F \circ G$ is not Lebesgue measurable. Consider the Lebesgue measurable function 1_B (the characteristic function of the set B, which is obviously Lebesgue measurable because B is Lebesgue measurable, and the only possible preimages are $\varnothing, B, \mathbb{R} \setminus B, \mathbb{R}$), and the continuous function h (we know it's continuous from (a)). Then, using the Borel set $\{1\}$ (closed sets are Borel), we see that

$$(1_B \circ h)^{-1}(\{1\}) = h^{-1}(1_B^{-1}(\{1\})) = h^{-1}(B) = A,$$

which recall is a Lebesgue non-measurable set, and so $(1_B \circ h)$ is NOT Lebesgue measurable.

Problem 4 - 2.1.10

Consider the measure space $(X, \mathscr{M}, \mu)$.

(a) We want to show that μ is complete if and only if for f measurable and $g = f$ μ-a.e., g is also measurable.

($\implies$) We are given f measurable and g s.t. $f = g$ on $X \setminus E$ where $E \in \mathscr{M}$, and $\mu(E) = 0$. That means that $g - f = 0$ on $X \setminus E$, and so the preimage $\{x \in X : (g-f)(x) < a\}$ is either some set $F \subseteq E$ if $a \leq 0$, or some set $(X \setminus E) \cup F$ for some set $F \subseteq E$ if $a > 0$. But by completeness of μ, these $F \in \mathscr{M}$ because they are subsets of the μ-null set E. Recall that $E \in \mathscr{M} \implies X \setminus E \in \mathscr{M}$, so the union $(X \setminus E) \cup F$ would also be in $\mathscr{M}$, so in either case, the preimage in question is a measurable set. Because $\{(-\infty, a)\}_{a \in \mathbb{R}}$ generate the Borel σ-algebra, we have shown that $(g - f)$ is measurable.

Recall that a sum of measurable functions is measurable, and so $f + (g - f) = g$ is measurable, as desired.

($\impliedby$) To show that μ is complete, we need to show that if $F \subseteq E$ for some μ-null set E (i.e. $\mu(E) = 0$), then $F \in \mathscr{M}$. Let $f = 0$, and let $g = 1_F$ (the characteristic function on F) for some F s.t. $F \subseteq E$ where E is a μ-null set. Then, $f = g$ μ-a.e. because $f = g$ on $X \setminus E$ where E is a μ-null set. Because $f = 0$ is obviously measurable, the assumption then gives that g is measurable, and so $F = g^{-1}(\{1\}) \in \mathscr{M}$.

(b) We want to show that μ is complete if and only if for f_n measurable (for all $n \in \mathbb{N}$) and $f_n \to f$ μ-a.e., f is also measurable.

($\implies$) From page 9 of Lec 10/19, we have that the $\limsup_{n\to\infty} f_n$ of measurable functions f_n is measurable. $f_n \to f$ μ-a.e. means that for $x \in X \setminus E$ for E s.t. $\mu(E) = 0$, $f(x) = \lim_{n\to\infty} f_n(x) = \limsup_{n\to\infty} f_n(x)$; in other words $f = \limsup_{n\to\infty} f_n$ μ-a.e. where $\limsup_{n\to\infty} f_n$ is measurable. Thus by (a), f must be measurable too.

($\impliedby$) If we let $f_1, f_2, \ldots$ all be some measurable function f, and we let g be any measurable function s.t. $f = g$ μ-a.e., then obviously $f_n \to g$ μ-a.e., and so by the assumption, g is measurable. In other words, the condition we are assuming here implies the RHS condition from (a), which we previously showed implied that μ is complete.

Problem 5

Let $(X, \mathscr{M})$ be a measurable space, and let $f : X \to \mathbb{R}^n, f = (f_1, \ldots, f_n)$. We want to show that f is $(\mathscr{M}, \mathscr{B}_{\mathbb{R}^n})$-measurable iff each $f_i : X \to \mathbb{R}$ is $(\mathscr{M}, \mathscr{B}_{\mathbb{R}})$-measurable.

($\implies$) If f is $(\mathscr{M}, \mathscr{B}_{\mathbb{R}^n})$-measurable, that means that for every $B \in \mathscr{B}_{\mathbb{R}^n}$ (the σ-algebra generated by the open sets of $\mathbb{R}^n$), $f^{-1}(B) \in \mathscr{M}$. Now pick any one of the f_i and let U be some arbitrary open set in $\mathbb{R}$; I claim that $f_i^{-1}(U) = f^{-1}(\mathbb{R} \times \ldots \times \mathbb{R} \times U \times \mathbb{R} \times \ldots \times \mathbb{R})$, where the U is in the ith spot. This is because the RHS $f^{-1}(\mathbb{R} \times \ldots \times \mathbb{R} \times U \times \mathbb{R} \times \ldots \times \mathbb{R})$ is equal to

$$= \{x \in X : f_1(x) \in \mathbb{R}, \ldots, f_i(x) \in U, \ldots, f_n(x) \in \mathbb{R}\} = \{x \in X : f_i(x) \in U\} = f_i^{-1}(U),$$

where we could forget about the conditions on the f_j, $j \neq i$ because $f_j(x)$ is always in $\mathbb{R}$ no matter the x. Furthermore, $\mathbb{R} \times \ldots \times \mathbb{R} \times U \times \mathbb{R} \times \ldots \times \mathbb{R}$ is an open subset of $\mathbb{R}^n$, and so by $(\mathscr{M}, \mathscr{B}_{\mathbb{R}^n})$-measurability, $f_i^{-1}(U)$ is in $\mathscr{M}$. Because U was an arbitrary open subset of $\mathbb{R}$, and open subsets generate $\mathscr{B}_{\mathbb{R}}$, we have by a proposition from Lec 10/19 that f_i is $(\mathscr{M}, \mathscr{B}_{\mathbb{R}})$-measurable. Finally, because f_i was arbitrary, we have that all the $f_1, \ldots, f_n$ are $(\mathscr{M}, \mathscr{B}_{\mathbb{R}})$-measurable.

($\impliedby$) First, from Theorem 1.4 from Stein & Shakarchi, every open subset of $\mathbb{R}^d$ ($d \geq 1$) can be written as a countable union of closed cubes. In other words, because $\mathscr{B}_{\mathbb{R}^n}$ is generated by the open subsets of $\mathbb{R}^n$, and the all such open subsets are countable unions of closed cubes, $\mathscr{B}_{\mathbb{R}^n}$ is in fact generated by closed cubes. Thus, we only need to prove that f^{-1} of a closed cube is an element of $\mathscr{M}$:

$$\begin{aligned} f^{-1}([a_1, b_1], \ldots, [a_n, b_n]) &= \{x \in X : f_1(x) \in [a_1, b_1], \ldots, f_n(x) \in [a_1, b_1]\} \\ &= \{x \in X : f_1(x) \in [a_1, b_1]\} \cap \ldots \cap \{x \in X : f_n(x) \in [a_n, b_n]\} \\ &= f_1^{-1}([a_1, b_1]) \cap \ldots \cap f_n^{-1}([a_n, b_n]) \end{aligned}$$

which is an element of $\mathscr{M}$ because it is the intersection of elements of $\mathscr{M}$ (each $f_i^{-1}([a_i, b_i]) \in \mathscr{M}$ because each f_i is $(\mathscr{M}, \mathscr{B}_{\mathbb{R}})$-measurable), and so we are done.

524 Homework 3

Daniel Rui - 10/21/20

Problem 1 - 1.3.9

Let $(X, \mathscr{M}, \mu)$ be a measure space (which reminder means that $\mathscr{M}$ is a σ-algebra of subsets of X, and $\mu : \mathscr{M} \to [0, \infty]$ is a measure), and consider arbitrary $E, F \in \mathscr{M}$. Because $\mathscr{M}$ is a σ-algebra, $E \setminus F, F \setminus E$, and $F \cap E$ are all in $\mathscr{M}$, and moreover, they are disjoint. That means that we can use the countable disjoint additivity property of measures to get that one hand:

$$\begin{aligned}\mu(E \setminus F) + \mu(F \cap E) + \mu(F \setminus E) + \mu(F \cap E) &= \mu((E \setminus F) \cup (F \cap E)) + \mu((F \setminus E) \cup (F \cap E)) \\ &= \mu(E) + \mu(F);\end{aligned}$$

while on the other hand:

$$\begin{aligned}\mu(E \setminus F) + \mu(F \cap E) + \mu(F \setminus E) + \mu(F \cap E) &= \mu((E \setminus F) \cup (F \cap E) \cup (F \setminus E)) + \mu(F \cap E) \\ &= \mu(E \cup F) + \mu(F \cap E),\end{aligned}$$

proving that $\mu(E) + \mu(F) = \mu(E \cup F) + \mu(F \cap E)$.

Problem 2 - 1.5.30

The next two problems will use the notation m as Lebesgue measure on $\mathbb{R}$, and $\mathscr{L}$ as the Lebesgue measurable sets, the domain of the function m. Folland also uses the notation $\mathscr{M}_\mu$ for the domain of the measure μ, so this may pop up. The problem asks us to show that for any $E \in \mathscr{L}$ s.t. $m(E) > 0$, and $\alpha < 1$, there is an open interval I s.t. $m(E \cap I) > \alpha m(I)$. Well for $\alpha \leq 0$, the claim is trivial since m is always non-negative, and we can take $I = \mathbb{R}$ to get that indeed $m(E \cap \mathbb{R}) = m(E) > 0$.

Now the case $\alpha \in (0, 1)$. Suppose the contrary to our desired claim: that there is some $\alpha \in (0, 1)$ s.t. for all open intervals I, $m(E \cap I) \leq \alpha m(I)$. Lemma 1.17 (applied to Lebesgue measure m) says that for any $E \in \mathscr{M}_m = \mathscr{L}$,

$$m(E) = \inf\left\{\sum_{i=1}^{\infty}(b_i - a_i) : E \subseteq \bigcup_{i=1}^{\infty}(a_i, b_i)\right\},$$

so in particular for our given E from above, we know that there is some cover $\{I_1, I_2, \ldots\}$ of open intervals whose total length is $< m(E)(\frac{1}{\alpha})$ (here we use the assumption that $0 < m(E) < \infty$; we address the $m(E) = \infty$ case in the last paragraph). Well now we combine all of the above inequalities (as well as use that $E \subseteq \bigcup_{i=1}^{\infty} I_i \implies E = E \cap (\bigcup_{i=1}^{\infty} I_i) = \bigcup_{i=1}^{\infty}(E \cap I_i)$, and subbadditivity) to get

$$m(E) = m\left(\bigcup_{i=1}^{\infty}(E \cap I_i)\right) \leq \sum_{i=1}^{\infty} m(E \cap I_i) \leq \alpha \sum_{i=1}^{\infty} m(I_i) < \alpha \cdot \left(m(E) \cdot \frac{1}{\alpha}\right) = m(E),$$

which is impossible, if $m(E)$ is finite. Thus it must be that at least for $E \in \mathscr{L}$ s.t. $0 < m(E) < \infty$, for all $\alpha \in (0, 1)$, there must be some open interval I s.t. $m(E \cap I) > \alpha m(I)$, as desired.

To combat the case when $m(E) = \infty$, observe that there must be some $N \in \mathbb{N}$ s.t. $E \cap (-N, N)$ has positive measure (because by property (iii) from Lecture 10/9, the measures of the increasing sequence of measurable sets $E \cap (-N, N)$ must approach $m(\bigcup_{N=1}^{\infty} E \cap (-N, N)) = m(E) = \infty$), and so we can take this $E \cap (-N, N)$ to be the "new E". The above proof holds for this "new E" which is bounded and hence has finite measure, and so we still have that for every $\alpha \in (0, 1)$, there is some open interval I s.t. $m(E \cap I) \geq m((E \cap (-N, N)) \cap I) > \alpha m(I)$.

REMARKS: note that in fact given $E \in \mathscr{L}$ s.t. $0 < m(E) < \infty$, and a fixed $\alpha \in (0, 1)$, the above proof shows that one of the intervals (say I_i) in the cover $\{I_1, \ldots\}$ of total length $< m(E)(\frac{1}{\alpha})$ satisfies $m(E \cap I_i) > \alpha m(I_i)$; and because the E in question is finite measure, and the cover $\{I_1, \ldots\}$ has total length $< m(E)(\frac{1}{\alpha})$, I_i must have finite measure too. And because the $m(E) = \infty$ piggybacks off the finite measure case, in all cases we can be more specific than the problem requests, and find an open interval I with FINITE MEASURE with the desired property. We will use this in Problem 3.

Problem 3 - 1.5.31

Given $E \in \mathscr{L}$ s.t. $m(E) > 0$, Problem 2 above gives that there is some open interval I of finite measure (see REMARKS under Problem 2) s.t. $m(E \cap I) > \frac{3}{4} m(I)$. We want to show that $E - E := \{x - y : x, y \in E\}$ contains $I_0 = (-\frac{1}{2} m(I), \frac{1}{2} m(I))$ (which is just I shifted over to center at 0). Notice first that 0 is obviously in $E - E$ because E is non-empty (otherwise it'd have measure 0), and for any $x \in E$, $x - x = 0$; and $E - E$ is symmetric across 0 because if $x - y \in E - E$, then $-(x - y) = y - x \in E - E$. That means that we only have to prove that $(0, \frac{1}{2} m(I))$ is in $E - E$.

Well, suppose that it isn't; then we have some $x_0 \in (0, \frac{1}{2} m(I))$ that isn't in $E - E$. This would mean that $x_0 + E$ doesn't intersect at all with E (because if it did, then there would be some $x, y \in E$ s.t. $x_0 + x = y \iff x_0 = y - x \in E - E$). Because translations preserve measurability/measure (because Lebesgue measure m is translation invariant — see Theorem 1.21), and because $(x_0 + E) \cap E = \varnothing \implies \big(x_0 + (E \cap I)\big) \cap (E \cap I) = \varnothing$, we get that

$$m\Big(\big((E + x_0) \cap (I + x_0)\big) \cup (E \cap I)\Big) = m\Big(\big((E \cap I) + x_0\big) \cup (E \cap I)\Big) = 2m(E \cap I) > 2 \cdot \frac{3}{4} m(I).$$

However, $\big((E + x_0) \cap (I + x_0)\big) \cup (E \cap I)\big) \subseteq (I + x_0) \cup I$, and if $I = (a, b)$, then $(I + x_0) \cup I \subseteq (a, b + \frac{b-a}{2})$, so putting the inequalities together (and using monotonicity of measure), we get that

$$\begin{aligned} \frac{3}{2} m(I) &< m\Big(\big((E + x_0) \cap (I + x_0)\big) \cup (E \cap I)\Big) \\ &\leq m\Big((I + x_0) \cup I\Big) \leq m\Big((a, b + \tfrac{b-a}{2})\Big) = \frac{3}{2}(b - a) = \frac{3}{2} m(I), \end{aligned}$$

which is impossible, because I has finite measure. Thus, it must be that all of $(0, \frac{1}{2} m(I))$ is contained within $E - E$, and so as stated before, by symmetry the interval $I_0 = (-\frac{1}{2} m(I), \frac{1}{2} m(I))$ is contained within $E - E$.

Problem 4

Suppose we have a set $E \subseteq \mathbb{R}^d$ and a sequence of open sets $\{O_n\}_{n=1}^{\infty}$ s.t. $O_n = \{x \in \mathbb{R}^d : d(x, E) < \frac{1}{n}\}$. We want to show that if E is compact, then $m(E) = \lim_{n\to\infty} m(O_n)$, where m here (and in the next problem) is Lebesgue measure in $\mathbb{R}^d$. Property (iv) proved in class on Friday 10/9 says that if we have a decreasing sequence $A_1 \supseteq \ldots \supseteq A_k \supseteq A_{k+1} \ldots$ of μ-measurable sets and $\mu(A_1) < \infty$, then $\lim_{k\to\infty} \mu(A_k) = \mu(\bigcap_{k=1}^{\infty} A_k)$. Here, the O_n are open and hence (Lebesgue) measurable, so we just need to prove that $\bigcap_{n=1}^{\infty} O_n = E$, and that $m(O_1) < \infty$.

$\bigcap_{n=1}^{\infty} O_n = E$: because all the $O_n \supseteq E$, $\bigcap_{n=1}^{\infty} O_n \supseteq E$. As for the other direction, suppose that there is some $x \in \bigcap_{n=1}^{\infty} O_n$ but not in E. Because E is compact (hence closed), $E^{\complement}$ is open and so there is some ball $B(x, \epsilon) \subseteq E^{\complement}$. But then $d(x, E) \geq \epsilon$ (because otherwise there would be $y \in E$ s.t. $|x-y| < \epsilon$, and hence in $B(x, \epsilon)$, which is impossible because $B(x, \epsilon) \subseteq E^{\complement} \implies B(x, \epsilon) \cap E = \varnothing$), which means that $x \notin O_n$ for all n large enough s.t. $\frac{1}{n} < \epsilon$; contradiction. Thus it must be that $\bigcap_{n=1}^{\infty} O_n \subseteq E$.

Now we just need to verify $m(O_1) < \infty$. Because E is compact (hence bounded), it can be contained within some $B(\mathbf{0}, R)$ for some radius R large enough. Then, I claim that $O_1 \subseteq B(\mathbf{0}, R+1)$. Suppose not: then there would be some $y \in O_1$ s.t. $|y - 0| \geq R + 1$, but by the definition of O_1, $d(y, E) = \inf\{|x - y| : x \in E\} < 1$, and so there must be some $x \in E \subseteq B(\mathbf{0}, R)$ s.t. $|x - y| < 1$. But then by the triangle inequality,

$$|y - 0| \leq |y - x| + |x - 0| < 1 + R,$$

which is a contradiction. Thus $O_1 \subseteq B(\mathbf{0}, R+1)$ is bounded and hence has finite measure, and so as stated in paragraph 1, we use property (iv) from Lecture 10/9 to get that

$$m(E) = m\left(\bigcap_{n=1}^{\infty} O_n\right) = \lim_{n\to\infty} m(O_n).$$

We now provide 2 examples where this conclusion fails; 1 in the case that E is closed and unbounded, and 1 in the case that E is open and bounded:

- E closed and unbounded: let $E = \mathbb{N}$ which is obviously closed and unbounded; then $O_n = \bigcup_{k=1}^{\infty}(-\frac{1}{n}+k, k+\frac{1}{n})$, and so for every O_n ($n \in \mathbb{N}$), $m(O_n) = \sum_{k=1}^{\infty} \frac{1}{2n} = \infty$, and so $\lim_{n\to\infty} m(O_n)$ most definitely does not converge to $m(\mathbb{N}) = 0$.
- E open and bounded: $\mathbb{Q} \cap [0, 1]$ would be great, because it has measure 0, while every open set containing it contains $[0, 1]$ (by denseness); however $\mathbb{Q} \cap [0, 1]$ is not open. We can tweak this a bit as follows: fixing any $\epsilon \in (0, 1)$, and enumerating $\mathbb{Q} \cap [0, 1]$ as $\{r_1, \ldots\}$, consider for each r_k the set $(-\frac{\epsilon}{2^k} \cdot \frac{1}{2} + r_k, r_k + \frac{1}{2}\frac{\epsilon}{2^k})$, and let E be the union of all these sets. Then, by subadditivity, we get that

$$\begin{aligned} m(E) &= m\left(\bigcup_{k=1}^{\infty}\left(-\frac{\epsilon}{2^k} \cdot \frac{1}{2} + r_k, r_k + \frac{1}{2}\frac{\epsilon}{2^k}\right)\right) \\ &\leq \sum_{k=1}^{\infty} m\left(\left(-\frac{\epsilon}{2^k} \cdot \frac{1}{2} + r_k, r_k + \frac{1}{2}\frac{\epsilon}{2^k}\right)\right) = \sum_{k=1}^{\infty} \frac{\epsilon}{2^k} = \epsilon. \end{aligned}$$

But it is still the case that any open set containing E (for example all the O_n) must contain $[0,1]$, because E contains $\mathbb{Q} \cap [0,1]$ and every open set containing that contains $[0,1]$ (by denseness). In other words, all the $O_n \supseteq [0,1]$, and so for all $n \in \mathbb{N}$, $m(O_n) \geq m([0,1]) = 1$. Thus, $\lim_{n\to\infty} m(O_n) \geq 1$, which is $> \epsilon \geq m(E)$, and so the limit does not converge to $m(E)$.

Problem 5

Let $\{E_k\}_{k=1}^{\infty}$ be a countable family of measurable subsets of $\mathbb{R}^d$ s.t. $\sum_{k=1}^{\infty} m(E_k) < \infty$, and consider $E = \{x \in \mathbb{R}^d : x \in E_k \text{ for infinitely many } k\}$. First we prove that $E = \bigcap_{n=1}^{\infty} \bigcup_{k\geq n} E_k$. The fact that x is in infinitely many of the E_k means that for any $n \in \mathbb{N}$, there is some $k \geq n$ s.t. $x \in E_k$. Thus for every $n \in N$, $x \in \bigcup_{k\geq n} E_k$, and so the "$\subseteq$" direction is proven. Now if $x \in \bigcap_{n=1}^{\infty} \bigcup_{k\geq n} E_k$, that means that for any $n \in \mathbb{N}$, there is some $k \geq n$ s.t. $x \in E_k$. Thus x must be in infinitely many of the E_k, because otherwise there would be a point N past which x is not in any of the E_k.

Now denoting $F_n = \bigcup_{k\geq n} E_k$, we see that all the F_n must be measurable because they are the countable union of measurable sets E_k, and measurable sets form a σ-algebra. Then $E = \bigcap_{n=1}^{\infty} F_n$ is measurable because it is the countable intersection of measurable sets F_n.

Finally, we have that $E \subseteq F_n$ for any $n \in \mathbb{N}$, which by monotonicity means that $m(E) \leq m(F_n)$ for all $n \in \mathbb{N}$. By countable subadditivity we have that $m(F_n) \leq \sum_{k=n}^{\infty} m(E_k)$, where the RHS must go to 0 as $n \to \infty$ (because we are given that the infinite sum $\sum_{k=1}^{\infty} m(E_k)$ evaluates to something finite, and hence the partial sums must converge, and so the differences $\sum_{k=1}^{\infty} m(E_k) - \sum_{k=1}^{n} m(E_k) = \sum_{k=n+1}^{\infty} m(E_k)$ must go to 0). To summarize, $m(E)$ is bounded below by 0 (because measures are non-negative), and must be $\leq$ any arbitrary small positive value, and so $m(E)$ must be exactly 0.

524 Homework 2

Daniel Rui - 10/14/20

Problem 1

For any closed rectangle $R = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_d, b_d] \subseteq \mathbb{R}^d$, we define its volume to be

$$\text{vol}(R) = \text{vol}\left(\prod_{i=1}^{d}[a_i, b_i]\right) = \prod_{i=1}^{d}(b_i - a_i)$$

and define the Lebesgue outer measure m^* to be

$$m^*(E) = \inf\left\{\sum_{i=1}^{\infty}\text{vol}(Q_i) : E \subseteq \bigcup_{i=1}^{\infty} Q_i \text{ for closed cubes } Q_i\right\}.$$

(a) We want to prove that for any closed cube Q, $m^*(Q) = \text{vol}(Q)$. Because $Q \subseteq Q$, and $m^*(Q)$ is an infimum taken over all closed-cube coverings of Q, $m^*(Q) \leq \text{vol}(Q)$ trivially.

For the other direction, suppose we have an arbitrary closed cube covering $\{Q_1, \ldots\}$ of Q. Then we can "fatten" up the Q_i by stretching the interval $[a, b]$ (one of the edges of Q_i) to $[a-\delta, b+\delta]$ so that the Q_i are now contained within the interior of the fattened Q_i, and the volume goes from $(b-a)^d$ to $(b-a+2\delta)^d$. For any $\epsilon > 0$, we can choose δ small enough s.t. $(b-a+2\delta) < (1+\epsilon)^{1/d}(b-a)$ to get that the fattened Q_i has volume $\leq (1+\epsilon)\,\text{vol}(Q_i)$. Doing this for all the Q_i, we get fattened $\hat{Q}_i$ s.t. the set of their interiors $\{\hat{Q}_1^\circ, \ldots\}$ is an open covering of Q (a compact set).

By the Heine-Borel property, there must be some finite number of these open sets, say $\{\hat{Q}_1^\circ, \ldots, \hat{Q}_n^\circ\}$ that cover Q. That is, we have that $Q \subseteq \bigcup_{k=1}^{n}\hat{Q}_k^\circ \subseteq \bigcup_{k=1}^{n}\hat{Q}_k$. Using Lemma 1.2 of Stein & Shakarchi (the proof of which I will sketch out in the "Problem 1 - appendix" below), we get that

$$\text{vol}(Q) \leq \sum_{k=1}^{n}\text{vol}(\hat{Q}_k) \leq \sum_{k=1}^{n}(1+\epsilon)\,\text{vol}(Q_k) \leq (1+\epsilon)\sum_{k=1}^{\infty}\text{vol}(Q_k).$$

Because $\epsilon > 0$ is arbitrary, we have that $\text{vol}(Q) \leq \sum_{k=1}^{\infty}\text{vol}(Q_k)$, and because the covering $\{Q_1, \ldots\}$ was arbitrary as well, everything $m^*(Q)$ is taking the infimum over is $\geq \text{vol}(Q)$, and so $m^*(Q)$ must be $\geq \text{vol}(Q)$ as well.

(b) We want to prove that if $E \subseteq \mathbb{R}^d$ is a countable union of closed cubes $\{Q_1, \ldots\}$ s.t. their interiors $\{Q_1^\circ, \ldots\}$ are disjoint, then $m^*(E) = \sum_{k=1}^{\infty}m^*(Q_k)$. From (a), we know that $m^*(Q_k) = \text{vol}(Q_k)$, so $m^*(E) \leq \sum_{k=1}^{\infty}\text{vol}(Q_k)$ follows easily from the infimum definition of m^*.

Now the other direction. First, let us establish that $m^*(Q_k^\circ) = \text{vol}(Q_k)$. Similar to what we did in (a) but in the other direction, we can "thin" Q_k to $\hat{Q}_k$ by pulling in the bounds of the intervals by δ (just like we pushed out the bounds of the intervals by δ in part (a)), carefully chosen s.t. $(1-\epsilon)\,\text{vol}(Q_k) \leq \text{vol}(\hat{Q}_k)$ (and of course $\hat{Q}_k \subseteq Q_k^\circ$). In other words, we have by monotonicity that

$$(1-\epsilon)\,\text{vol}(Q_k) \leq \text{vol}(\hat{Q}_k) = m^*(\hat{Q}_k) \leq m^*(Q_k^\circ) \leq m^*(Q_k) = \text{vol}(Q_k)$$

which proves the assertion because $\epsilon > 0$ is arbitrary.

In class on Monday 10/12, we closed by showing that metric outer measure (proof in Problem 2) implies that all closed sets are measurable. By complements, all open sets are measurable, and we proved in class on Friday 10/9 that disjoint measurable sets are additive. Thus, we've shown that

$$m^*(E) \geq m^*\left(\bigcup_{k=1}^{\infty} Q_k^\circ\right) = \sum_{k=1}^{\infty} m^*(Q_k^\circ) = \sum_{k=1}^{\infty} \text{vol}(Q_k).$$

Problem 1 - appendix

A sketch of Lemma 1.2: we prove that if we have a closed rectangle $R = [a_1, b_1] \times \ldots \times [a_d, b_d]$ and closed rectangles $R_1, \ldots, R_N$ ($R_j = [a_{j1}, b_{j1}] \times \ldots \times [a_{jd}, b_{jd}]$) that cover R, then $\text{vol}(R) \leq \sum_{j=1}^{N} \text{vol}(R_j)$. We first extend the edges of the rectangles to get a grid, where the grid points in the ith coordinate are $P_i := \{a_i, b_i, a_{1i}, b_{1i}, \ldots, a_{Ni}, b_{Ni}\}$. We can order this list to get $P_i = \{{}_ic_1, \ldots, {}_ic_{m_i}\}$ where ${}_ic_1 < \ldots < {}_ic_{m_i}$ and $m_i = |P_i|$. Let $\mathscr{I}_i = \{[{}_ic_1, {}_ic_2], \ldots, [{}_ic_{m_i-1}, {}_ic_{m_i}]\}$. Then consider $\mathscr{S}$, the set of all the partition rectangles:

$$\mathscr{S} := \{I_1 \times \ldots \times I_d : I_i \in \mathscr{I}_i\}$$

Lemma: let $\mathscr{S}_{R_j}$ be the set of all rectangles in $\mathscr{S}$ contained in R_j (could take R instead of R_j). Then $\sum_{S \in \mathscr{S}_{R_j}} \text{vol}(S) = \text{vol}(R_j)$. To prove this, we first prove a baby lemma (the stitching two rectangles into one lemma):

Suppose we have two rectangles where one adjacent to the other , and is just a shifted/stretched copy of of the other, where the shifts and stretches are in the direction of one of the coordinate. Formally, we can write this as two rectangles

$$T_1 = I_1 \times \ldots \times I_{i-1} \times ({}_ic_\alpha, {}_ic_\beta) \times I_{i+1} \times \ldots \times I_d \quad \text{and} \quad T_2 = I_1 \times \ldots \times I_{i-1} \times ({}_ic_\beta, {}_ic_\gamma) \times I_{i+1} \times \ldots \times I_d$$

The rectangle T_3 is these two rectangles stitched together:

$$T_3 = I_1 \times \ldots \times I_{i-1} \times ({}_ic_\alpha, {}_ic_\gamma) \times I_{i+1} \times \ldots \times I_d$$

Then because $({}_ic_\gamma - {}_ic_\beta) + ({}_ic_\beta, {}_ic_\alpha) = ({}_ic_\gamma, {}_ic_\alpha)$, we have that $\text{vol}(T_1) + \text{vol}(T_2) = \text{vol}(T_3)$. It is clear that this proof/baby lemma can be extended to more than just two rectangles like this; we'll call such a collection of rectangles (all adjacent, all shifted/stretched along direction of 1 of the coordinate axes) a "row" of rectangles aligned with the ith coordinate axes, or "row along i" for short.

Now the plan is to stitch together all "rows along 1", and then stitching along 2 and so on until we have the complete R_j, proving that the sum of the volumes of the partition rectangles is $\text{vol}(R_j)$ (because the baby lemma showed that stitching preserves volumes). Formally, a "row along 1" of rectangles $\in \mathscr{S}_{R_j}$ is a collection (where ${}_ic_\alpha = a_{j1}$ and ${}_ic_\beta = b_{j1}$):

$$\{[{}_ic_\alpha, {}_ic_{\alpha+1}] \times I_2 \times \ldots \times I_d, \ldots, [{}_ic_{\beta-1}, {}_ic_\beta] \times I_2 \times \ldots \times I_d\}$$

so in particular there is a "row along 1" for all

$$I_2, \ldots, I_d \text{ s.t. } I_i \in \mathscr{I}_i \text{ and } I_i \subseteq [a_{j1}, b_{j1}]$$

where the restrictions on I_i are the keep the rectangles $\subseteq R_j$. Stitching along such a row gives the rectangle

$$[a_{j1}, b_{j1}] \times I_2 \times \ldots \times I_d$$

(which to reiterate has the same volume as the sum of the rectangles from that "row along 1"). So this step turned all "rows along 1" into rectangles with 1st coordinate interval $= [a_{j1}, b_{j1}]$. As you can imagine, we keep doing this for all d coordinates until we end up with just 1 rectangle:

$$[a_{j1}, b_{j1}] \times \ldots \times [a_{jd}, b_{jd}] = R_j$$

Again, to reiterate the volumes are preserved through stitching so the lemma is proved. Now all rectangles in $\mathscr{S}$ are contained within one of the R_j, and so $\sum_{j=1}^{N} \text{vol}(R_j) = \sum_{j=1}^{N} \sum_{S \in \mathscr{S}_{R_j}} \text{vol}(S)$ counts each S at least once (more than once if that S is contained in more than 1 R_j), while $\text{vol}(R) = \sum_{S \in \mathscr{S}_R} \text{vol}(S)$ counts only the rectangles contained within R exactly once, and so obviously the sum counting all the $\text{vol}(S)$ at least once is $\geq$ than the sum counting some of the $\text{vol}(S)$ exactly once (and the rest zero times). Thus Lemma 1.2 is proven.

Problem 2

We want to show that if $E \subseteq \mathbb{R}^d$ is equal to $E_1 \cup E_2$ where $d(E_1, E_2) = \delta > 0$, then $m^*(E) = m^*(E_1) + m^*(E_2)$. By subadditivity, we get $m^*(E) \leq m^*(E_1) + m^*(E_2)$, so we just have to prove the other direction. We do this by constructing, for every cover $\{Q_i\}$ of E, covers $\{Q_i^1\}$ and $\{Q_i^2\}$ of E_1 and E_2 resp. s.t. the sum of the total volume of $\{Q_i^1\}$ and $\{Q_i^2\}$ is $\leq$ the total volume of $\{Q_i\}$ (implying that the infimum of the sum of the total volumes of covers of E_1 and E_2 is $\leq$ the infimum of the total volume of covers of E).

First, if $m^*(E) = \infty$, then certainly the claim $m^*(E) = \infty \geq m^*(E_1) + m^*(E_2)$ follows. Thus, we only need to consider the case where $m^*(E) < \infty$ (i.e. we only need to consider the case when the total volume of $\{Q_i\}$ is bounded $\iff$ each of the Q_i are bounded). Alright; now I claim that given any particular Q_i in the cover $\{Q_i\}$, we can bisect each of the sides to get 2^d cubes each with sidelength $\frac{1}{2}$ that of Q_i, and so volume $\frac{1}{2^d}$ the volume of Q_i (call this operation "halving" the cube Q_i) – note that the volume of all these subcubes summed is equal to the original volume of Q_i ($2^d \cdot \frac{1}{2^d} \text{vol}(Q_i) = \text{vol}(Q_i)$). Also, these 2^d smaller cubes obviously cover Q_i. Thus replacing Q_i with these 2^d halved-length cubes in the cover, we still have a (countable) cover of E, and the total volume of this new cover is the same as the old. Given any cube Q_i, we can "halve" it finitely many times until the sidelengths s of the resulting cubes are small enough that $s\sqrt{d}$ (the length of the longest diagonal = diameter of the cube) is $< \delta$ (which we defined above to be $d(E_1, E_2)$).

Because there are countably many cubes, and we only do finitely many "halving" processes per cube,

and each "halving" process turns 1 cube into 2^d cubes (a finite number), and every time the new cover has the same total volume as the original cover, we now have a countable cover $\{Q'_i\}$ with the same total volume as the cover $\{Q_i\}$, and with cubes whose diameters are $< \delta$. This means that none of the $Q'_i \in \{Q'_i\}$ have non-trivial intersection with both E_1 and E_2, because if they did then their diameter would necessarily be $\geq \delta = d(E_1, E_2)$. That means that we can split the cover $\{Q'_i\}$ into two covers $\{Q^1_i\}$ and $\{Q^2_i\}$, where $Q'_i \in \{Q'_i\}$ belongs in $\{Q^1_i\}$ iff it has non-trivial intersection with E_1 and belongs in $\{Q^2_i\}$ iff it has non-trivial intersection with E_2 (so no Q'_i belongs in both, whereas maybe some Q'_i might not belong in neither). Then the sum of the total volumes of $\{Q^1_i\}$ and $\{Q^2_i\}$ must be $\leq$ the total volume of $\{Q'_i\}$ (which equals the total volume of $\{Q_i\}$), because each $\text{vol}(Q'_i)$ is counted exactly once in the latter, whereas either once or zero times in the former.

Problem 3

In class on Monday 10/12, we defined the Cantor set by defining C_0 to be $[0, 1]$, and defining C_1 to be the disjoint union of the 2 intervals of length 3^{-1} we get after removing the middle third, and defining C_k to be the disjoint union of 2^k intervals of length 3^{-k} we get after cutting out the middle third of each of the 2^{k-1} disjoint intervals of length $3^{-(k-1)}$ in C_k. What we mean by removing the middle third of an interval $[a, b]$ is removing the open interval $(a + \frac{b-a}{3}, b - \frac{b-a}{3})$.

Now this problem asks us to consider a similar construction, except the middle "third" we remove is now of length $\alpha 3^{-k}$ for some fixed $\alpha \in (0, 1)$. That is to say, removing the middle of the interval $[a, b]$ in F_{k-1} is now taken to mean removing $(\frac{a+b}{2} - \frac{1}{2}\alpha 3^{-k}, \frac{a+b}{2} + \frac{1}{2}\alpha 3^{-k})$. F_0 is still $[0, 1]$, but F_k is now the disjoint union of 2^k intervals of length L_k, where $L_k = \frac{1}{2}(L_{k-1} - \alpha 3^{-k})$; to better see this pattern let's write out a couple terms: $L_0 = 1$, $L_1 = \frac{1}{2}(1) - \frac{1}{2}\alpha\frac{1}{3}$, $L_2 = \frac{1}{2}(\frac{1}{2}(1) - \frac{1}{2}\alpha\frac{1}{3}) - \frac{1}{2}\alpha\frac{1}{3^2} = (\frac{1}{2^2} - \frac{1}{2^2}\frac{\alpha}{3}) - \frac{1}{2}\frac{\alpha}{3^2}$, $L_3 = \frac{1}{2^3} - \frac{1}{2^3}\frac{\alpha}{3} - \frac{1}{2^2}\frac{\alpha}{3^2} - \frac{1}{2}\frac{\alpha}{3^3} = \frac{1}{2^3} - \frac{\alpha}{2^3}\frac{1}{3}(1 + 2^1\frac{1}{3^1} + 2^2\frac{1}{3^2})$. The general term can be written as

$$L_k = \frac{1}{2^k} - \frac{\alpha}{3 \cdot 2^k}\left(1 + \frac{2}{3} + \ldots + \frac{2^{k-1}}{3^{k-1}}\right) = \frac{1}{2^k} - \frac{\alpha}{3 \cdot 2^k}\left(\frac{1 - (\frac{2}{3})^k}{1 - \frac{2}{3}}\right)$$

(one could easily prove this rigorously by induction). Now for the properties of $F = \bigcap_{k=0}^{\infty} F_k$:

- F is closed, because F_0 is closed, and each iteration we remove/subtract only open sets, and so each of the F_k are closed, and any arbitrary intersection of closed sets is closed.
- Pick any $x \in [0, 1]$, and any $\epsilon > 0$ and consider $B(x, \epsilon) = (x - \epsilon, x + \epsilon)$. Now since $L_k \to 0$ as $k \to \infty$, there must be some K s.t. $k \geq K \implies L_k < \epsilon$. Now it can't be that $B(x, \epsilon)$ is completely contained in one of the 2^k disjoint closed intervals making up F_k, because if that were the case $m^*(B(x, \epsilon)) \leq L_k < \epsilon$ (using the fact that the length or "vol" of one of the 2^k intervals is L_k, and using Problem 1(a) which says that m^* of one of those closed intervals (i.e. cubes) is exactly equal to the "vol", which is L_k). However this can not be the case because $m^*(B(x, \epsilon)) = \text{vol}([x - \epsilon, x + \epsilon]) = 2\epsilon$ (see the proof that m^* of the interior of a closed cube is equal to m^* of the closed cube itself in the 2nd paragraph of Problem 1(b)). Thus, it is not the case that $B(x, \epsilon) \subseteq F_k$, because if so it would have be completely contained within one of the

2^k closed intervals that make up F_k, which we just proved isn't the case. Thus, there must be some point in $B(x,\epsilon)$ outside of F_k, so in $F_k^{\complement}$. But $F \subseteq F_k \implies F^{\complement} \supseteq F_k^{\complement}$, and so $B(x,\epsilon)$ has non-trivial intersection with $F^{\complement}$. Because $\epsilon > 0$ was arbitrary, we have just shown by Proposition 0.22 in Folland that $x \in \overline{F^{\complement}}$, and because $x \in [0,1]$ was arbitrary, this shows that $[0,1] \subseteq \overline{F^{\complement}}$ (i.e. all points in $[0,1]$ are limit points of $F^{\complement}$), and so $F^{\complement}$ is dense in $[0,1]$.

- As we proved in class Monday and Wednesday 10/12 and 10/14, m^* a metric outer measure $\implies$ closed sets are m^*-measurable, and we proved that m^* is a metric outer measure in Problem 2, and so all the F_k and F are m^*-measurable. Furthermore, we have that $F_0 \supset F_1 \supset \ldots$ and $m^*(F_0) \leq \text{vol}([0,1]) = 1 < \infty$ (because $[0,1]$ is a closed cube covering F_0), and so we can use property (iv) proved in class on Friday 10/9 to get that

$$m^*(F) = m^*\left(\bigcap_{k=0}^{\infty} F_k\right) = \lim_{k\to\infty} m^*(F_k)$$

But F_k is the disjoint union of 2^k closed intervals of length (also known as "vol") L_k, and so by additivity of disjoint measurable sets and by Problem 1(a) which tells us that m^* of each of those 2^k closed intervals of "vol" L_k is exactly L_k, we get that $m^*(F_k) = 2^k L_k = 1 - \frac{\alpha}{3}\left(\frac{1-(\frac{2}{3})^k}{1-\frac{2}{3}}\right)$ and so

$$m^*(F) = \lim_{k\to\infty} 1 - \frac{\alpha}{3}\left(\frac{1-(\frac{2}{3})^k}{1-\frac{2}{3}}\right) = 1 - \frac{\alpha}{3}\left(\frac{1-0}{1-\frac{2}{3}}\right) = 1 - \frac{\alpha}{3}\cdot(3) = 1-\alpha$$

Problem 4

Let (X,ρ) be a non-empty complete metric space, and suppose there is $f : X \to X$ s.t. there is some $\lambda \in (0,1)$ s.t. $\rho(f(x),f(y)) \leq \lambda\rho(x,y)$ for all $x,y \in X$. Fix any $x \in X$ and consider the sequence $\{x, f(x), f^{\circ 2}(x), f^{\circ 3}(x), \ldots\} := \{x, f(x), f(f(x)), f(f(f(x))), \ldots\}$. Consider now the distance $\rho(f^{\circ m}(x), f^{\circ n}(x))$ for arbitrary m,n (let's have $n > m$): iterating the identity over and over again we get that

$$\rho(f^{\circ m}(x), f^{\circ n}(x)) \leq \lambda\rho(f^{\circ(m-1)}(x), f^{\circ(n-1)}(x)) \leq \ldots \leq \lambda^m \rho(f^{\circ 0}(x), f^{\circ(n-m)}(x))$$

This bound on the right can be simplified further: for any $k \in \mathbb{N}$, we have

$$\begin{aligned}
\rho(x, f^{\circ k}(x)) &\leq \rho(x, f^{\circ 1}(x)) + \rho(f^{\circ 1}(x), f^{\circ 2}(x)) + \ldots + \rho(f^{\circ(k-1)}(x), f^{\circ k}(x)) \\
&\leq \rho(x, f^{\circ 1}(x)) + \lambda\rho(x, f^{\circ 1}(x)) + \lambda^{k-1}\rho(x, f^{\circ 1}(x)) \\
&= \rho(x, f(x))(1 + \lambda + \lambda^2 + \ldots + \lambda^{k-1}) \\
&\leq \rho(x, f(x))\left(\sum_{k=0}^{\infty} \lambda^k\right) = \frac{1}{1-\lambda}\rho(x, f(x))
\end{aligned}$$

Putting the inequalities together, we get that for any $m,n \in \mathbb{N}$ where $n \geq m$,

$$\rho(f^{\circ m}(x), f^{\circ n}(x)) \leq \frac{\lambda^m}{1-\lambda}\rho(x, f(x))$$

Now because $\lambda \in (0,1)$, $m \to \infty \implies \lambda^m \to 0$, which implies $\frac{\lambda^m}{1-\lambda}\rho(x, f(x)) \to 0$ as well because the other stuff are just constants. Thus, for every $\epsilon > 0$, there is M big enough that $n \geq m \geq M \implies \rho(f^{\circ m}(x), f^{\circ n}(x)) \leq \frac{\lambda^m}{1-\lambda}\rho(x, f(x)) < \epsilon$, and so the sequence is Cauchy. Because X is complete, this sequence converges to some $u \in X$. That is to say, fixing any $\epsilon > 0$, there is $N \in \mathbb{N}$ s.t. $n \geq N \implies \rho(u, f^{\circ n}(x)) < \frac{\epsilon}{2}$. Now consider $\rho(u, f(u))$. We have that for any $n \geq N$,

$$\rho(u, f(u)) \leq \rho(u, f^{\circ(n+1)}(x)) + \rho(f^{\circ(n+1)}(x), f(u)) \leq \rho(u, f^{\circ(n+1)}(x)) + \lambda\rho(f^{\circ n}(x), u) < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon.$$

Because $\epsilon > 0$ was arbitrary, $\rho(u, f(u))$ must be 0, and so $u = f(u)$. Lastly, such a fixed point must be unique, because if we have some $u' \in X$ (not equal to u) also a fixed point, then the identity would yield

$$\rho(u, u') = \rho(f(u), f(u')) \leq \lambda\rho(u, u') \iff 1 \leq \lambda$$

where we divided by $\rho(u, u')$ because we assumed $u \neq u' \iff \rho(u, u') > 0$. This is a contradiction, because $\lambda \in (0, 1)$.

Problem 5

As proven in Theorem 0.26 (as well as in class on 10/7, on page 2 of the slides), given a compact set E in a metric space (X, ρ) and an open cover of E, there is some $\epsilon > 0$ s.t. every ball of radius ϵ with non-trivial intersection with E is completely contained within one of the open sets of the cover. I will give a quick overview: suppose not. Then for every $\frac{1}{n}$, $n \in \mathbb{N}$, there is some ball $B(x_n, \frac{1}{n})$ s.t. $B(x_n, \frac{1}{n}) \cap E \neq \varnothing$ and $B(x_n, \frac{1}{n})$ is not contained within any one set of the open cover. Define a sequence by choosing some $x'_n \in B(x_n, \frac{1}{n}) \cap E$. Using the Bolzano-Weierstrass definition of compactness (which we know to be equivalent to the more standard Heine-Borel definition on metric spaces), we know there is a convergent subsequence of the $\{x'_1, \ldots\} \subseteq E$, converging to some $x \in E$. Then x is contained in one of the open sets U from the cover, and by definition of open set, there is some $N \in \mathbb{N}$ s.t. $B(x, \frac{1}{N})$ is also contained in U. But by the definition of convergence, going far enough out in the subsequence all x' in the subsequence satisfy $\rho(x, x') < \frac{1}{2N}$. Because there are infinitely many "far enough out" x', and the radii corresponding to the x' are $\in \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ (the set of reciprocals of the natural numbers), there must be some x' in particular with corresponding radius $< \frac{1}{2N}$. Thus by the triangle inequality, the ball corresponding to this particular x' is contained in $B(x, \epsilon) \subseteq U$, contradicting our initial definition of the $B(x_n, \frac{1}{n})$.

Ok. Now given any compact set $K \subseteq X$ and an open cover $\{G_i\}_{i \in I}$ of K. From above, we know there is some $\epsilon > 0$ s.t. ALL balls $B(x, \epsilon)$ with non-trivial intersection with K are contained completely within one of the G_i. Now consider a subset S of K with diameter $< \epsilon$. Then pick any $x \in S$, and consider $B(x, \epsilon)$. All $y \in S$ must be in $B(x, \epsilon)$ because by definition of diameter, $\rho(x, y) \leq \text{diam}(S) < \epsilon$ for all $x, y \in S$. So we have that given any subset S of diameter $< \epsilon$, $S \subseteq B(x, \epsilon) \subseteq G_i$ where x was any element in S, and G_i was one set from the open cover.

524 Homework 1

Daniel Rui - 10/7/20

Problem 1

Let (X, ρ) be a metric space, and let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence of points in X that has a cluster point $x \in X$. By the definition of cluster point, this means that for all $\epsilon > 0$ and all $N \in \mathbb{N}$, there is $n \geq N$ s.t. $\rho(x, x_n) < \epsilon$. We want to show that this guarantees that the sequence $\{x_n\}$ converges to x; i.e. that for all $\epsilon > 0$, there is $N \in \mathbb{N}$ s.t. $n \geq N \implies \rho(x, x_n) < \epsilon$.

Fix $\epsilon > 0$. By Cauchy, we know that there is some N s.t. $n, m \geq N \implies \rho(x_n, x_m) < \frac{\epsilon}{2}$. Because x is a cluster point, we know that there must exist some $N_0 \geq N$ s.t. $\rho(x, x_{N_0}) < \frac{\epsilon}{2}$. Well then we know that for all $n \geq N$, $\rho(x_n, x_{N_0}) < \frac{\epsilon}{2}$, and so by the triangle inequality, we know that for all $n \geq N$, $\rho(x_n, x) \leq \rho(x_n, x_{N_0}) + \rho(x_{N_0}, x) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$, as desired.

Problem 2

Letting $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers and $x \in \mathbb{R}$, we want to show that $x = \lim_{n\to\infty} x_n \iff$ every subsequence of $\{x_n\}_{n=1}^{\infty}$ has a subsubsequence that converges to x.

($\implies$) Consider any subsequence $\{x_{n_k}\}_{k=1}^{\infty}$; that is, the indices $\{n_1, n_2, \ldots\}$ are a subset of $\mathbb{N}$. Obviously, the $\{n_k\}$ are strictly increasing, and so because $n_1 \geq 1$, we have that $n_k \geq k$. Then because $x = \lim_{n\to\infty} x_n$, or in math notation, for all $\epsilon > 0$, there is $N \in \mathbb{N}$ s.t. $n \geq N \implies |x_n - x| < \epsilon$, we have that $k \geq N \implies n_k \geq N \implies |x_{n_k} - x| < \epsilon$ and so the subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ also converges to x. Thus we can take the subsubsequence to be exactly the subsequence, and so all subsequences have a subsubsequence converging to x.

($\impliedby$) Suppose not; i.e. suppose that somehow, we have that all subsequences have a subsubsequence converging to x, and yet the $\{x_n\}_{n=1}^{\infty}$ do not converge to x. By the definition of convergence, this means that for some $\epsilon > 0$, there will always be $n \geq N$ s.t. $|x_n - x| \geq \epsilon$ no matter how big we choose $N \in \mathbb{N}$. This means in particular that there must be infinitely many n s.t. $|x_n - x| \geq \epsilon$, and so we can create a subsequence $\{n_k\}_{k=1}^{\infty}$ consisting of the 1st, 2nd, 3rd, etc. indices s.t. $|x_{\text{that index}} - x| \geq \epsilon$. Well by construction, for all $n_k \in \{n_k\}_{k=1}^{\infty}$, $|x_{n_k} - x| \geq \epsilon$, and so it is impossible for any subsubsequence of the $\{x_{n_k}\}_{k=1}^{\infty}$ to converge; contradiction.

Problem 3

We want to prove that $S \subset \mathbb{R}^n$ is compact if and only if S is closed and bounded.

($\implies$) Let S be compact. First, S must be bounded: consider the open cover $\{(-k, k)^n\}_{k=1}^{\infty}$, which necessarily covers S because $\bigcup_{k=1}^{\infty}(-k, k)^n = \mathbb{R}^n$. By compactness, S is covered by $(-1, 1)^n \cup \ldots \cup (-N, N)^n$, and so $S \subseteq (-N, N)^n$ and so it's bounded.

S must also be closed: suppose not. Then there is $p \notin S$ that is on the boundary. Consider the open cover $\{\mathbb{R}^n \setminus \overline{B(p, 1/k)}\}_{k=1}^{\infty}$ where $B(p, 1/k)$ is the open ball centered at p with radius $1/k$ (and the overline is "closure of"), which must necessarily cover S because

$$\bigcup_{k=1}^{\infty} U_k := \bigcup_{k=1}^{\infty} \mathbb{R}^n \setminus \overline{B(p, 1/k)} = \mathbb{R}^n \setminus \{p\} \supseteq S$$

By compactness, S can be covered by $U_1 \cup \ldots \cup U_M = U_M$, and we know $(\mathbb{R}^n \setminus \overline{B(p, 1/M)})$ and $B(p, 1/M)$ are disjoint, so we know S and $B(p, 1/M)$ must be disjoint. This means that p is actually not on the boundary of S (it is in the exterior); contradiction.

$(\impliedby)$ This direction was covered by Prop 0.26 in Folland; however I will give a quick overview: first Folland proved Theorem 0.25 which states the following equivalences for subsets S of a metric space: S is complete and totally bounded (can be covered by finitely many balls of radius ϵ) $\iff$ the Bolzano-Weierstrass property (every sequence in S has a convergent subsequence to a point in S) $\iff$ the Heine-Borel property (every open cover of S has a finite subcover, i.e. the most common definition of compactness). This theorem proves that all closed and bounded subsets of $\mathbb{R}^n$ are compact because all closed subsets of $\mathbb{R}^n$ are complete, and all bounded subsets of $\mathbb{R}^n$ are totally bounded.

Problem 4

We want to prove that for a metric space (X, ρ), X is complete if and only if for any decreasing sequence $\{A_n\}_{n=1}^{\infty}$ of non-empty closed subsets of X s.t. $\lim_{n\to\infty} \operatorname{diam} A_n := \sup\{\rho(x, y) : x, y \in A_n\} = 0$, $\bigcap_{n=1}^{\infty} A_n = \{x\}$ for some $x \in X$.

$(\implies)$ Suppose that we have an arbitrary decreasing sequence $\{A_n\}_{n=1}^{\infty}$ of non-empty closed subsets of X s.t. $\lim_{n\to\infty} \operatorname{diam} A_n := \sup\{\rho(x, y) : x, y \in A_n\} = 0$, and suppose further that X is complete.

For each of the A_n, pick some $x_n \in A_n$ (the A_n are all non-empty). Because the A_n are decreasing, all the x_i for $i \geq n$ are in A_n, and because $\lim_{n\to\infty} \operatorname{diam} A_n = 0 \iff \forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. $n \geq N \implies \operatorname{diam} A_n < \epsilon$, we have that $n, m \geq N \implies \rho(x_n, x_m) \leq \operatorname{diam} A_n < \epsilon$ and so the $\{x_n\}_{n=1}^{\infty}$ form a Cauchy sequence.

By completeness of X, this Cauchy sequence converges to some $x \in X$. Furthermore, each A_k contains the subsequence $\{x_n\}_{n=k}^{\infty}$ (which obviously also converges to x), and so by Prop 0.22, $x \in \overline{A_k} = A_k$ (closed sets are equal to their closure — the closure of S is defined as the smallest closed set containing S, and obviously the smallest closed set containing a given closed set is itself). Thus, we have shown that $x \in A_k$ for all $k \in \mathbb{N}$.

Finally, say there was some $y \in X$ also in all the A_k, where $y \neq x$. This is impossible, because $\rho(y, x) > 0$ (a metric must satisfy $\rho(x, y) = 0 \iff x = y$), and $\lim_{n\to\infty} \operatorname{diam} A_n = 0$ means that all n sufficiently large will have $\operatorname{diam} A_n < \rho(y, x)/2$, forcing y to not be in any of the A_n for n sufficiently large. Thus the intersection $\bigcap_{n=1}^{\infty} A_n$ contains x, and only x.

($\Longleftarrow$) Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence of points in X. Define A_k to be the closure of the set $\bigcup_{n=k}^{\infty} B(x_n, \frac{1}{k})$. These A_k are decreasing, because

$$B\Big(x_n, \tfrac{1}{k+1}\Big) \subseteq B\big(x_n, \tfrac{1}{k}\big) \implies \bigcup_{n=k+1}^{\infty} B\Big(x_n, \tfrac{1}{k+1}\Big) \subseteq \bigcup_{n=k}^{\infty} B\big(x_n, \tfrac{1}{k}\big)$$

and because closures are monotone (i.e. $A \subseteq B \implies \overline{A} \subseteq \overline{B}$ — proof: the closure is defined as the intersection of all the closed sets containing the set, and all closed sets containing B contain A).

Furthermore, for any fixed $\epsilon > 0$, $\operatorname{diam} A_k < \epsilon$ for all k sufficiently large: because $\{x_n\}_{n=1}^{\infty}$ is Cauchy, we know that there is N s.t. $n, m \geq N \implies \rho(x_n, x_m) < \frac{\epsilon}{4}$; this means that choosing K big enough so that $K \geq N$ and $\frac{1}{K} < \frac{\epsilon}{3}$, we get that for any $k \geq K$ and any $y_1, y_2 \in A_k = \bigcup_{n=k}^{\infty} B(x_n, \frac{1}{k})$, y_1 and y_2 must each be in some $B(x_n, \frac{1}{k})$ (by definition of A_k), say $B(x_{n_1}, \frac{1}{k})$ and $B(x_{n_2}, \frac{1}{k})$ resp. (where $n_1, n_2 \geq k \geq K \geq N$), and

$$\rho(y_1, y_2) \leq \rho(y_1, x_{n_1}) + \rho(x_{n_1}, x_{n_2}) + \rho(x_{n_2}, y_2) \leq \frac{1}{k} + \frac{\epsilon}{4} + \frac{1}{k} < \frac{\epsilon}{3} + \frac{\epsilon}{4} + \frac{\epsilon}{3} = \frac{11}{12}\epsilon.$$

Because $\operatorname{diam} A_k$ is the supremum of all such $\rho(y_1, y_2)$, it must be $\leq \frac{11}{12}\epsilon < \epsilon$. To summarize, for every $\epsilon > 0$, we showed there is some K s.t. $k \geq K \implies \operatorname{diam} A_k < \epsilon$.

By the initial assumption, the intersection of all these A_k must therefore be some single element $x \in X$. From above, we know that for every $\epsilon > 0$, there is some K s.t. $k \geq K \implies \operatorname{diam} A_k < \epsilon$, and so in particular for $n \geq K$, $x_n \in B(x_n, \frac{1}{k}) \subseteq A_k$ and $x \in A_k$, so $\rho(x_n, x) < \epsilon$. In other words, the $\{x_n\}$ (an arbitrary Cauchy sequence) converge to $x \in X$, and so X is complete.

Problem 5a

We are given a non-decreasing function $g : \mathbb{R} \to [0, 1]$, and we want to prove that the set of discontinuities is countable. Suppose $x_0 \in \mathbb{R}$ is a point at which g is discontinuous. That means that for some $\epsilon_{x_0} > 0$, no matter what $\delta > 0$ one chooses, there is always some x' within δ of x s.t. $|f(x_0) - f(x')| \geq \epsilon_{x_0}$.

Now notice that because g is non-decreasing, for any $x \in \mathbb{R}$, $x' < x \implies g(x') \leq g(x)$, and so we must have that $\sup_{x'<x}\{g(x')\} \leq g(x)$. Similarly, $\inf_{x'>x}\{g(x')\} \geq g(x)$. Putting these together, we have that $a_x := \sup_{x'<x}\{g(x')\} \leq \inf_{x'>x}\{g(x')\} =: b_x$. We would like to prove that for our point of discontinuity x_0 above, $a_{x_0} < b_{x_0}$: suppose NOT; then $a_{x_0} = b_{x_0}$. By definition, this means that in particular, there must be $x_{-1} < x_0$ s.t. $g(x_{-1}) \geq g(x_0) - \frac{\epsilon_{x_0}}{3}$ and $x_1 > x_0$ s.t. $g(x_1) \leq g(x_0) + \frac{\epsilon_{x_0}}{3}$. But that would mean that

$$\begin{aligned} & |x' - x_0| < \min\{x_0 - x_{-1}, x_1 - x_0\} \\ \implies & x_0, x' \in (x_{-1}, x_1) \\ \implies & |g(x') - g(x_0)| \leq |g(x_1) - g(x_{-1})| = g(x_1) - g(x_{-1}) \leq 2 \cdot \frac{\epsilon_{x_0}}{3} < \epsilon_{x_0}, \end{aligned}$$

contradicting the above claim that "there is always some x' within δ of x s.t. $|f(x_0) - f(x')| \geq \epsilon_{x_0}$". Thus, it must be that $a_{x_0} < b_{x_0}$. Finally, pick some rational $r_{x_0} \in (a_{x_0}, b_{x_0})$ (which we know we can always do because rationals are dense). If we do this for all points of discontinuity, we get a set of rationals that satisfy $x_0 < x_0' \implies r_{x_0} < b_{x_0} := \inf_{x'>x_0}\{g(x')\} \leq \sup_{x'<x_0'}\{g(x')\} =: a_{x_0'} < r_{x_0'}$, and so we have an injective mapping from the points of discontinuity to the rationals which we know are countable, implying that the cardinality of the set of points of discontinuities is countable.

Problem 5b

We are given non-decreasing functions $f_1, f_2, \ldots : \mathbb{R} \to [0,1]$, and we want to prove that there is some $g : \mathbb{R} \to [0,1]$ s.t. there is a subsequence $\{n'(1), n'(2), \ldots\} \subset \mathbb{N}$ s.t. the subsequence $\{f_{n'(k)}\}_{k=1}^{\infty}$ converges pointwise to g everywhere on $\mathbb{R}$ except for maybe a countable subset $A \subset \mathbb{R}$.

First, let's construct a subsequence that converges pointwise to some function on $\mathbb{Q}$ (where $\mathbb{Q}$ was chosen because it's both countable and dense). Let $\{r_1, r_2, \ldots\}$ be a list of all rational numbers (indexed by all natural numbers). Now, consider the sequence $\{f_1(r_1), f_2(r_1), \ldots\} \subset [0,1]$. Because $[0,1]$ is closed and bounded, i.e. compact, it must have a convergent subsequence (the Bolzano-Weierstrass property from Theorem 0.25), say indexed by $N_1 := \{n_1(1), n_1(2), \ldots\} \subset \mathbb{N}$. We can then construct a subsubsequence $N_2 := \{n_2(1), n_2(2), \ldots\} \subseteq N_1$ of functions from the first subsequence that converge pointwise on r_2 (and also r_1 because it's is a subsequence of N_1). Thus it is clear that we can continue to this to construct a decreasing chain of subsequences $N_1 \supseteq N_2 \supseteq \ldots$ s.t. $\{f_{n_k(i)}\}_{i=1}^{\infty}$ converges pointwise on $\{r_1, \ldots, r_k\}$.

Now it'd be nice if we could say "continue this forever until we get a subsequence N_∞ contained within all the other N_k", but we always have to be very careful when dealing with ∞; for example if $N_k = \{z \in \mathbb{Z} : z \geq k\}$, then N_∞ would have to be $\varnothing$! So this approach of trying to find a special subsequence contained in all the N_k will not work; however, we actually do not need the subsequence to be completely contained within all the N_k! We only ever care about the "tail" behavior of the subsequence, so we can reframe the challenge to be to come up with a subsequence $\{n'(1), n'(2), \ldots\}$ whose "tail" will be in all the N_k (intuitively speaking). This can be accomplished by taking $n'(k)$ to be some index in N_k s.t. $n'(k) > n'(k-1) > \ldots > n'(2) > n'(1)$, to guarantee that the tail $\{n'(k), n'(k+1), \ldots\} \subseteq N_k$ — as an explicit example, we can take $n'(k) = n_k(k)$ (this satisfies the criteria because by nature of taking subsequences we have that $n_{k-1}(k) \leq n_k(k)$, and of course $n_{k-1}(k-1) < n_{k-1}(k)$).

Because $\lim_{i\to\infty} f_{n_k(i)}(r_k)$ exists (and say equal to some $\hat{g}(r_k)$), that means that for any $\epsilon > 0$, there is $I \in \mathbb{N}$ s.t. $i \geq I \implies |f_{n_k(i)}(r_k) - \hat{g}(r_k)| < \epsilon$. Furthermore, because $\{n'(k), n'(k+1), \ldots\}$ is a subsequence of N_k starting with $n'(k) = n_k(k)$, it must be that for any $i \geq k$, $n'(i) \geq n_k(i)$, implying that $n'(i) = n_k(i')$ for some $i' \geq i$. Putting everything together, we get that for $i \geq \max\{I, k\}$, $|f_{n'(i)}(r_k) - \hat{g}(r_k)| = |f_{n_k(i')}(r_k) - \hat{g}(r_k)|$ for some $i' \geq i \geq I$, meaning that $|f_{n'(i)}(r_k) - \hat{g}(r_k)| = |f_{n_k(i')}(r_k) - \hat{g}(r_k)| < \epsilon$; i.e. $\lim_{i\to\infty} f_{n'(i)}(r_k) = \hat{g}(r_k)$ as well.

Thus we now have a new function $\hat{g} : \{r_1, r_2, \ldots\} \to [0,1]$, which happens to be increasing. We can see this by contradiction: if $r_i < r_j$ yet $\hat{g}(r_i) > \hat{g}(r_j)$, then by the definition of pointwise convergence, there is $K_1, K_2 \in \mathbb{N}$ s.t.

$$k \geq K_1 \implies |f_{n'(k)}(r_i) - \hat{g}(r_i)| < \frac{\hat{g}(r_i) - \hat{g}(r_j)}{2} \implies f_{n'(k)(r_i)} > \hat{g}(r_i) - \frac{\hat{g}(r_i) - \hat{g}(r_j)}{2}$$

and

$$k \geq K_2 \implies |f_{n'(k)}(r_j) - \hat{g}(r_j)| < \frac{\hat{g}(r_i) - \hat{g}(r_j)}{2} \implies f_{n'(k)(r_j)} < \hat{g}(r_j) + \frac{\hat{g}(r_i) - \hat{g}(r_j)}{2}.$$

By monotonocity, $f_{n'(k)}(r_i) \leq f_{n'(k)}(r_j)$, and so putting everything together we have that for $k \geq \max\{K_1, K_2\}$,

$$\frac{\hat{g}(r_i) + \hat{g}(r_j)}{2} < f_{n'(k)}(r_i) \leq f_{n'(k)}(r_j) < \frac{\hat{g}(r_i) + \hat{g}(r_j)}{2}$$

which is very much impossible.

Now IF one could extend this $\hat{g}$ to some non-decreasing $g : \mathbb{R} \to [0,1]$ (extension in that $g(r_k) = \hat{g}(r_k)$), that would be great because from part (a), we know that non-decreasing functions are continuous everywhere except perhaps a countable set, and monotonicity forces pointwise convergence anywhere g is continuous! Let us prove this latter assertion first: suppose that x_0 is a point at which g is continuous; we want to show then that $\lim_{k\to\infty} f_{n'(k)}(x_0) = g(x_0)$. Alright then — let's fix some arbitrary $\epsilon > 0$. By continuity, there must be some $\delta > 0$ s.t. $|x - x_0| < \delta \implies |g(x) - g(x_0)| < \frac{\epsilon}{2}$. Because $\{r_1, r_2, \ldots\}$ is dense, there must be some $r_i \in (x_0 - \delta, x_0)$ and some $r_j \in (x_0, x_0 + \delta)$. By pointwise convergence, there must be $K \in \mathbb{N}$ s.t. $k \geq K \implies |f_{n'(k)}(r_i) - g(r_i)|, |f_{n'(k)}(r_j) - g(r_j)| < \frac{\epsilon}{2}$. Stitching together all these claims, we get that for our arbitrary chosen $\epsilon > 0$, we have K s.t. for $k \geq K$,

$$\left(g(x_0) - \frac{\epsilon}{2}\right) - \frac{\epsilon}{2} < g(r_i) - \frac{\epsilon}{2} < f_{n'(k)}(r_i) \leq f_{n'(k)}(x_0) \leq f_{n'(k)}(r_j) < g(r_j) + \frac{\epsilon}{2} < \left(g(x_0) + \frac{\epsilon}{2}\right) + \frac{\epsilon}{2}$$

which is just the definition of convergence rephrased a little bit.

We're almost done — we just have to fill in that big "IF" in the previous paragraph to finish the proof that $\{f_{n'(k)}\}_{k=1}^{\infty}$ converges to g at every point of continuity of g (so everywhere except perhaps a countable subset of $\mathbb{R}$). Fortunately, this is quite easy; we will show that there is a non-decreasing extension of $\hat{g}$ to the real line by explicitly constructing one such extension: define $g(x) = \sup_{r_i \leq x}\{\hat{g}(r_i)\}$. Then, for any $r \in \{r_1, r_2, \ldots\}$, $g(r) = \hat{g}(r)$ because by definition $g(r) \geq \hat{g}(r)$, and by monotonicity of $\hat{g}$, $r_i \leq r \implies \hat{g}(r_i) \leq \hat{g}(r)$ so the supremum must be $\leq \hat{g}(r)$. Lastly, it is obvious that g is non-decreasing because $A \subseteq B \implies \sup\{A\} \leq \sup\{B\}$.