# 525 Homework 9

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## Problem 1 - 6.3.31

Suppose that $1 \leq p_j \leq \infty$, and $\sum_{j=1}^n p_j^{-1} = r^{-1} \leq 1$. If $f_j \in L^{p_j}$ for $j \in [n]$, then we want to show that $\prod_{j=1}^n f_j \in L^r$ and $\| \prod_{j=1}^n f_j \|_r \leq \prod_{j=1}^n \|f_j\|_{p_j}$ (a generalized Hölder inequality). As Folland suggests, we start with the case $n = 2$.

For $n = 2$, we have $p_1, p_2 \in [1, \infty]$ s.t. $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{r} \leq 1$ (so $r \geq 1$). Note that $r = \infty \iff p_1, p_2 = \infty$, in which case for $f_1, f_2 \in L^\infty$, $|f_1 f_2| \leq_{\text{a.e.}} \|f_1\|_\infty \|f_2\|_\infty \implies \|f_1 f_2\|_\infty \leq \|f_1\|_\infty \|f_2\|_\infty$. This argument holds for all $n \in \mathbb{N}$, not just $n = 2$, so we will only be considering $r < \infty \iff$ at least one $p_j < \infty$ from now on.

Anyways, $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{r} \leq 1$ means that $\frac{r}{p_1}$ and $\frac{r}{p_2}$ are $\geq 0$ and $\leq 1$ and sum to 1, implying that $\frac{p_1}{r}$ and $\frac{p_2}{r}$ are $\geq 1$ and conjugate exponents (where at least one of them is $< \infty$). Thus, we can apply Hölder's inequality (Theorem 6.2 and Theorem 6.8(a), which we proved in 525hw7), which tells us that for $f_j \in L^{p_j}$,

$$\begin{aligned} \|f_1 f_2\|_r^r = \||f_1 f_2|^r\|_1 &= \int_X |f_1 f_2|^r \, d\mu \leq \||f_1|^r\|_{p_1/r} \||f_2|^r\|_{p_2/r} \\ &= \left( \int_X |f_1|^{r \cdot p_1/r} \, d\mu \right)^{r/p_1} \left( \int_X |f_1|^{r \cdot p_2/r} \, d\mu \right)^{r/p_2} = \|f_1\|_{p_1}^r \|f_2\|_{p_2}^r \end{aligned}$$

which yields that indeed $f_1 f_2 \in L^r$ with $\|f_1 f_2\|_r \leq \|f_1\|_{p_1} \|f_2\|_{p_2}$ (taking $p$th roots of the above inequality).

We proceed by induction. The base case $n = 2$ is finished, so let us fix some $N \in \mathbb{N}$ and assume (the induction hypothesis) that we have proved the claim for $n < N$. We have that $\frac{r}{p_N} + r(\sum_{j=1}^{N-1} \frac{1}{p_j}) = 1$, so we have $\frac{p_N}{r}$ and $(\sum_{j=1}^{N-1} \frac{1}{p_j})^{-1}/r$ are $\geq 1$ and conjugate exponents (where at least one of them is $< \infty$, since we are assuming at least one $p_j < \infty \iff r < \infty$). Thus, by the above $n = 2$ case (taking "$p_1$" to be $p_N$ and "$p_2$" to be $(\sum_{j=1}^{N-1} \frac{1}{p_j})^{-1}$), we have for $f_j \in L^{p_j}$ $(j \in [N])$:

$$\left\| \prod_{j=1}^N f_j \right\|_r \leq \|f_N\|_{p_N} \left\| \prod_{j=1}^{N-1} f_j \right\|_{(\sum_{j=1}^{N-1} \frac{1}{p_j})^{-1}} =: \|f_N\|_{p_N} \left\| \prod_{j=1}^{N-1} f_j \right\|_{r'} \leq \prod_{j=1}^N \|f_j\|_{p_j},$$

where the last inequality is from taking $r' = (\sum_{j=1}^{N-1} \frac{1}{p_j})^{-1} \geq 1$ s.t. $\sum_{j=1}^{N-1} \frac{1}{p_j} = \frac{1}{r'} \leq 1$ and applying the induction hypothesis for $n = N - 1$.

## Problem 2 - 6.3.34

Let $f$ be absolutely continuous on $[\epsilon, 1]$ for all $\epsilon \in (0,1)$ and $M := \int_0^1 x|f'(x)|^p\,dx < \infty$. We want to show that $\lim_{x\to 0} f(x)$ exists and is finite for $p > 2$; $|f(x)|/|\log x|^{1/2} \to 0$ as $x \to 0$ for $p = 2$; and $|f(x)|/x^{1-(2/p)} \to 0$ for $p \in [1, 2)$.

For $p \in [1, \infty)$, Hölder's inequality tells us that for $q$ the conjugate exponent (which we can apply because $f'$ is measurable on (0,1), by $f$ being absolutely continuous on every $[\epsilon, 1] \implies f' \in L^1([\epsilon, 1])$ by Theorem 3.35(c) in Folland; and because $x$ and $x^{-1/p}$ are measurable on (0,1) by continuity), $\int_0^1 |f'|\,dx \le \|x^{1/p}|f'(x)|\|_{L^p((0,1))}\|x^{-1/p}\|_{L^q((0,1))}$. For $p > 2$, $q = p/(p-1) \implies q/p < 1 \iff -q/p > -1$, so $\|x^{-1/p}\|^q_{L^q((0,1))} = \int_0^1 x^{-q/p}\,dx$ is finite. Of course, we have by our assumption that $\|x^{1/p}|f'(x)|\|_{L^p((0,1))} = M^{1/p} < \infty$, so in summary we now know that for $p > 2$, $\int_0^1 |f'|\,dx < \infty$ so $f' \in L^1([0,1])$ (if $p = \infty$, then our assumption that $\int_0^1 x|f'(x)|^p\,dx < \infty$ forces $|f'| \le_{\text{a.e.}} 1$, so in this case $f'$ is also in $L^1([0,1])$).

Then, defining $g$ on $[0,1]$ as $g(t) = f(1) + \int_1^t f'\,dx = g(1) - \int_t^1 f'\,dx$, we have by the Lebesgue FTC (see Theorem 3.35 in Folland) that $g$ is absolutely continuous on $[0,1]$. But because $f$ is absolutely continuous on every $[\epsilon, 1]$ for $\epsilon \in (0,1)$, it is the case that for any $t \in (0,1]$, $f(t)$ also equals $f(1) - \int_t^1 f'\,dx$; i.e. $g$ and $f$ agree on $(0,1]$. Thus, because $g$ absolutely continuous implies continuous on $[0,1]$, $\lim_{x\to 0} g(x)$ exists and equals $g(0)$, which is $< \infty$. But as we said, $g$ and $f$ agree on $(0,1]$, so $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x)$, so the limit exists and is finite, as desired.

Now **we consider the case** $p = 2$. Note that for any $t, s \in (0,1]$, $t < s$, the above application of Hölder's inequality still works with $t$ replacing "0" and $s$ replacing "1"; i.e. we have $\int_t^s |f'|\,dx \le (\int_t^s x|f'(x)|^2\,dx)^{1/2}(\int_t^s x^{-1}\,dx)^{1/2}$. Let us now fix arbitrary $\epsilon > 0$. Because we assumed $\int_0^1 x|f'(x)|^p\,dx < \infty$, i.e. $x|f'(x)|^2 \in L^1([0,1])$, Corollary 3.6 in Folland (or the Ex. 2.2.14 (i.e. 524hw5 Problem 2) and subsequently continuity of measure from above, Theorem 1.8(d)) tells us that there is some $\delta \in (0,1)$ s.t. for $s \in (0,\delta)$, $\int_t^s x|f'(x)|^2\,dx \le \int_0^s x|f'(x)|^2\,dx < (\frac{\epsilon}{2})^2$. Let us now fix such an $s \in (0,\delta)$. Again by the Lebesgue FTC, we have $f(t) = f(s) - \int_t^s f'\,dx$, and so

$$\begin{aligned} 0 \le \frac{|f(t)|}{|\log t|^{1/2}} &\le \frac{|f(s)| + \int_t^s |f'|\,dx}{|\log t|^{1/2}} \\ &\le \frac{|f(s)| + \frac{\epsilon}{2}(\log(s) - \log(t))^{1/2}}{|\log t|^{1/2}} \\ &\le \frac{|f(s)| + \frac{\epsilon}{2}(\log(1) - \log(t))^{1/2}}{|\log t|^{1/2}} = \frac{|f(s)|}{|\log t|^{1/2}} + \frac{\epsilon}{2}. \end{aligned}$$

Since $s \in (0,\delta)$ is fixed and we know $f$ is absolutely continuous (hence continuous) on any $[\eta, 1]$ for $\eta \in (0,1)$, $|f(s)|$ is some finite quantity, and as $t \to 0$, $\frac{|f(s)|}{|\log t|^{1/2}} \to 0$. This tells us that for some $\delta' > 0$, $\frac{|f(s)|}{|\log t|^{1/2}} < \frac{\epsilon}{2}$, and so for $t \in (0, \min\{s, \delta'\})$, we have that $0 \le \frac{|f(t)|}{|\log t|^{1/2}} < \epsilon$. Since $\epsilon > 0$ was fixed arbitrarily above, we have proved that $|f(x)|/|\log x|^{1/2} \to 0$ as $x \to 0$.

For **the case** $p \in (1,2)$, it is much the same, but this time Hölder's inequality says (for $t, s \in (0,1)$, $t < s$): $\int_t^s |f'|\,dx \le (\int_t^s x|f'(x)|^p\,dx)^{1/p}(\int_t^s x^{-1/(p-1)}\,dx)^{(p-1)/p}$. Fixing $\epsilon > 0$ arbitrary and finding a corresponding $s \in (0,\delta)$ s.t. $(\int_t^s x|f'(x)|^p\,dx)^{1/p} \le (\int_0^s x|f'(x)|^p\,dx)^{1/p} < \epsilon$ (completely analogous to what we did above), we have

$$\begin{aligned}
0 \le \frac{|f(t)|}{t^{1-(2/p)}} &\le \frac{|f(s)| + \int_t^s |f'|\,dx}{t^{1-(2/p)}} \\
&\le \frac{|f(s)| + \epsilon[-(1-\frac{1}{p-1})^{-1}(t^{1-1/(p-1)} - s^{(1-1/(p-1))})]^{(p-1)/p}}{t^{1-(2/p)}} \\
&\le \frac{|f(s)| + \epsilon[-(1-\frac{1}{p-1})^{-1}(t^{1-1/(p-1)} - 0)]^{(p-1)/p}}{t^{1-(2/p)}} \\
&= \frac{|f(s)| + \epsilon(\frac{p-1}{2-p})^{(p-1)/p} \cdot t^{(p-2)/p}}{t^{1-(2/p)}} = \frac{|f(s)|}{t^{1-(2/p)}} + \epsilon(\tfrac{p-1}{2-p})^{(p-1)/p}.
\end{aligned}$$

Like above, we know that $|f(s)|$ is a finite number, and again we can find $\delta' > 0$ s.t. for $t \in (0, \min\{s, \delta'\})$, $\frac{|f(s)|}{t^{1-(2/p)}} < \epsilon$. Then, the expression $\epsilon + \epsilon(\frac{p-1}{2-p})^{(p-1)/p})$ goes to 0 as $\epsilon > 0$, telling us that for any $\eta > 0$, we can find $\epsilon > 0$ sufficiently small s.t. we get $0 \le \frac{|f(t)|}{t^{1-(2/p)}} < \eta$ for all small enough $t \in (0,1)$. This is exactly the definition of convergence, so we are done with this case.

**Finally, we have just** $p = 1$. This time Hölder's inequality says (for $t, s \in (0,1)$, $t < s$): $\int_t^s |f'|\,dx \le \int_t^s x|f'(x)|\,dx \cdot \|1_{[t,s]}x^{-1}\|_\infty = \frac{1}{t}\int_t^s x|f'(x)|\,dx$. Fixing $\epsilon > 0$ arbitrary and finding a corresponding $s \in (0,\delta)$ s.t. $\int_t^s x|f'(x)|\,dx < \frac{\epsilon}{2}$ (completely analogous to what we did above), we have $0 \le t|f(t)| \le t|f(s)| + t\frac{1}{t}\frac{\epsilon}{2} = t|f(s)| + \frac{\epsilon}{2}$. Like above, we know that $|f(s)|$ is a finite number, so choosing $\delta' = \frac{\epsilon}{2|f(s)|}$ we have $t \in (0, \min\{s, \delta'\}) \implies 0 \le t|f(t)| < \epsilon$. Thus convergence is proven, and we are done with Problem 2.

## Problem 3

Suppose $f \in L^1([0,1])$. We want to show that if $\int_0^1 f(x)(\sin x)^n\,dx = 0$ for all $n \in \mathbb{N}$, then $f =_{\text{a.e.}} 0$. Let us define $S$ to be the set of finite linear combinations (coefficients in $\mathbb{R}$) and products of the $\{(\sin x)^n\}_{n=1}^\infty$ (i.e. $S$ is the set of polynomials in $(\sin x)$, $\mathbb{R}[\sin x]$). $S$ is an algebra, since such "polynomials" clearly form vector subspace of $C([0,1])$ with products of these "polynomials" also "polynomials" of this type. Also, because $\sin x$ is strictly increasing on $[0,1]$, $S$ does indeed separate points (for all $x, y \in [0,1]$, $x < y$, we have $\sin(x) < \sin(y)$).

By Stone-Weierstrass (or rather the corollary, Corollary 4.50 in Folland), which we can use because $[0,1]$ is a compact metric space, we have that $S$ is dense in $\{f \in C([0,1],\mathbb{R}) : f(0) = 0\}$, since for every $n \in \mathbb{N}$, $(\sin 0)^n = 0$. We know that $C([0,1])$ is dense in $L^1([0,1])$ (Lec 2/19/21 proved that in particular that $C_c(\mathbb{R})$ is dense in $L^1(\mathbb{R})$, so restricting to $[0,1]$ gives us denseness of $C([0,1])$ in $L^1([0,1])$). We show that $\{f \in C([0,1]) : f(0) = 0\}$ is also dense in $L^1([0,1])$, by showing for all $\epsilon > 0$ and $g \in C([0,1])$, there is $f \in \{f \in C([0,1]) : f(0) = 0\}$ s.t. $\|f - g\|_1 < \frac{\epsilon}{2}$ (this suffices since for every $h \in L^1([0,1])$, denseness of $C([0,1])$ gives $g \in C([0,1])$ s.t. $\|h-g\|_1 < \frac{\epsilon}{2}$, so $\|f-h\|_1 < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$).

Well, for any $g \in C([0,1])$, it is a continuous function on a compact set, so it's bounded: there is $M \in \mathbb{N}$ s.t. $|g| \leq M$ on $[0,1]$. Defining $\phi$ as $\phi(x) = \frac{2M}{\epsilon} x \cdot 1_{[0,\epsilon/2M]} + 1_{(\epsilon/2M,1]}$, we have that $\phi$ is continuous, and $0 \leq \int_0^1 |g - \phi g|\, dx = \int_0^{\epsilon/2M} |1-\phi| \cdot |g|\, dx \leq \int_0^{\epsilon/2M} 1 \cdot M\, dx = \frac{\epsilon}{2}$. Thus, $\phi g$ is in $\{f \in C([0,1]) : f(0) = 0\}$ and satisfies $\|g - \phi g\|_1 \leq \frac{\epsilon}{2}$, so we have successfully proven that $\{f \in C([0,1]) : f(0) = 0\}$ is dense in $L^1([0,1])$.

Now let us consider $h := \text{sign}(f) \in L^1([0,1])$ (the fixed $f \in L^1([0,1])$ from sentence 1). By the result we just proved, there is some sequence of $s_n \in S$ that converge to $h$ in $L^1$. By Corollary 2.32 in Folland, there is a subsequence $\{s_{n_k}\}_{k=1}^{\infty} \subseteq \{s_n\}_{n=1}^{\infty}$ that converges to $h$ a.e. on [0,1]. By Egoroff's theorem (Theorem 2.33), there is some $E \subseteq [0,1]$ s.t. $s_{n_k} \to h$ uniformly on $E$, and $m(E^{\complement}) < \eta$ for some fixed $\eta > 0$ (that we'll specify later). Let us now fix arbitrary $\epsilon > 0$. Uniform convergence gives that there is $N \in \mathbb{N}$ s.t. $n \geq N \implies |h - s| < \frac{\epsilon}{3\|f\|_1}$ on $E$, where $\|f\|_1$ is just some finite number because $f \in L^1([0,1])$. Corollary 3.6 in Folland says that because $f \in L^1([0,1]) \iff |f| \in L^1([0,1])$, there is some $\delta_1 > 0$ s.t. $m(A) < \delta_1 \implies \int_A |f|\, dx < \frac{\epsilon}{3}$. Because $s_N$ is continuous on a compact set, it is bounded, so $fs_N$ is also in $L^1([0,1])$, implying that there is some $\delta_2 > 0$ s.t. $m(A) < \delta_2 \implies \int_A |fs_N|\, dx < \frac{\epsilon}{3}$. Let $\delta = \min\{\delta_1, \delta_2\}$.

Because all the $s_n \in S$ (i.e. are linear combinations of powers of $\sin x$) and we know $\int_0^1 f(x)(\sin x)^n\, dx = 0$ for all $n \in \mathbb{N}$, $\int_0^1 f s_n\, dx$ also $= 0$ for all $n \in \mathbb{N}$. Thus, choosing $\eta < \delta$, we have

$$\begin{aligned}
0 \leq \int_0^1 |f|\, dx &= \left|\int_0^1 |f|\, dx\right| = \left|\int_{E^{\complement}} |f|\, dx + \int_E |f|\, dx\right| \\
&= \left|\int_{E^{\complement}} |f|\, dx + \int_E fh\, dx - \int_E fs_N\, dx - \int_{E^{\complement}} fs_N\, dx\right| \\
&\leq \frac{\epsilon}{3} + \left|\int_E f(h - s_N)\, dx\right| + \int_{E^{\complement}} |fs_N|\, dx \\
&\leq \frac{\epsilon}{3} + \int_E |f| \cdot |h - s_N|\, dx + \frac{\epsilon}{3} \leq \int_E |f| \cdot \frac{\epsilon}{3\|f\|_1}\, dx + \frac{2\epsilon}{3} = \epsilon
\end{aligned}$$

Because $\epsilon > 0$ was arbitrarily chosen, this proves that $\int_0^1 |f|\, dx = 0$, so by Prop. 2.23 in Folland, $f =_{\text{a.e.}} 0$.

*Remark:* this proof (in particular the part starting with "Now let us consider $h$...") is general enough to prove that if an set $S$ is dense in $L^1(X,\mu)$ (or dense in $C_c(X)$), where $X$ is some subset of some Euclidean space, and each $s \in S$ is bounded, $\mu(X) < \infty$ (required for Egoroff and perhaps various other things, like $\text{sign}(f) \in L^1(X)$), and $\int_X fs\, d\mu = 0$ for all $s \in S$, then $f =_{\text{a.e.}} 0$. Simply replace all references to $[0,1]$ above by $X$ and all integrals $\int_0^1$ by integrals $\int_X$.

## Problem 4

Let $f$ be an arbitrary fixed Lebesgue measurable function on $\mathbb{R}$. For $p \in \mathbb{R}$, define $F(p) = \int_{\mathbb{R}} |f|^p\, dt$, where we use the convention $0^0 = 0$. We want to show that $I := \{p \in \mathbb{R} : F(p) < \infty\}$ is a connected subset of $\mathbb{R}$ and that $\log F$ is a convex function on this set.

We will prove that $I$ is an interval; i.e. for any $a, b \in I$, $a < b$, then for all $c \in (a, b)$, $c \in I$ as well (it is an elementary/undergraduate analysis fact that the connected subsets of $\mathbb{R}$ are exactly the intervals). For any $r \in [1, \infty)$, the function $r^x$ defined on $\mathbb{R}$ is an increasing function (strictly increasing if $r > 0$, and weakly increasing/constant for $r = 1$), and for $r \in [0, 1)$, the function $r^x$ is decreasing (if $r = 0$, $r^x = \infty$ for $x < 0$ and 0 for $x \geq 0$).

Thus, supposing $a, b \in I$, $a < b$, then we have for any $c \in (a, b)$ that $|f|^c \leq |f|^b$ on $[|f| \geq 1]$, and $|f|^c \leq |f|^a$ on $[|f| < 1]$. Thus, $\int_{\mathbb{R}} |f|^c \, dt = \int_{[|f|\geq 1]} |f|^c \, dt + \int_{[|f|<1]} |f|^c \, dt \leq \int_{[|f|\geq 1]} |f|^b \, dt + \int_{[|f|<1]} |f|^a \, dt \leq \int_{\mathbb{R}} |f|^b \, dt + \int_{\mathbb{R}} |f|^a \, dt < \infty$. This proves that $I$ is an interval as desired, so now we prove log convexity.

We want to show that for any $a, b \in I$, $a < b$, and $c \in (a, b)$ i.e. $c = \lambda a + (1 - \lambda)b$ for some $\lambda \in (0, 1)$, $\log(F(c)) \leq \lambda \log(F(a)) + (1 - \lambda) \log(F(b)) = \log(F(a)^\lambda F(b)^{1-\lambda}) \iff F(c) \leq F(a)^\lambda F(b)^{1-\lambda}$. This follows from Hölder's inequality applied for $p = \frac{1}{\lambda}, q = \frac{1}{1-\lambda} \in (1, \infty)$:

$$F(c) = \int_{\mathbb{R}} |f|^{a\lambda} |f|^{(1-\lambda)b} \, dx \leq \left( \int_{\mathbb{R}} |f|^{a\lambda(1/\lambda)} \, dx \right)^\lambda \left( \int_{\mathbb{R}} |f|^{(1-\lambda)b(1/(1-\lambda))} \, dx \right)^{1-\lambda} = F(a)^\lambda F(b)^{1-\lambda}.$$

## Problem 5

Suppose $K \in L^p(X)$ for $X = [0, 1] \times [0, 1]$ for some $p \in (1, \infty)$. Let $q$ be the conjugate exponent of $p$.

(a) For $f \in L^q([0, 1])$, let $[Af](x) = \int_0^1 K(x, y) f(y) \, dy$. We want to show that $[Af](x)$ exists for almost every $x \in [0, 1]$ (w.r.t 1-dim Lebesgue measure), and that $A$ is a bounded linear map from $L^q([0, 1])$ to $L^p([0, 1])$.

We **first show** $K(x, y)f(y) \in L^1(X)$. If we show this, then Fubini (Theorem 2.37(b) in Folland) tells us that $[Af](x) = \int_{[0,1]} K(x, y) f(y) \, dy$ is in $L^1([0, 1])$, which in particular implies that $[Af](x)$ exists for a.e. $x \in [0, 1]$. So we now consider the integral $\int_X |K(x, y)f(y)| \, d[m \times m]$, where $K(x, y)f(y) \in L^+(X)$ because $K(x, y) \in L^p(X)$ is measurable, and the inverse image of a Borel set $B \in \mathcal{B}_{\mathbb{R}}$ w.r.t $f(x, y) := f(y)$ is just $[0, 1] \times f^{-1}(B)$, i.e. a measurable rectangle in $X$ (see opening paragraphs of §2.5 in Folland). By Tonelli (Theorem 2.37(a)), $\int_X |K(x, y)f(y)| \, d[m^2] = \int_0^1 |f(y)| (\int_0^1 |K(x, y)| \, dx) \, dy$. By Hölder's inequality, this is $\leq \|f\|_q (\int_0^1 (\int_0^1 |K(x, y)| \, dx)^p \, dy)^{1/p}$.

Minkowski's inequality for integrals (Theorem 6.19) gives that $(\int_0^1 (\int_0^1 |K(x, y)| \, dx)^p \, dy)^{1/p} \leq \int_0^1 (\int_0^1 |K(x, y)|^p \, dx)^{1/p} \, dy$. Since the measure of $[0, 1]$ is 1 and $g(x) := x^{1/p}$ is concave, reverse Jensen's inequality (which can be easily proven by applying Jensen's inequality (which I proved on Problem 5 of the 524 final) to $-g$) says that $\int_0^1 (\int_0^1 |K(x, y)|^p \, dx)^{1/p} \, dy \leq (\int_0^1 \int_0^1 |K(x, y)|^p \, dx \, dy)^{1/p}$. But the double integral inside the parentheses is exactly $\int_X |K|^p \, d[m^2]$ by Tonelli, so combining all these inequalities together we get that $\int_X |K(x, y)f(y)| \, d[m^2] \leq \|f\|_{L^q([0,1])} \|K\|_{L^p(X)}$. Since $K \in L^p(X)$ and $f \in L^q([0, 1])$, the RHS is finite, so indeed $K(x, y)f(y) \in L^1(X)$. To reiterate, this proves that $[Af](x) \in L^1([0, 1])$.

Now **for the bounded linear part**. By linearity of the integral for $L^1$ functions and distributativity of multiplication, it is clear that $A$ is a linear map on $L^q([0,1])$. The above two paragraphs show that $A$ is a bounded linear map from $L^q([0,1])$ to $L^1([0,1])$, so this step is to strengthen that to be to $L^p([0,1])$ instead.

Look now at the second part of Minkowski's inequality for integrals, Theorem 6.19(b): if we prove that for a.e. $y \in [0,1]$, the function $x \mapsto K(x,y)f(y) \in L^p([0,1])$ and the function $y \mapsto (\int_0^1 |K(x,y)f(y)|^p\,dx)^{1/p}$ is in $L^1([0,1])$, then $\|Af\|_{L^p([0,1])} \le \int_0^1 (\int_0^1 |K(x,y)f(y)|^p\,dx)^{1/p}\,dy = \int_0^1 |f(y)|(\int_0^1 |K(x,y)|^p\,dx)^{1/p}\,dy$. Now applying Hölder's inequality, we see that this quantity is $\le \|f\|_q(\int_0^1 |(\int_0^1 |K(x,y)|^p\,dx)^{1/p}|^p\,dy)^{1/p} = \|f\|_q(\int_0^1\int_0^1 |K(x,y)|^p\,dx\,dy)^{1/p} = \|f\|_q\|K\|_{L^p(X)}$. This shows that for any $f \in L^q([0,1])$, $Af$ is indeed in $L^p([0,1])$, and $\|Af\|_p \le \|f\|_q\|K\|_{L^p(X)}$, implying that $A$ is a bounded (with operator norm $\|A\| \le \|K\|_{L^p(X)}$) linear map from $L^q([0,1]) \to L^p([0,1])$. We are almost done; we just have to prove those pesky "ifs" from above.

**Pesky "if" 1:** since $K \in L^p(X) \implies |K(x,y)|^p \in L^1(X)$, one of the conclusions of Fubini's theorem (Theorem 2.37) say that for a.e. fixed $y \in [0,1]$, $|K(x,y)|^p$ (as a function of $x$ on $[0,1]$) is in $L^1([0,1])$, i.e. for a.e. fixed $y \in [0,1]$ (say $y \in E_1$, for $E_1 \subseteq [0,1]$ with $E_1^{\complement}$ having measure 0), $K(x,y) \in L^p([0,1])$. Since $f(y) \in L^1([0,1])$, for a.e. $y \in [0,1]$ (say $y \in E_2$, for $E_2 \subseteq [0,1]$ with $E_2^{\complement}$ having measure 0) $f(y)$ is just some finite number, so for all fixed $y \in E_1 \cap E_2$ (which is a.e. $y \in [0,1]$ since $(E_1 \cap E_2)^{\complement} = E_1^{\complement} \cup E_2^{\complement}$ also has measure 0), the function $x \mapsto K(x,y)f(y)$ is in $L^p([0,1])$.

**Pesky "if" 2:** to check whether the function $y \mapsto (\int_0^1 K(x,y)f(y)\,dx)^{1/p}$ is in $L^1([0,1])$, we consider the integral $\int_0^1 (\int_0^1 |K(x,y)f(y)|^p\,dx)^{1/p}\,dy = \int_0^1 |f(y)|(\int_0^1 |K(x,y)|^p\,dx)^{1/p}\,dy$. But we have already considered this integral, two paragraphs ago! In that paragraph, we showed that this integral is $\le \|f\|_q\|K\|_{L^p(X)}$, which is indeed $< \infty$ since $f \in L^1([0,1])$ and $K \in L^p(X)$. Thus, all "if"s are checked, and we are done.

(b) If for every $f \in L^q([0,1])$, $[Af](x) = 0$ for almost every $x \in [0,1]$ (w.r.t. 1-dim Lebesgue measure), we want to show that $K = 0$ for a.e. $(x,y) \in [0,1]\times[0,1]$ (w.r.t. 2-dim Lebesgue measure).

We will use the "*Remark*" from the end of Problem 3 above. As we have been doing, let $X = [0,1] \times [0,1]$. This is a compact metric space, so we can use Stone-Weierstrass. Define the set $S \subseteq C(X)$ to be the set of all finite linear combinations of products $\{g(x)h(y) : g, h \in C([0,1])$, we see that $S$ is an algebra because such finite linear combinations of these $g(x)h(y)$ do indeed form a vector space over $\mathbb{R}$ (subspace of $C(X)$), and products of such functions are again functions of this form. Note that $S$ also separates points, since the functions $f_1(x,y) = x$ and $f_2(x,y) = y$ separate points with different $x$ and $y$ coordinates respectively. Since $S$ also contains the constant functions, Corollary 4.50 gives that $S$ is dense in $C(X)$. Note that all $s \in S \subseteq C(X)$ are continuous functions on a compact set, hence bounded, and also for all $g \in C([0,1])$, we have $g \in C([0,1]) \subseteq L^\infty([0,1]) \subseteq L^q([0,1])$ (see Prop. 6.12 in Folland).

Checking the "*Remark*" from the end of Problem 3 again, the only thing we have yet to prove is that $\int_X Ks\, d[m^2] = 0$ for all $s \in S$. By linearity of the integral, we only need to prove that for any $g, h \in C([0,1]) \subseteq L^1([0,1])$, $\int_X K(x,y)g(x)h(y)\, d[m^2] = 0$. Since $K \in L^p(X) \subseteq L^1(X)$ and $g, h$ (continuous functions on a compact set, hence bounded) are both bounded in absolute value by say some $M < \infty$, $K(x,y)g(x)h(y) \in L^1(X)$, so by Fubini, we have $\int_X K(x,y)g(x)h(y)\, d[m^2] = \int_0^1 g(x)(\int_0^1 K(x,y)h(y)\, dy)\, dx = \int_0^1 g(x)[Ah](x)\, dx$. But recall we are assuming that for every $f \in L^1([0,1])$, $[Af](x) = 0$ for a.e. $x \in [0,1]$, and so $\int_0^1 g(x)[Ah](x)\, dx = \int_0^1 0\, dx = 0$. Thus, we have proved that $\int_X Ks\, d[m^2] = 0$ for all $s \in S$, and having fulfilled all the other requirements I set in the Problem 3 "*Remark*", we finally reach the conclusion that $K =_{\text{a.e.}} 0$, as desired.

# 525 Homework 8

Daniel Rui - 3/2/21

## Problem 1 - 6.2.20

Let us suppose we have some functions $f_n \in L^p$ satisfying $M := \sup_{n\in\mathbb{N}} \|f_n\|_p < \infty$ and $f_n \to f$ a.e. (this suffices to show $f \in L^p$ with $\|f\|_p \le M$, by Fatou: $\int |f|^p \, d\mu \le \liminf_{n\to\infty} \int |f_n|^p \, d\mu \le M^p < \infty$).

(a) If $p \in (1,\infty)$, we show that $f_n \to f$ weakly in $L^p$. Recall from Lec 2/24/21 that $f_n \to f$ weakly in $L^p$ means that for every $g \in L^q$ (for $q$ conjugate to $p$), $\int f_n g \, d\mu \to \int fg \, d\mu$. By Theorem 6.15, this is equivalent (since we are only considering $p \in (1,\infty)$) to $\phi(f_n) \to \phi(f)$ for every $\phi \in (L^p)^*$. We first prove the three little lemmas that Folland suggests proving: given $g \in L^q$ where $q$ is the conjugate exponent to $p$, and arbitrary $\epsilon > 0$, we show

(i) there is $\delta > 0$ s.t. $\mu(E) < \delta \implies \int_E |g|^q \, d\mu < \epsilon$. This is just Corollary 3.6 in Folland, which applies since $|g|^q \in L^1(\mu)$.

(ii) there is $A \subseteq X$ s.t. $\mu(A) < \infty$ and $\int_{A^\complement} |g|^p \, d\mu < \epsilon$. By Prop. 2.20 in Folland (which we proved in 524hw5 Problem 1), we have that since $|g|^p \in L^1$, the set $E_0 := [|g|^p > 0]$ is $\sigma$-finite, i.e. it equals $\bigcup_{n=1}^\infty E_n$ for measurable sets $E_n$ all with finite measure. Moreover, by Ex. 2.2.14 (i.e. 524hw5 Problem 2), $\lambda(E) := \int_E |g|^p \, d\mu$ is a measure. We know that $\bigcup_{n=1}^N E_n$ is an increasing sequence of sets (as $N \to \infty$) s.t. $\lambda(\bigcup_{n=1}^\infty E_n) = \lambda(E_0) = \int_X |g|^p \, d\mu$, so by continuity of measure from below (see Theorem 1.8(c) in Folland), we have that $\lim_{N\to\infty} \lambda(\bigcup_{n=1}^N E_n) = \int_X |g|^p \, d\mu < \infty$, where the $< \infty$ furthermore gives us that there is $N \in \mathbb{N}$ s.t. $\lambda(\bigcup_{n=1}^N E_n) \ge \int_X |g|^p \, d\mu - \epsilon$. Taking $A$ to be exactly this set $\bigcup_{n=1}^N E_n$, indeed $\mu(A) \le \sum_{n=1}^N \mu(E_n) < \infty$ and $\int_{A^\complement} |g|^p \, d\mu = \lambda(X \setminus A) = \lambda(X) - \lambda(A) < \epsilon$.

(iii) there is $B \subseteq A$ s.t. $\mu(A \setminus B) < \delta$ and $f_n \to f$ uniformly on $B$. This is exactly the conclusion of Egoroff's theorem (which we can apply since $\mu(A) < \infty$ and $f_n \to f$ a.e.).

With those little results in mind, we can begin the problem. Let us fix $g \in L^q$ and arbitrary $\epsilon > 0$. We use (i), (ii), and (iii) above to get $\delta$, $A$, and $B$ with the above properties. By uniform convergence of $f_n \to f$ on $B$, we have some $N \in \mathbb{N}$ s.t. $n \ge N \implies |f_n - f| < \epsilon$ on $B$ (everywhere). Then, we have for $n \ge N = N_\epsilon$,

$$\begin{aligned}
\left|\int_X f_n g \, d\mu - \int_X fg \, d\mu\right| &\le \int_X |f_n - f||g| \, d\mu = \||f_n - f| \cdot |g|\|_1 \\
&\le \||f_n - f||g|1_{A^\complement}\|_1 + \||f_n - f||g|1_{A\setminus B}\|_1 + \||f_n - f||g|1_B\|_1 \\
&\le \|f_n - f\|_p \||g|1_{A^\complement}\|_q + \|f_n - f\|_p \||g|1_{A\setminus B}\|_q + \||f_n - f|1_B\|_p \|g\|_q \\
&\le \big(\|f_n\|_p + \|f\|_p\big)\epsilon^q + \big(\|f_n\|_p + \|f\|_p\big)\epsilon^q + \epsilon^p \|g\|^p \le 4M\epsilon^q + \epsilon^p \|g\|_q.
\end{aligned}$$

Because $M$ and $\|g\|_q$ are finite, the quantity on the RHS goes to 0 as $\epsilon \to 0$. Thus for any $\eta > 0$, we can find $\epsilon > 0$ small in enough s.t. for $n \ge N_\epsilon$, we get that $|\int_X f_n g \, d\mu - \int_X fg \, d\mu| < \eta$. This is exactly what it means for $\int_X f_n g \, d\mu$ to converge to $\int_X fg \, d\mu$, and because $g \in L^q$ was arbitrary, we have indeed shown that $f_n \to f$ weakly in $L^p$.

(b) The result in part (a) is not true in general for $p = 1$; we give two counterexamples, one in $L^1(\mathbb{R}, m)$ and one in $l^1(\mathbb{N})$. For the counterexample in $L^1(\mathbb{R}, m)$, we showed in Problem 3(b) below that the sequence $f_n(x) = n \cdot 1_{(0,1/n)}(x)$ converges a.e. (actually everywhere) to 0, but $f_n \not\to 0$ weakly in $L^p$ for any $p \in [1, \infty]$ (see the details below); moreover it is clear that for any $n \in \mathbb{N}$, $\|f_n\|_1 = n \cdot \frac{1}{n} = 1$, and so $\sup_{n\in\mathbb{N}} \|f_n\|_1 = 1 < \infty$.

Now for the $l^1(\mathbb{N})$ counterexample. Consider $f_n(x) = 1_{\{n\}}(x)$. For any $N \in \mathbb{N}$, for any $n > N$ we have $f_n(N) = 0$, so indeed $f_n \to 0$ everywhere on $\mathbb{N}$. Also, for every fixed $n \in \mathbb{N}$, $\|f_n\|_1 = \sum_{m=1}^{\infty} f_n(m) = 1$, so $\sup_{n\in\mathbb{N}} \|f_n\|_1 = 1 < \infty$. To show $f_n \not\to 0$ weakly in $L^1$, it suffices to find $\phi \in (l^1)^*$ s.t. $\phi(f_n) \not\to 0$. By Theorem 6.15 (which we can apply because counting measure on $\mathbb{N}$ is $\sigma$-finite), this is equivalent to finding $g \in l^\infty$ s.t. $\int f_n g \not\to \int 0 \cdot g = 0$; simply taking $g = 1$ on $\mathbb{N}$ (which is in $l^\infty$) we get $\int f_n g = \int f_n = 1$ for all $n \in \mathbb{N}$, which indeed does not converge to 0 as $n \to \infty$. And so, we are done.

## Problem 2 - 6.2.21

If $p \in (1, \infty)$, we want to show that $f_n \to f$ weakly in $l^p(A) \iff \sup_{n\in\mathbb{N}} \|f_n\|_p < \infty$ and $f_n \to f$ pointwise (everywhere). Note that the ( $\Longleftarrow$ ) direction is covered by Problem 1(a) (above), since $l^p(A)$ for any set $A$ is just $L^p$ on $A$ with counting measure, and of course pointwise everywhere convergence implies (pointwise) a.e. convergence. Thus, we only need to focus on ( $\Longrightarrow$ ):

So we are given a sequence $\{f_n\}_{n=1}^{\infty}$ that converges weakly in $l^p(A)$ to some $f$. That is to say, for every $\phi \in (L^p)^*$ (or equivalently every $h \in l^q(A)$), $\phi(f_n) \to \phi(f)$ (or equivalently $\int f_n h = \sum_{a\in A} f_n(a)h(a) \to \int fh = \sum_{a\in A} f(a)h(a)$; see Theorem 6.15 in Folland). In particular, for every $a \in A$, we can take $h = 1_{\{a\}} \in l^q(A)$ to get that $f_n(a) \to f(a)$ for every $a \in A$. That gives pointwise everywhere convergence, so we all that's left is bounded sup of norms.

Recall (from pg. 159 of Folland) that we can define (for each $n \in \mathbb{N}$) $\widehat{f_n} \in (L^p)^{**}$ by $\widehat{f_n}(\phi) = \phi(f_n)$. Theorem 5.8 (also on pg. 159) tells us that $\|\widehat{f_n}\| = \|f_n\|_p$ (where the norm on the LHS is the operator norm defined on $(L^p)^{**}$). Anyways, we have for all fixed $\phi \in (L^p)^*$ that $|\widehat{f_n}(\phi)| = |\phi(f_n)|$; but because $\phi(f_n) \to \phi(f)$ (convergence of real/complex numbers), we have $\lim_{n\to\infty} |\phi(f_n)| = |\phi(f)| < \infty$ (by continuity of $|\cdot|$), and so $\sup_{n\in\mathbb{N}} |\phi(f_n)| < \infty$ (since $|\phi(f_n)|$ is finite for each $n \in \mathbb{N}$, and we know the limit is $< \infty$). Thus, we now know (again for any fixed $\phi \in (L^p)^*$) that $\sup_{n\in\mathbb{N}} |\widehat{f_n}(\phi)| < \infty$. By Theorem 5.13 (the uniform boundedness principle), this implies that $\sup_{n\in\mathbb{N}} \|\widehat{f_n}\| = \sup_{n\in\mathbb{N}} \|f_n\|_p$ is finite, as desired.

## Problem 3 - 6.2.22

Let $X = [0, 1]$ with Lebesgue measure.

(a) Let $f_n(x) = \cos(2\pi n x)$. We want to show that $f_n \to 0$ weakly in $L^2$, but $f_n \not\to 0$ a.e. or in measure. As Folland suggests, we use Ex. 5.5.63 in Folland which says that any orthonormal sequence in an infinite dimensional Hilbert space $\mathscr{H}$ converges weakly to 0.

We first prove this exercise. As I wrote above in Problem 1(a), $f_n \to 0$ weakly in $\mathscr{H} := L^2(X, m)$ means that for any $\phi \in \mathscr{H}^*$, $\phi(f_n) \to \phi(0) = 0$ as $n \to \infty$; but by the Riesz representation theorem (Theorem 5.25 in Folland), we have that all $\phi \in \mathscr{H}^*$ can be written as as $\phi(f) = \langle f, h\rangle$ for some $h \in \mathscr{H}$. That means that Ex. 5.5.63 is equivalent to showing that for an orthonormal set $\{u_n\}_{n=1}^\infty$, for every $h \in \mathscr{H}$, $\langle u_n, h\rangle \to \langle 0, h\rangle = 0$. This is quite easy, since by Bessel's inequality (Theorem 5.26), we have that $\sum_{n=1}^\infty |\langle u_n, h\rangle|^2 = \sum_{n=1}^\infty |\langle h, u_n\rangle|^2 \le \|h\|^2 < \infty$ (where the $< \infty$ is from $h \in L^2$, and the first equality is because $|\langle x, y\rangle| = |\langle y, x\rangle|$. As $\sum_{n=1}^\infty |\langle u_n, h\rangle|^2$ is a convergent series of numbers, it must be that $|\langle u_n, h\rangle| \to 0$ as $n \to \infty$, which proves that $\langle u_n, h\rangle \to 0$ as $n \to \infty$.

With this exercise in mind, we now see if the $f_n$ defined above are an orthonormal sequence in $L^2([0,1])$. Observe that (using the elementary trig identities $\cos(x)\cos(y) = \frac{1}{2}[\cos(x-y) + \cos(x+y)]$ and for any $n \in \mathbb{N}$ not equal to 0, $\int_0^1 \cos(2\pi n x)\,dx = (\frac{1}{2\pi n}\sin(2\pi n x))|_0^1 = 0$) for any $n, m \in \mathbb{N}$,

$$\begin{aligned}\int_0^1 f_n f_m\,dx &= \int_0^1 \tfrac{1}{2}[\cos(2\pi(n-m)x) + \cos(2\pi(n+m)x)]\,dx \\ &= \tfrac{1}{2}\int_0^1 \cos(2\pi(n-m)x)\,dx = \tfrac{1}{2}\cdot 1_{[n=m]}.\end{aligned}$$

Thus, although $\{f_n\}_{n=1}^\infty$ is not quite an orthonormal set, the $g_n := \sqrt{2}f_n$ are. By the exercise, this means that the $g_n$ weakly converge to 0, i.e. for all $\phi \in (L^2)^*$, $\phi(g_n) = \phi(\sqrt{2}f_n) \to 0$, but by linearity of linear functionals we have $\sqrt{2}\phi(f_n) \to 0 \implies \phi(f_n) \to 0$, implying that the $f_n$ are weakly convergent to 0 as well.

As for the **non-convergence claims**, first recall that a.e. convergence implies convergence in measure (if $X$ has finite measure, which in our case $X = [0,1]$ has $m([0,1]) = 1 < \infty$), by Egoroff's theorem (Theorem 2.33 in Folland), meaning that if we show $f_n \not\to 0$ in measure then $f_n \not\to 0$ a.e. too. We proceed by contradiction: let us assume that $f_n \to 0$ in measure. By Theorem 2.30 (and the fact that convergence in measure $\implies$ Cauchy in measure; see my proof of Problem 2 from 524hw7), convergence in measure implies that a subsequence converges almost everywhere. So now we have some subsequence $\{f_{n_k}\}_{k=1}^\infty \subseteq \{f_n\}_{n=1}^\infty$ s.t. $f_{n_k} \to 0$ a.e. as $k \to \infty$. This is equivalent to $f_{n_k}^2 \to_{\text{a.e.}} 0$ (since $x \to 0 \iff x^2 \to 0$) or $\lim_{k\to\infty} f_{n_k}(x) = 0$ for a.e. $x \in [0,1]$, and because $|f_{n_k}^2| = f_{n_k}^2 = \cos^2(2\pi n_k x) \le 1$ everywhere on $[0,1]$ for all $n_k$, where $1 \in L^1([0,1])$, the DCT tells us that

$$\lim_{k\to\infty}\int_0^1 f_{n_k}^2\,dx = \int_0^1 \lim_{k\to\infty} f_{n_k}^2\,dx \int_0^1 0\,dx = 0.$$

But recall from the display equation from the previous paragraph, which tells us that $\int_0^1 f_{n_k}^2\,dx = \frac{1}{2}$ for all $k \in \mathbb{N}$! This implies that $\frac{1}{2} = 0$, which is clearly a contradiction. Thus, it must be that our initial assumption that $f_n \to 0$ in measure was wrong, and so it must in fact be that $f_n \not\to 0$ a.e. nor in measure.

(b) Let us define $f_n(x) = n \cdot 1_{(0,1/n)}(x)$. We show that $f_n \to 0$ a.e. and in measure, but $f_n \not\to 0$ weakly in $L^p$ for any $p \in [1, \infty]$.

$f_n \to 0$ a.e. because for any $x > 0$, we have that for any $n > \frac{1}{x} \iff \frac{1}{n} < x$, $f_n(x) = 0$, implying that $f_n \to 0$ for $x > 0$. For $x \leq 0$, all the $f_n(x) = 0$, so they also converge to 0; thus we actually have everywhere pointwise convergence of the $f$ to 0. Convergence in measure is because for any $\epsilon > 0$, $m([f_n > \epsilon]) \leq m([f_n > 0]) = m((0, 1/n)) = \frac{1}{n}$, which goes to 0 as $n \to \infty$. Finally, to prove $f_n \not\to 0$ in $L^p$ for any $p \in [1, \infty]$ it suffices to find for each $p \in [1, \infty]$ a $\phi \in (L^p)^*$ s.t. $\phi(f_n) \not\to 0$. Well, note that $\phi$ defined as $\phi(f) = \int_0^1 f\, dx$ is in $(L^1)^*$, i.e. it is a bounded linear functional on $L^1([0, 1])$, and hence by Prop. 6.12 in Folland it is in $(L^p)^*$, i.e. a bounded linear function on $L^p([0, 1])$ for all $p \in (1, \infty]$ as well. Furthermore, $\phi(f_n) = \int_0^1 f_n\, dx = n \cdot \frac{1}{n} = 1$ for all $n \in \mathbb{N}$, so indeed $\phi(f_n) \not\to 0$ as $n \to \infty$, and we are done.

## Problem 4

Suppose that $\mu$ is a Borel measure on $\mathbb{R}^d$, with furthermore a constant $C > 1$ s.t. for all $x \in \mathbb{R}^d$ and $r > 0$, $C^{-1} r^d \leq \mu(B(x, r)) \leq C r^d$.

**Part 1:** we want to show that $\mu$ is absolutely continuous w.r.t. Lebesgue measure $m$ on $\mathbb{R}^d$; i.e. for $A \in \mathscr{L}^d$ (the set of Lebesgue measurable sets in $\mathbb{R}^d$), $m(A) = 0 \implies \mu(A) = 0$. By Theorem 2.40 in Folland, we have that for any $A \in \mathscr{L}^d$, $m(A) = \inf\{m(U) : U \supseteq A, U \text{ open}\}$. Thus, assuming we have $A \in \mathscr{L}^d$ s.t. $m(A) = 0$, and picking arbitrary $\epsilon > 0$, we see that there is open $U \subseteq \mathbb{R}^d$ s.t. $m(U) < m(A) + \epsilon = \epsilon$.

Let us pick some $D \in \mathbb{N}$. For any open set $U \subseteq \mathbb{R}^d$, for any $x \in U$, there is some $r_x \in (0, D)$ s.t. $B(x, r) \subseteq U$, by the definition of openness, and so $U = \bigcup_{x \in U} B(x, r_x)$. Letting $\mathscr{F} := \{\overline{B(x, r_x)}\}_{x \in U}$, we see that $\mathscr{F}$ is a collection of non-degenerate closed balls in $\mathbb{R}^d$ with $\sup_{B \in \mathscr{F}} \{\text{diam}(B)\} \leq D < \infty$. Thus, we can use the Vitali covering theorem (see Lec 11/23/20), which tells us that there is a countable family $\mathscr{G}$ of disjoint balls in $\mathscr{F}$ s.t. $U = \bigcup_{x \in U} B(x, r_x) \subseteq \bigcup_{x \in U} \overline{B(x, r_x)} = \bigcup_{B \in \mathscr{F}} B \subseteq \bigcup_{B \in \mathscr{G}} 5B$. As $\mathscr{G}$ is a disjoint collection of balls, we have that $m(\bigcup_{B \in \mathscr{G}} 5B) = 5^d m(\bigcup_{B \in \mathscr{G}} B) = 5^d \sum_{B \in \mathscr{G}} m(B)$, or $5^d \sum_{n=1}^{\infty} m(\overline{B(x_n, r_{x_n})})$, where the $x_n$ are the corresponding $x$'s (center points) of the $B \in \mathscr{G}$.

But note that defining $\omega_d := m(B(x, 1))$ (for any $x \in \mathbb{R}^d$ by translation invariance of $m$), we have that $\omega_d r^d = m(B(x, r)) \leq m(\overline{B(x, r)}) \leq m(B(x, r + \epsilon)) = \omega_d (r + \epsilon)^d$ for any $r > 0$ and $\epsilon > 0$, meaning that we can take the limit as $\epsilon \to 0$ to get that $m(\overline{B(x, r)}) = \omega_d r^d$. A similar argument can be made for $\mu$ to show that $C^{-1} r^d \leq \mu(\overline{B(x, r)}) \leq C r^d$ for any $x \in \mathbb{R}^d$ and $r > 0$. Anyways, connecting this to the chain of equalities at the end of the previous paragraph, we have that for $E := \bigcup_{B \in \mathscr{G}} 5B$,

$$m(E) = 5^d \sum_{n=1}^{\infty} m(B(x_n, r_{x_n})) = 5^d m\left(\bigcup_{n=1}^{\infty} B(x_n, r_{x_n})\right) \leq 5^d m\left(\bigcup_{x \in U} B(x, r_x)\right) = 5^d m(U) < 5^d \epsilon,$$

and also

$$\mu(E) \leq \sum_{B \in \mathcal{G}} \mu(5B) \leq \sum_{n=1}^{\infty} C(5r_{x_n})^d = 5^d \frac{C}{\omega_d} \sum_{n=1}^{\infty} \omega_d r_{x_n}^d = 5^d \frac{C}{\omega_d} m\left( \bigcup_{n=1}^{\infty} B(x_n, r_{x_n}) \right) < 5^d \frac{C}{\omega_d} \epsilon.$$

As $A \subseteq U \subseteq E$, by monotonicity of measures, $\mu(A) \leq \mu(E) < 5^d \frac{C}{\omega_d} \epsilon$. This result is true for any $\epsilon > 0$ (since we chose $\epsilon > 0$ arbitrary above), so indeed $\mu(A) = 0$.

**Part 2:** now that we know $\mu \ll m$, the Radon-Nikodym theorem (see Theorem 3.8 in Folland) says that there is some (unique $m$-a.e.) $f : \mathbb{R}^d \to \mathbb{R}$ s.t. for any $A \in \mathscr{B}_{\mathbb{R}^d}$, $\mu(A) = \int_A f \, dm$ (we can apply the R-N theorem because $\mu, m$ are both $\sigma$-finite because $\mu, m$ of any ball is finite, and $\mathbb{R}^d = \bigcup_{n=1}^{\infty} B(0, n)$). We want to show that $\frac{C^{-1}}{\omega_d} \leq f \leq \frac{C}{\omega_d}$ $m$-a.e. on $\mathbb{R}^d$.

Again, as I said above, $\mu$ of any ball is a finite number, so $f \in L^1_{\text{loc}}(\mathbb{R}^d)$. By (a slightly weaker version of) the Lebesgue differentiation theorem for $L^1_{\text{loc}}$ functions (see pg. 4 of Lec 12/2/20), we have that for $m$-a.e. $x \in \mathbb{R}^d$, $\lim_{r \to 0} \frac{1}{m(B(x_0, r))} \int_{B(x,r)} f(y) \, dm(y) = f(x_0)$. Simplifying the expression a little bit, we have $\text{LHS}(r) := \frac{1}{m(B(x,r))} \int_{B(x,r)} f(y) \, dm(y) = \frac{\mu(B(x,r))}{\omega_d r^d}$. But because $C^{-1} r^d \leq \mu(B(x, r)) \leq C r^d$ for any $x \in \mathbb{R}^d$ and $r > 0$, we have that $C^{-1}/\omega^d \leq \text{LHS}(r) \leq C/\omega_d$, which holds as we take $r \to 0$: $\frac{C^{-1}}{\omega_d} \leq \lim_{r \to 0} \text{LHS}(r) \leq \frac{C}{\omega_d}$. But again recall from two sentences ago that $\lim_{r \to 0} \text{LHS}(r) =_{\text{a.e.}} f$, so indeed $\frac{C^{-1}}{\omega_d} \leq f \leq \frac{C}{\omega_d}$ $m$-a.e. on $\mathbb{R}^d$.

# 525 Homework 7

Daniel Rui - 2/24/20

## Problem 1 - 6.1.2

We want to prove the following facts about $L^\infty$.

(a) If $f$ and $g$ are measurable functions on $X$, then $\|fg\|_1 \leq \|f\|_1\|g\|_\infty$. Furthermore, if $f \in L^1$ and $g \in L^\infty$, then $\|fg\|_1 = \|f\|_1\|g\|_\infty \iff |g(x)| = \|g\|_\infty$ a.e. on the set where $f(x) \neq 0$. Notice that $|fg| = |f| \cdot |g| \leq |f| \leq_{\text{a.e.}} \|g\|_\infty$, and so (by monotonicity and scalar multiplication of the integral for $L^+$ functions; see end of Prop. 2.13 in Folland) we have $\|fg\|_1 = \int_X |fg|\, d\mu \leq \int_X |f|\|g\|_\infty\, d\mu = \|g\|_\infty\|f\|_1$ (regardless of whether the integrals are finite or infinite).

Assuming $f \in L^1$ and $g \in L^\infty$, we get that $\|f\|_1, \|g\|_\infty$ are finite, and so all the integrals will be finite (allowing us to e.g. subtract). Because $|g|$ is a.e. bounded by $\|g\|_\infty$, we have that $\|g\|_\infty - g$ is a.e. $\geq 0$. Thus, we see that $\|fg\|_1 = \|f\|_1\|g\|_\infty \iff \int_X |f| \cdot \|g\|_\infty - |f| \cdot |g|\, d\mu = 0$, which happens if and only if $|f|(\|g\|_\infty - |g|) = 0$ a.e. (see Prop. 2.16 in Folland). This shows that $\|fg\|_1 = \|f\|_1\|g\|_\infty$ happens if and only if $|g| = \|g\|_\infty$ a.e. on the set where $f \neq 0$, as desired.

(b) We want to show that $\|\cdot\|_\infty$ is a norm on $L^\infty$. We check the three criterion:

- $\|\cdot\|_\infty = 0$ implies 0: if $\|f\|_\infty = 0$, that means that $|f| \leq \|f\|_\infty = 0$ a.e., i.e. $f$ must equal 0 a.e., where "0 a.e." is the "0" element in $L^\infty$.
- Scalar multiplication: note that $\|f\|_\infty$ can be phrased as $\inf\{a \geq 0 : |f| \leq_{\text{a.e.}} a\}$, and so because $|\lambda f| = |\lambda| \cdot |f| \leq_{\text{a.e.}} |\lambda| \cdot \|f\|_\infty$ (for any $\lambda \in \mathbb{R}$), it must be that $\|\lambda f\|_\infty \leq |\lambda| \cdot \|f\|_\infty$. For the other direction, we have that $|\lambda||f| = |\lambda f| \leq_{\text{a.e.}} \|\lambda f\|_\infty$; if $\lambda \neq 0$, we can divide to get $|f| \leq_{\text{a.e.}} \|\lambda f\|_\infty/|\lambda| \implies \|f\|_\infty \leq \|\lambda f\|_\infty/|\lambda| \implies |\lambda| \cdot \|f\|_\infty \leq \|\lambda f\|_\infty$ (and if $\lambda = 0$, then indeed $\|\lambda f\|_\infty = \|0\|_\infty = \inf\{a \geq 0 : |0| \leq a\} = 0 = |\lambda| \cdot \|f\|_\infty$).
- Triangle inequality: we have $|f + g| \leq |f| + |g|$ by the ordinary triangle inequality, but $|f| + |g| \leq_{\text{a.e.}} \|f\|_\infty + \|g\|_\infty$, implying that $\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$.

(c) We want to show that $\|f_n - f\|_\infty \to 0 \iff$ there is $E \in \mathscr{M}$ s.t. $\mu(E^\complement) = 0$ and $f_n \to f$ uniformly on $E$. ($\implies$): by definition of $\|\cdot\|_\infty$, we know that for every $n \in \mathbb{N}$, $|f_n - f| \leq \|f_n - f\|_\infty$ everywhere on a set $E_n$ s.t. $E_n^\complement$ is a null set. Well, taking $E = \bigcap_{n\in\mathbb{N}} E_n$, we have that $E^\complement = \bigcup_{n\in\mathbb{N}} E_n^\complement$ is also a null set (a countable union of null sets is a null set), and moreover for all $n \in \mathbb{N}$, $|f_n - f| \leq \|f_n - f\|_\infty$ everywhere on $E$. Since $\|f_n - f\|_\infty \to 0$, we have that on $E$, $|f_n - f| \to 0$ as $n \to \infty$, i.e. $f_n \to f$ uniformly on $E$.

($\impliedby$): $f_n \to f$ uniformly on $E$ means that for any $\epsilon > 0$, we can find $N \in \mathbb{N}$ s.t. $n \geq N \implies |f_n - f| < \epsilon$ everywhere on $E$. But since $E^\complement$ is a null set, we have that $|f_n - f| < \epsilon$ a.e., so again by $\|f\|_\infty = \inf\{a \geq 0 : |f| \leq_{\text{a.e.}} a\}$, we have that $n \geq N \implies \|f_n - f\|_\infty \leq \epsilon$. As we can find such an $N \in \mathbb{N}$ for any arbitrary fixed $\epsilon > 0$, we have that $\|f_n - f\|_\infty \to 0$ as $n \to \infty$.

(d) We want to show that $L^\infty$ is a Banach space. Clearly, $L^\infty$ is a vector space (if $f, g \in L^\infty$, then $af + bg \in L^\infty$ for any $a, b \in \mathbb{R}$ because $af + bg$ is measurable and is bounded a.e. by $|af + bg| \leq a|f| + b|g| \leq_{\text{a.e.}} a\|f\|_\infty + b\|g\|_\infty < \infty$), and in part (b) we established that $\|\cdot\|_\infty$ is a norm on $L^\infty$. Thus, we just need to show that $L^\infty$ is complete w.r.t. $\|\cdot\|_\infty$. Theorem 5.1 tells us that we can show this by showing that every absolutely convergent series in $L^\infty$ converges. So let $\{f_n\}_{n\in\mathbb{N}} \subseteq L^\infty$ be an absolutely convergent sequence of functions; i.e. we have $B := \sum_{n=1}^\infty \|f_n\|_\infty < \infty$.

Define $E_n$ the same way as we did in part (c), and similarly let $E = \bigcap_{n\in\mathbb{N}} E_n$ (so again $E^\complement$ is a null set). Then, for any $n \in \mathbb{N}$, $|f_n| \leq \|f_n\|_\infty$ everywhere on $E$. Thus, we have that for every $x \in E$, $\sum_{n=1}^\infty |f_n(x)| \leq \sum_{n=1}^\infty \|f_n\|_\infty = B$, which we know to be finite. An absolutely convergent series of real numbers also converges (in the ordinary fashion), so for every $x \in E$, we know $\sum_{n=1}^\infty f_n(x)$ exists, and moreover we have that $|\sum_{n=1}^\infty f_n(x)| \leq \sum_{n=1}^\infty |f_n(x)| \leq B$. Thus, $f := \sum_{n=1}^\infty f_n$ is a.e. equal to a bounded (by $B$) measurable function, and hence in $L^\infty$. Finally, we check that $\sum_{n=1}^N f_n$ converges to $f$ in the $\|\cdot\|_\infty$ norm as $N \to \infty$.

We have that for any $N \in \mathbb{N}$, $|\sum_{n=N+1}^\infty f_n| \leq \sum_{n=N+1}^\infty |f_n| \leq_{\text{a.e.}} \sum_{n=N+1}^\infty \|f_n\|_\infty$, implying (again by $\|f\|_\infty = \inf\{a \geq 0 : |f| \leq_{\text{a.e.}} a\}$) that $\|f - \sum_{n=1}^N f_n\|_\infty = \left\|\sum_{n=N+1}^\infty f_n\right\|_\infty \leq \sum_{n=N+1}^\infty \|f_n\|_\infty$, where the RHS goes to 0 as $N \to \infty$ because $\sum_{n=1}^\infty \|f_n\|_\infty$ is bounded, and so indeed all absolutely convergent series in $L^\infty$ converges to some $L^\infty$ function in the $\|\cdot\|_\infty$ norm, i.e. $L^\infty$ is indeed a Banach space.

(e) We want to show that simple functions (i.e. functions of the form $\sum_{i=1}^n a_i 1_{E_i}$ where all the $E_i$ have finite measure) are dense in $L^\infty$ (they're in $L^\infty$ since $a_1, \ldots, a_n \in \mathbb{R} \implies \max\{|a_1|, \ldots, |a_n|\} < \infty$). Let $f$ be arbitrary in $L^\infty$; then by Theorem 2.10 in Folland, we have a sequence of simple functions $\phi_n$ that converge to $f$ uniformly on any set on which $f$ is bounded. Since $f \in L^\infty$, it is bounded (by $\|f\|_\infty < \infty$) a.e., meaning that the $\phi_n$ converge to $f$ uniformly on a set $E$ s.t. $\mu(E^\complement) = 0$. By part (c), this means that $\|\phi_n - f\|_\infty \to 0$. Thus, we've shown that for any $f \in L^\infty$, there indeed exist simple functions that are arbitrarily close to it in the $L^\infty$ norm.

## Problem 2 - 6.1.3

If $1 \leq p < r \leq \infty$, we want to show that $L^p \cap L^r$ is a Banach space with norm $\|f\| = \|f\|_p + \|f\|_r$, and if $q \in (p, r)$, the inclusion map $L^p \cap L^r \to L^q$ is continuous. First, $L^p \cap L^r$ is a vector space, because if we have $f, g \in L^p \cap L^r$, then because $L^p$ and $L^r$ are both individually Banach spaces, for any $a, b \in \mathbb{R}$, $af + bg$ is in $L^p$ (since $f, g \in L^p$) and in $L^r$ (since also $f, g \in L^r$), implying that $af + bg \in L^p \cap L^r$. Now to show $\|\cdot\|$ is indeed a norm:

- Norm 0 implies 0: if $f \in L^p \cap L^r$ and $0 = \|f\| = \|f\|_p + \|f\|_r$, then $\|f\|_p, \|f\|_r$ must both be 0 (since norms are non-negative); but since $\|\cdot\|_p$ is the norm of the Banach space $L^p$, $\|f\|_p = 0 \implies f = 0$.
- Scalar multiplication: for any $\lambda \in \mathbb{R}$ and $f \in L^p \cap L^r$, we have $\|\lambda f\| = \|\lambda f\|_p + \|\lambda f\|_r = \lambda\|f\|_p + \lambda\|f\|_r = \lambda\|f\|$, as desired.

- Triangle inequality: for any $f, g \in L^p \cap L^r$, we have $\|f+g\| = \|f+g\|_p + \|f+g\|_r \leq \|f\|_p + \|g\|_p + \|f\|_r + \|g\|_r = \|f\| + \|g\|$, as desired.

Finally, we show that $L^p \cap L^r$ is complete w.r.t. the norm. Again, we use Theorem 5.1, which tells us it suffices to show that every absolutely convergent series in $L^p \cap L^r$ converges to something in $L^p \cap L^r$ in the $\|\cdot\|$ norm. So, let's say we have a sequence $\{f_n\} \subseteq L^p \cap L^r$ s.t. $\sum_{n=1}^{\infty} \|f_n\| < \infty$. But because $\|f\|_p, \|f\|_r \leq \|f\|$ for any $f \in L^p \cap L^r$, we also have that $\sum_{n=1}^{\infty} \|f_n\|_p, \sum_{n=1}^{\infty} \|f_n\|_r \leq \sum_{n=1}^{\infty} \|f_n\| < \infty$. Reading through the proof of Theorem 6.6, we see that this implies that $\sum_{n=1}^{\infty} f_n(x)$ converges a.e. (convergence of a series of real numbers), and that $f(x) := \sum_{n=1}^{\infty} f_n(x)$ (defined for a.e. $x$) is in $L^p$ and in $L^r$ (i.e. $f \in L^p \cap L^r$), and $\sum_{n=1}^{N} f_n$ converges to $f$ in both the $L^p$ and $L^r$ norms as $N \to \infty$. That is to say, $\|f - \sum_{n=1}^{N} f_n\|_p = \|\sum_{n=N+1}^{\infty} f_n\|_p$ and $\|\sum_{n=N+1}^{\infty} f_n\|_r$ both go to 0 as $N \to \infty$. But then we have that

$$\left\| \sum_{n=N+1}^{\infty} f_n \right\| = \left\| \sum_{n=N+1}^{\infty} f_n \right\|_p + \left\| \sum_{n=N+1}^{\infty} f_n \right\|_r$$

goes to 0 as $N \to \infty$, implying that the sequence $\{f_n\}_{n=1}^{\infty}$ converges to $f \in L^p \cap L^r$ in the $\|\cdot\|$ norm. As we said before, this and Theorem 5.1 yield that $L^p \cap L^r$ is indeed a Banach space w.r.t. $\|\cdot\|$.

Finally, observe that for any $q \in (p, r)$, Proposition 6.10 tells us that $L^p \cap L^r \subseteq L^q$ and $\|f\|_q \leq \|f\|_p^{\lambda} \|f\|_r^{1-\lambda}$ for any measurable $f$ and where $\lambda \in (0,1)$. Thus, if $\phi : L^p \cap L^r \to L^q$ is the inclusion map (mapping $f \in L^p \cap L^r$ to the same $f$, but now thought of as in $L^q$ instead), we have that $\|f-g\| < \epsilon$ implies that $\|f-g\|_q \leq \|f-g\|_p^{\lambda} \|f-g\|_r^{1-\lambda} \leq \|f-g\|^{\lambda} \|f-g\|^{1-\lambda} = \|f-g\| < \epsilon$ (where $\epsilon > 0$ is arbitrary), and so $\phi$ is indeed continuous.

## Problem 3 - 6.1.10

Let $p \in [1, \infty)$, and suppose we have a sequence $\{f_n\}_{n=1}^{\infty} \subseteq L^p$ and $f \in L^p$ s.t. $f_n \to_{\text{a.e.}} f$. We want to show that $\|f_n - f\|_p \to 0 \iff \|f_n\|_p \to \|f\|_p$. The ( $\implies$ ) direction is super easy: by the reverse triangle inequality, we have that $|\|f_n\|_p - \|f\|_p| \leq \|f_n - f\|_p$, and so $\|f_n - f\|_p \to 0$ implies that $|\|f_n\|_p - \|f\|_p| \to 0$, which is equivalent to $\|f_n\|_p \to \|f\|_p$ as $n \to \infty$.

For the ( $\impliedby$ ) direction, the hint suggests to use Ex. 20 in Folland §2.3, i.e. Problem 1 of 524 HW 6, so we copy the proposition we proved then: suppose that we have $f_n, g_n, f, g \in L^1(\mu)$, and $f_n \to f$ and $g_n \to g$ (both $\mu$-a.e.), and $|f_n| \leq g_n$ $\mu$-a.e. and $\int g_n \, d\mu \to \int g \, d\mu$; then, $\int f_n \, d\mu \to \int f \, d\mu$. We use this as follows: let "$f_n$" be $|f_n - f|^p$, "$f$" be 0, "$g_n$" be $2^p(|f_n|^p + |f|^p)$, and "$g$" to be $2^{p+1}|f|^p$.

All of these functions are in $L^1$, since $f_n, f \in L^p \implies |f_n - f|, |f_n|, |f| \in L^p \implies |f_n - f|^p, |f_n|^p, |f|^p \in L^1$, and because $f_n \to_{\text{a.e.}} f$, $|f_n - f| \to_{\text{a.e.}} 0 \implies |f_n - f|^p \to_{\text{a.e.}} 0$ and $2^p(|f_n|^p + |f|^p) \to_{\text{a.e.}} 2^p(|f|^p + |f|^p) = 2^{p+1}|f|^p$. According to pg. 181 in Folland, $|f_n - f|^p \leq 2^p(|f_n|^p + |-f|^p)$, and moreover $\int_X 2^p(|f_n|^p + |f|^p) \, d\mu = 2^p \int_X |f_n|^p \, d\mu + 2^p \int_X |f|^p \, d\mu$ converges to $2^p \int_X |f|^p \, d\mu + 2^p \int_X |f|^p \, d\mu = \int_X 2^{p+1}|f|^p \, d\mu$ because $\|f_n\|_p \to \|f\| \implies \int_X |f_n|^p \, d\mu \to \int_X |f|^p \, d\mu$. Thus, all the hypothesis for our above proposition are checked, so we can conclude that $\int_X |f_n - p|^p \, d\mu \to \int_X 0 \, d\mu = 0$, which is equivalent to the desired assertion that $\|f_n - f\|_p \to 0$.

## Problem 4

Let $\mathscr{X}$ be a separable Banach space, and let $\{T_n\}_{n=1}^{\infty} \subseteq \mathscr{X}^*$ be s.t. $\sup_{n\in\mathbb{N}} \|T_n\| = M < \infty$. We want to show that there exists a subsequence $\{T_{n_k}\}_{k=1}^{\infty}$ s.t. $\lim_{k\to\infty} T_{n_k}(x)$ exists for every $x \in \mathscr{X}$ (denote this limit by $T(x)$), and that moreover $T$ is linear and satisfies $\|T\| \leq M$ (we call $T$ the "weak limit" of $\{T_n\}_{n=1}^{\infty}$).

Let the countable dense set of $\mathscr{X}$ be $\mathscr{D}$. We know that there is a subsequence $\{T_{n_k}\}_{k=1}^{\infty}$ of the sequence $\{T_n\}_{n=1}^{\infty}$ s.t. $\lim_{k\to\infty} T_{n_k}(x)$ exists for every $x \in \mathscr{D}$. Since we have already performed this argument many times (at least twice — once during our proof of Arzelá-Ascoli and once in Problem 5(b) of 524 HW 1), I will only give a sketch of how to prove this: enumerating $\mathscr{D}$ as $\{x_1, x_2, \ldots\}$, we see that $|T_n(x_1)|$ is bounded (by $\|T_n\|\|x_1\| \leq M\|x_1\|$) for all $n \in \mathbb{N}$, so by Bolzano-Weierstrass, there is a subsequence $\{T_{n_1(k)}\}_{k=1}^{\infty}$ ($n_1 : \mathbb{N} \to \mathbb{N}$ strictly increasing) s.t. $T_{n_1(k)}(x_1)$ converges as $k \to \infty$. By the same reasoning, we can take a subsequence of this $\{T_{n_2(k)}\}_{k=1}^{\infty}$ ($n_2$ strictly increasing and $n_2(k) = n_1(K)$ for some $K \geq k$) s.t. $T_{n_2(k)}(x_2)$ converges as $k \to \infty$ (since it is a subsequence of the $n_1$ subsequence $T_{n_2(k)}(x_1)$ still converges). In this manner, we can define $n_m : \mathbb{N} \to \mathbb{N}$ for every $m \in \mathbb{N}$ s.t. $T_{n_m(k)}(x_i)$ converges as $k \to \infty$ for all $i \leq m$. Then, the diagonal subsequence $\{T_{n_k(k)}\}_{k=1}^{\infty}$ satisfies that $T_{n_k(k)}(x_i)$ converges as $k \to \infty$ for all $x_i \in \mathscr{D}$.

We now show that if a subsequence $\{T_{n_k}\}_{k=1}^{\infty} \subseteq \{T_n\}_{n=1}^{\infty}$ converges for all $x \in \mathscr{D}$, then it converges for all $x \in \mathscr{X}$. Since the $T_n$ output in $\mathbb{C}$ (or I guess $\mathbb{R}$), and $\mathbb{C}$ (or $\mathbb{R}$) is complete, it suffices to show that for any $x \in \mathscr{X}$, $\{T_{n_k}(x)\}_{k=1}^{\infty}$ is a Cauchy sequence. Well, we have that for any $k, m \in \mathbb{N}$ and $x' \in \mathscr{X}$,

$$|T_{n_k}(x) - T_{n_m}(x)| \leq |T_{n_k}(x) - T_{n_k}(x')| + |T_{n_k}(x') - T_{n_m}(x')| + |T_{n_m}(x') - T_{n_m}(x)|.$$

The first and third terms on the RHS are bound respectively by $\|T_{n_k}\|\|x-x'\|$ and $\|T_{n_m}\|\|x-x'\|$, but both of these are bounded by $M\|x-x'\|$ (since all the $T_n$ are linear and satisfy $\sup_{n\in\mathbb{N}} \|T_n\| = M < \infty$). Let us now fix arbitrary $\epsilon > 0$. By denseness of $\mathscr{D}$, we know we can find $x' \in \mathscr{D}$ s.t. $\|x - x'\| < \frac{\epsilon}{3M}$ (if $M = 0$, then all the $T_n$ are identically 0 so the result is trivial). Now since $x' \in \mathscr{D}$, and we know $\{T_{n_k}(x)\}_{k=1}^{\infty}$ converges (i.e. is a Cauchy sequence) for all $x \in \mathscr{D}$, we know that there is $N \in \mathbb{N}$ s.t. $k, m \geq N \implies |T_{n_k}(x') - T_{n_m}(x')| < \frac{\epsilon}{3}$. Thus, the big inequality above now tells us that $k, m \geq N \implies |T_{n_k}(x) - T_{n_m}(x)| < M\frac{\epsilon}{3M} + \frac{\epsilon}{3} + M\frac{\epsilon}{3M} = \epsilon$. As $\epsilon > 0$ was chosen arbitrarily, this is exactly what it means for $\{T_{n_k}(x)\}_{k=1}^{\infty}$ to be a Cauchy sequence, so we have indeed shown that any subsequence $\{T_{n_k}\}_{k=1}^{\infty} \subseteq \{T_n\}_{n=1}^{\infty}$ that converges for all $x \in \mathscr{D}$ converges for all $x \in \mathscr{X}$.

Alright. We now have the existence of a subsequence $\{T_{n_k}\}_{k=1}^{\infty} \subseteq \{T_n\}_{n=1}^{\infty}$ s.t. $\lim_{k\to\infty} T_{n_k}(x)$ exists for every $x \in \mathscr{X}$ (as we said in the introduction we shall call this limit $T(x)$). We only need to show that $T$ is linear and satisfies $\|T\| \leq M$ to finish the problem. Well, for any $x, x' \in \mathscr{X}$ and $a, b \in \mathbb{C}$ (or $\mathbb{R}$), we have $T(ax + bx') = \lim_{k\to\infty} T_{n_k}(ax + bx') = \lim_{k\to\infty}[aT_{n_k}(x) + bT_{n_k}(x')] = a\lim_{k\to\infty} T_{n_k}(x) + b\lim_{k\to\infty} T_{n_k}(x') = aT(x) + bT(x')$, so $T$ is indeed linear; and lastly for any $x \in \mathscr{X}$, $|T(x)| = |\lim_{k\to\infty} T_{n_k}(x)| = \lim_{k\to\infty} |T_{n_k}(x)| \leq \sup_{k\in\mathbb{N}} |T_{n_k}(x)| \leq \sup_{k\in\mathbb{N}} \|T_{n_k}\|\|x\| \leq M\|x\|$, so indeed $\|T\| \leq M$.

## Problem 5

Let $p \in (1,\infty)$ and $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions in $C([0,1]) \cap C^1((0,1))$ satisfying $\int_0^1 |f_n(x)|^p\,dx + \int_0^1 |f_n'(x)|^p\,dx \le 1$ for every $n \in \mathbb{N}$.

(a) We want to show that there exists a subsequence $\{f_{n_k}\}_{k=1}^{\infty} \subseteq \{f_n\}_{n=1}^{\infty}$ and $f \in C([0,1]) \subseteq L^p([0,1])$ s.t. $f_{n_k} \to f$ uniformly on $[0,1]$ and $f_{n_k} \to f$ in $L^p([0,1])$ as $k \to \infty$. The Arzelá-Ascoli theorem from Lec 1/13 says that because $[0,1]$ is a compact separable metric space, if we can show that $\{f_n\}_{n=1}^{\infty}$ is equicontinuous (for all fixed $x \in [0,1]$, for all $\epsilon > 0$ there is $\delta$ s.t. for any $n$, $|x-y| < \delta \implies |f_n(x) - f_n(y)| < \epsilon$) and pointwise bounded ($\{f_n(x)\}_{n=1}^{\infty}$ is bounded subset of $\mathbb{R}$ for every $x \in [0,1]$), then there is a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ and $f \in C([0,1])$ s.t. $f_{n_k} \to f$ uniformly.

**Equicontinuous:** since the $f_n$ are in $C([0,1]) \cap C^1((0,1))$, we have that for any $n \in \mathbb{N}$, $f_n(x) = f_n(0) + \int_0^x f_n'(y)\,dy$ (by even the basic version of the FTC, no need for the Lebesgue FTC involving absolute continuity). Then, for any $n \in \mathbb{N}$ and $x, y \in [0,1]$ ($x \le y$) we have that $|f_n(x) - f_n(y)| = |\int_x^y f_n'(t)\,dt| \le \int_x^y f_n(t)\,dt = \int_0^1 |f_n'(t)| \cdot 1_{[x,y]}(t)\,dt = \|f_n' \cdot 1_{[x,y]}\|_1$. By Holder's inequality, this is bounded by $\|f_n'\|_p \|1_{[x,y]}\|_q$ (where $q \in (1,\infty)$ is s.t. $\frac{1}{p} + \frac{1}{q} = 1$).

But we assumed that $\int_0^1 |f_n(x)|^p\,dx + \int_0^1 |f_n'(x)|^p\,dx \le 1$ for every $n \in \mathbb{N}$, so in particular (because both terms on the LHS are $\ge 0$) we have that $\int_0^1 |f_n'|^p\,dx \le 1 \implies (\int_0^1 |f_n'|^p\,dx)^{1/p} = \|f_n'\|_p \le 1$. Furthermore, $\|1_{[x,y]}\|_q = (y-x) = |y-x|$, so overall we get that $|f_n(x) - f_n(y)| \le |y-x|$. This proves equicontinuity, because we know that for all $\epsilon > 0$, choosing $\delta = \epsilon$ gives that for any $n \in \mathbb{N}$ and $x, y \in [0,1]$, $|x-y| < \delta \implies |f_n(x) - f_n(y)| < \epsilon$.

**Pointwise bounded:** note that we have $(\int_0^1 |f_n(x)|\,dx)^p \le \int_0^1 |f_n(x)|^p\,dx$ by Jensen (which I proved in Problem 5 of the 524 Final), and so $\int_0^1 |f_n(x)|\,dx \le (\int_0^1 |f_n(x)|^p\,dx)^{1/p} \le 1$ (alternatively one could use Prop. 6.12 in Folland). **Note that we can do the same argument for $\int_0^1 |f_n'(x)|\,dx$!** Now by the reverse triangle inequality, we have that $|f_n(x)| = |f_n(0) + \int_0^x f_n'(y)\,dy| \ge |f_n(0)| - |\int_0^x f_n'(y)\,dy|$, and so by monotonicity we have

$$\int_0^1 \left(|f_n(0)| - \left|\int_0^x f_n'(y)\,dy\right|\right) dx \le \int_0^1 |f_n(x)|\,dx \le 1,$$

implying that

$$\begin{aligned} |f_n(0)| = \int_0^1 |f_n(0)|\,dx &\le 1 + \int_0^1 \left|\int_0^x f_n'(y)\,dy\right| dx \le 1 + \int_0^1 \int_0^x |f_n'(y)|\,dy\,dx \\ &\le 1 + \int_0^1 \int_0^1 |f_n'(y)|\,dy\,dx \le 1 + \int_0^1 1\,dx = 1 + 1 = 2. \end{aligned}$$

Thus, we have that for all $n \in \mathbb{N}$ and $x \in [0,1]$,

$$|f_n(x)| = \left|f_n(0) + \int_0^x f_n'(y)\,dy\right| \le |f_n(0)| + \int_0^x |f_n'(y)|\,dy \le 2 + \int_0^1 |f_n'(y)|\,dy \le 2 + 1 = 3.$$

Thus, Arzelá-Ascoli now tells us that there is a subsequence $\{f_{n_k}\}_{k=1}^{\infty} \subseteq \{f_n\}_{n=1}^{\infty}$ that converges uniformly to some $f \in C([0,1])$ on $[0,1]$. For convergence in $L^p([0,1])$, observe that $\|f_{n_k} - f\|_p = (\int_0^1 |f_{n_k} - f|^p \, dx)^{1/p} \leq (\int_0^1 \|f_{n_k} - f\|_\infty^p \, dx)^{1/p} = \|f_{n_k} - f\|_\infty$ (alternatively one could use Prop 6.12 in Folland). But recall from Problem 1(c) above that uniform convergence $\implies \|f_{n_k} - f\|_\infty \to 0$, and so indeed $\|f_{n_k} - f\|_p \to 0$.

(b) We want to show that there exists a subsubsequence $\{f_{n_{k_j}}\}_{j=1}^{\infty} \subseteq \{f_{n_k}\}_{k=1}^{\infty}$ and $h \in L^p([0,1])$ s.t. for every $\phi \in C_c((0,1))$ (i.e. $\phi$ is continuous with compact support in $(0,1)$),

$$\lim_{j\to\infty} \int_0^1 f'_{n_{k_j}}(x)\phi(x)\,dx = \int_0^1 h(x)\phi(x)\,dx.$$

Let us define a sequence of operators $T_k : L^q([0,1]) \to \mathbb{R}$ by $T_k(\phi) = \int_0^1 f'_{n_k}(x)\phi(x)\,dx$, where $q$ is the conjugate exponent to $p$ (i.e. $q \in (1,\infty)$ s.t. $\frac{1}{p} + \frac{1}{q} = 1$). Then, it is clear that each $T_k$ is linear (by linearity of the integral and the distributive property of multiplication), and moreover we have $|T_k(\phi)| \leq \int_0^1 |f'_{n_k}| \cdot |\phi| \, dx \leq \|f'_{n_k}\|_p \|\phi\|_q$. Recall from part (a) (2nd paragraph of "Equicontinuous" section) that the inequality we assumed at the introduction gives that $\|f'_n\|_p \leq 1$ for every $n \in \mathbb{N}$, and so we have that for any $k \in \mathbb{N}$, $|T_k(\phi)| \leq 1 \cdot \|\phi\|_q \implies \sup_{k\in\mathbb{N}} \|T_k\| =: M \leq 1$.

Since $L^q([0,1])$ is a separable Banach space (see Theorem 6.6 in Folland and the Problem 5 Appendix below), we can now use Problem 4, which tells us that there exists a subsequence $\{T_{k_j}\}_{j=1}^{\infty} \subset \{T_k\}_{k=1}^{\infty}$ s.t. $T(\phi) := \lim_{j\to\infty} T_{k_j}(\phi)$ (i.e. the limit exists for all $\phi \in L^q([0,1])$) is linear and satisfies $\|T\| \leq M \leq 1$. By Theorem 6.15, because $T \in (L^q([0,1]))^*$, there exists some $h \in L^p([0,1])$ s.t. $T(\phi) = \int_0^1 h\phi \, dx$ for all $\phi \in L^q([0,1])$. That is to say, we've proved that subsubsequence $\{f_{n_{k_j}}\}_{j=1}^{\infty} \subseteq \{f_{n_k}\}_{k=1}^{\infty}$ and $h \in L^p([0,1])$ s.t. for every $\phi \in L^q([0,1])$, $\lim_{j\to\infty} \int_0^1 f'_{n_{k_j}}(x)\phi(x)\,dx = \int_0^1 h(x)\phi(x)\,dx$.

To conclude, we just observe that $C_c((0,1)) \subseteq L^q([0,1])$, because any $\phi \in C_c((0,1))$ is continuous on a compact set (hence bounded on it), and 0 outside of that compact set (hence it is in fact bounded everywhere on $[0,1]$), and again from Prop. 6.12 in Folland we have that $\|\phi\|_q \leq \|\phi\|_\infty$, which we just said was finite.

(c) We want to show that for every $\phi \in C_c^1((0,1))$ (i.e. $\phi$ is $C^1$ with compact support in $(0,1)$), $\int_0^1 h(x)\phi(x)\,dx = -\int_0^1 f(x)\phi'(x)\,dx$. Let us fix arbitrary $\phi \in C_c^1((0,1))$. Recall that all the $f_n \in C^1((0,1))$, and so for every $n \in \mathbb{N}$, $f'_n$ exists and is continuous on $(0,1)$; similarly we are taking $\phi \in C_c^1((0,1))$, so $\phi'$ also exists and is continuous on $(0,1)$ (and $\phi(0) = \phi(1) = \phi'(0) = \phi'(1) = 0$ by being compactly supported in $(0,1)$). By product rule for derivatives, we have for any $n \in \mathbb{N}$ that $(f_n\phi)' = f'_n\phi + f_n\phi'$ on $(0,1)$, so $\int_0^1 (f_n\phi)'\,dx = \int_0^1 f'_n\phi\,dx + \int_0^1 f_n\phi'\,dx$.

Because $(f_n\phi)'$ is continuous on $(0,1)$ and $\phi$ is 0 near the endpoints $x = 0, 1$ (by compact support, there must be open sets around $x = 0, 1$ on which $\phi$ is identically 0), $(f_n\phi)'$ is in fact continuous on $[0,1]$ and so by the basic FTC we have that $\int_0^1 (f_n\phi)'\,dx = f_n(1)\phi(1) - f_n(0)\phi(0) = 0 - 0 = 0$, and so for any $n \in \mathbb{N}$ we have that $\int_0^1 f'_n\phi\,dx = -\int_0^1 f_n\phi'\,dx$.

This equality and part (b) now tells us that

$$-\lim_{j\to\infty}\int_0^1 f_{n_{k_j}}(x)\phi'(x)\,dx = \lim_{j\to\infty} -\int_0^1 f_{n_{k_j}}(x)\phi'(x)\,dx = \int_0^1 h(x)\phi(x)\,dx.$$

Recall from the end of the "Pointwise bounded" section that for any $n \in \mathbb{N}$ and $x \in [0,1]$, $|f_n(x)| \leq 3$, and that $\phi$ is continuous on $[0,1]$ and hence bounded by say $B$ (so $|\phi| \leq B$ on $[0,1]$). Recall also from part (a) that $f_{n_k} \to f$ uniformly as $k \to \infty$, so in particular $f_{n_{k_j}}\phi \to_{\text{a.e.}} f\phi$ on $[0,1]$ as $j \to \infty$. Then because $|f_{n_{k_j}}\phi| \leq 3B$ on $[0,1]$ and $\int_0^1 3M\,dx = 3B < \infty$, the DCT tells us that $\lim_{j\to\infty}\int_0^1 f_{n_{k_j}}(x)\phi'(x)\,dx = \int_0^1 f(x)\phi'(x)\,dx$. Putting everything together, we indeed get that for any $\phi \in C_c^1((0,1))$ (because the one we chose during the argument was arbitrary),

$$-\int_0^1 f(x)\phi'(x)\,dx = -\lim_{j\to\infty}\int_0^1 f_{n_{k_j}}(x)\phi'(x)\,dx = \int_0^1 h(x)\phi(x)\,dx,$$

and at last, we are finished.

## Problem 5 Appendix

Here we prove that $L^p([0,1])$ (and actually also $L^p(\mathbb{R})$ is separable for any $p \in (0,\infty)$. We begin with some elementary lemmas:

**Lemma 1:** if $U \subseteq \mathbb{R}$ is open, then there are disjoint open intervals $I_1, I_2, \ldots$ such that $U = \bigcup_{n=1}^{\infty} I_n$.
*Proof:* (recall that an interval $(a,b) := \{x \in \mathbb{R} : a < x < b\}$, and that $(a,b) = \bigcup_{a<\alpha<\beta<b}(\alpha,\beta)$). For all rational $x \in U$, define

$$I_x = \bigcup_{x\in(a,b)\subseteq U} (a,b)$$

which is itself an open interval (because all the $(a,b)$ are open and share the point $x$ — more explicitly, for all $\alpha, \beta \in I_x$, $\alpha < \beta$, there must be intervals such that $\alpha \in (a,b)$ and $\beta \in (c,d)$; but both intervals contain $x$, and so $a < x < b$ and $c < x < d$, implying that $c < d$ and so $(\alpha,\beta) \subseteq (a,d) = (a,b) \cup (c,d) \subseteq I_x$, where again $\alpha$ and $\beta$ were arbitrary in $I_x$), and a subset of $U$.
For all irrational $x \in U$, by openness, there is some $r_x$ such that $B(x,r_x) \subseteq U$, and so picking any rational $y$ in $B(x,r_x)$, we see that $B(x,r_x) \subseteq I_y$ (by definition of $I_y$). In other words, for all $x \in U$, there is $q \in U \cap \mathbb{Q}$ such that $x \in I_q$, so

$$U \subseteq \bigcup_{q\in U\cap\mathbb{Q}} I_q.$$

For the other direction, note that $I_q \subseteq U$ for all $q \in U \cap \mathbb{Q}$. Lastly, for all $p, q \in U \cap \mathbb{Q}$, either $I_p = I_q$ or $I_p \cap I_q = \varnothing$ because case one, $x \in I_p \cap I_q \implies$ that $I_p \cup I_q$ is an open interval (because $I_p$ and $I_q$ are both open intervals sharing a point $x$), and so by the definition of $I_p$ and $I_q$, $I_p \cup I_q \subseteq I_p, I_q \implies I_p = I_q$. Case two is $\nexists x \in I_p \cap I_q$, i.e. $I_p \cap I_q = \varnothing$. If we want to be more rigorous, we can replace $\{I_k\}$ with $\{J_k\}$, where we discard all duplicate $I_k$'s.

**Lemma 2:** if $E \in M_\mu$ ($\mu$ is a Borel measure on $\mathbb{R}$, e.g. Lebesgue measure $m$) and $\mu(E) < \infty$, then for every $\epsilon > 0$ there is a set $A$ that is a finite union of open intervals with rational endpoints s.t. $\mu(E \triangle A) < \epsilon$ (variant of Prop. 1.20 in Folland, i.e. Exercise 1.5.26 in Folland that we did not do earlier).

*Proof:* Theorem 1.18 in Folland tells us that for $E \in \mathcal{M}_\mu$, $\mu(E) = \inf\{\mu(U) : E \subseteq U, U \text{ open}\}$. Fix arbitrary $\epsilon > 0$. By the definition of infimum, there must be some open $U \subseteq \mathbb{R}$ s.t. $\mu(U) < \mu(E) + \frac{\epsilon}{3}$. Lemma 1 says that $U = \bigcup_{n=1}^{\infty} I_n$, where the $I_n$ are disjoint open intervals. By countable disjoint additivity of measures, we have that $\mu(U) = \sum_{n=1}^{\infty} \mu(I_n)$. As $\mu(E) < \infty$, $\mu(U) < \infty$ and so this infinite sum converges. That is to say, there is some $N \in \mathbb{N}$ s.t. $\sum_{n=N+1}^{\infty} \mu(I_n) < \frac{\epsilon}{3}$, or equivalently $\sum_{n=1}^{\infty} \mu(I_n) < \sum_{n=1}^{N} \mu(I_n) + \frac{\epsilon}{3}$. Letting $V = \bigcup_{n=1}^{N} I_n$ (so $V \subseteq U$), the above inequality says $\mu(U) < \mu(V) + \frac{\epsilon}{3}$. Now because rationals are dense in $\mathbb{R}$, we can shrink the open intervals $I_n$ for $n \in \{1, \ldots, N\}$ an arbitrarily small amount to open intervals $J_n$ with rational endpoints. So let $A = \bigcup_{n=1}^{N} J_n$ satisfy $\mu(V) < \mu(A) + \frac{\epsilon}{3}$ (and keep in mind that $A \subseteq V \subseteq U$). Then,

$$\mu(E \triangle A) = \mu(A \setminus V \cup V \setminus A) \leq \mu(A \setminus V) + \mu(V \setminus A) \leq \mu(U \setminus V) + \mu(U \setminus A) < \tfrac{\epsilon}{3} + \tfrac{2\epsilon}{3} = \epsilon,$$

as desired.

Finally we get to the **central claim**. Let $f$ be arbitrary in $L^p([0,1])$. Being in $L^p([0,1])$ forces $f$ to be bounded a.e., so by Theorem 2.10 in Folland, there is a sequence of simple functions $\phi_N$ that converge to $f$ uniformly on the set on which $f$ is bounded, so almost everywhere. This means in particular that $\|f - \phi_n\|_p^p = \int_{[0,1]} |f - \phi_n|^p \, dm \to 0$, and so $\|f - \phi_n\|_p \to 0$.††† We now fix arbitrary $\epsilon > 0$. Then, we know that there is some $N \in \mathbb{N}$ s.t. $\|f - \phi_N\|_p < \frac{\epsilon}{2}$. By the definition of simple function, we have that $\phi_N = \sum_{i=1}^{m} a_i 1_{E_i}$ for some $m \in \mathbb{N}$, $a_i \in \mathbb{R}$, and measurable $E_i$ with finite measure. We now use Lemma 2 and denseness of rationals to get $A_i$ (a finite union of disjoint open intervals with rational endpoints) and $b_i \in \mathbb{Q}$ s.t. $\|a_i 1_{E_i} - b_i 1_{A_i}\|_p = |a_i - b_i| \mu(E \triangle A) < \frac{\epsilon}{2m}$. Having done this for all $i \in \{1, \ldots, m\}$, we now have the "super simple function" $\psi = \sum_{i=1}^{m} b_i 1_{A_i}$ s.t. $\|f - \psi\|_p \leq \|f - \phi_N\|_p + \|\phi_N - \psi\|_p < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

Now consider the set of all functions of the form $\sum_{i=1}^{M} b_i 1_{J_i}$ where $M \in \mathbb{N}$, the $b_i \in \mathbb{Q}$, and the $J_i$ are open intervals with rational endpoints (the above $\psi$ is of this form). We'll call this set the set of "super simple functions". For fixed $M$, we have countably $(\prod_{i=1}^{M} \mathbb{Q} \times (\mathbb{Q} \times \mathbb{Q}))$ many functions, and a countable union of countable sets is still countable. Thus finally, we have proved that the set of "super simple functions" is a countable dense set of $L^p([0,1])$, and so $L^p([0,1])$ is separable.

†††: this is the only place we use $[0,1]$ specifically; clearly any bounded set works. In fact if we were proving this for $\mathbb{R}$, we would see that $f_n := f \cdot 1_{B(0,n)}$ converges a.e. to $f$, are bounded in absolute value by $|f| \in L^p$, and hence by DCT satisfy $\lim_{n\to\infty} \int_{\mathbb{R}} f_n \, dm = \int_{\mathbb{R}} f \, dm$. This all means that even on all of $\mathbb{R}$, performing the above argument on a big enough $B(0,N)$ (a bounded set) gives an arbitrarily close approximation by a "super simple function" (0 outside $B(0,N)$), yielding indeed denseness of the "super simple functions".

# 525 Homework 6

Daniel Rui - 2/17/20

## Problem 1 - 5.5.56

I actually did most of this last week (in Problem 2(c)), but I reproduce it here with some more details. We show that for any subset $E \subseteq \mathscr{H}$, $(E^\perp)^\perp$ is the smallest closed subspace containing $E$. To prove "containing $E$" and "closed subspace", we clearly have $E \subseteq (E^\perp)^\perp$ (since $y \in E^\perp \iff \langle x, y\rangle = 0$ for all $x \in E$, and $(E^\perp)^\perp = \{z \in \mathscr{H} : \langle y, z\rangle = 0 \text{ for all } y \in E^\perp\}$), and we know from pg. 173 of Folland that $E^\perp$ is a closed subspace for any $E \subseteq \mathscr{H}$.

To prove "smallest", suppose on the contrary we have a smaller closed subspace $\mathscr{M}$ than $(E^\perp)^\perp$ containing $E$; i.e. there is $x_0 \in (E^\perp)^\perp$ but not in $\mathscr{M}$. By Theorem 5.8(a), we know there exists $f \in \mathscr{H}^*$ s.t. $f(x_0) \neq 0$, but $f|_{\mathscr{M}} = 0$. By Riesz representation (Theorem 5.25), there is a unique $y_0 \in \mathscr{H}$ s.t. $f(x) = \langle x, y_0\rangle$ for all $x \in \mathscr{H}$. Thus in particular, $\langle x, y_0\rangle = 0$ for all $x \in \mathscr{M}$ (so in particular for all $x \in E$, since $E \subseteq \mathscr{M}$), meaning that $y_0 \in E^\perp$. But $\langle x_0, y_0\rangle = f(x_0) \neq 0$, so $x_0$ can't be in $(E^\perp)^\perp$; contradiction.

## Problem 2

Let $\mathscr{H}$ be a separable Hilbert space, and $A \in L(\mathscr{H}, \mathscr{H})$ with $A^*$ as the adjoint of $A$. We want to show that for every pair $a, b \in \mathscr{H}$, the system of equations (1) $x + A^*y = a$, (2) $Ax - y = b$ has a unique solution $x, y \in \mathscr{H}$. That is to say, for every $(a, b) \in \mathscr{H} \times \mathscr{H}$, we want a unique decomposition $(a, b) = (x, Ax) + (A^*y, -y)$, where we add elements of $\mathscr{H} \times \mathscr{H}$ component-wise. The outline for this proof is to show that (1) $\mathscr{H} \times \mathscr{H}$ can be considered a Hilbert space with the inner product as $\langle (x, x'), (y, y')\rangle = \langle x, y\rangle + \langle x', y'\rangle$, and (2) the set $\mathscr{M} := \{(x, Ax) : x \in \mathscr{H}\}$ is a closed subspace of $\mathscr{H} \times \mathscr{H}$ with $\mathscr{M}^\perp = \{(A^*y, -y) : y \in \mathscr{H}\}$, so that we can use Theorem 5.24 in Folland to get the desired unique decomposition.

$\mathscr{H} \times \mathscr{H}$ is Hilbert space: defining $\lambda(x, x') = (\lambda x, \lambda x')$ for $\lambda \in \mathbb{C}$ and $(x, x') + (y, y') = (x + y, x' + y')$, we get that $\mathscr{H} \times \mathscr{H}$ can be thought of as a vector space. Defining $\langle (x, x'), (y, y')\rangle = \langle x, y\rangle + \langle x', y'\rangle$, this is actually an inner product because

- Observe that

$$\begin{aligned}\langle a(x, x') + b(y, y'), (z, z')\rangle &= \langle (ax + by, ax' + by'), (z, z')\rangle = \langle ax + by, z\rangle + \langle ax' + by', z'\rangle \\ &= a(\langle x, z\rangle + \langle x', z'\rangle) + b(\langle y, z\rangle + \langle y', z'\rangle) = a\langle (x, x'), (z, z')\rangle + b\langle (y, y'), (z, z')\rangle\end{aligned}$$

  for all $(x, x'), (y, y'), (z, z') \in \mathscr{H} \times \mathscr{H}$ and $a, b \in \mathbb{C}$.
- $\langle (y, y'), (x, x')\rangle = \langle y, x\rangle + \langle y', x'\rangle = \overline{\langle x, y\rangle} + \overline{\langle x', y'\rangle} = \overline{\langle (x, x'), (y, y')\rangle}$ for all $(x, x'), (y, y') \in \mathscr{H} \times \mathscr{H}$.
- and $\langle (x, x'), (x, x')\rangle = \langle x, x\rangle + \langle x', x'\rangle$, which is in $(0, \infty)$ for all $(x, x') \in \mathscr{H} \times \mathscr{H}$ that aren't $= (0, 0)$, because $\langle x, x\rangle \in (0, \infty)$ for all nonzero $x \in \mathscr{H}$.

Finally, $\mathscr{H}\times\mathscr{H}$ is complete w.r.t. the norm $\|(x,x')\| = \sqrt{\langle (x,x'),(x,x')\rangle}$. Note that the norm satisfies $\|(x,x')\|^2 = \langle x,x\rangle + \langle x',x'\rangle = \|x\|^2 + \|x'\|^2$. Now suppose that $\{x_n,y_n\}_{n=1}^\infty$ is a Cauchy sequence in $\mathscr{H}\times\mathscr{H}$; then we have that for any $\epsilon\in(0,1)$ there is $N\in\mathbb{N}$ s.t. $n,m\geq N \implies \|(x_n,y_n)-(x_m,y_m)\| < \epsilon$, but because $\epsilon<1$, we have that $\|(x_n,y_n)-(x_m,y_m)\|^2 \leq \|(x_n,y_n)-(x_m,y_m)\|$, so we have $\|x_n-x_m\|^2, \|y_n-y_m\|^2 \leq \|x_n-x_m\|^2+\|y_n-y_m\|^2 = \|(x_n,y_n)-(x_m,y_m)\|^2 \leq \|(x_n,y_n)-(x_m,y_m)\| < \epsilon$, implying that $\{x_n\}_{n=1}^\infty, \{y_n\}_{n=1}^\infty$ are both Cauchy sequences in $\mathscr{H}$.

Because $\mathscr{H}$ is complete w.r.t. the $\mathscr{H}$ norm, these two sequences converge to $x,y\in\mathscr{H}$ respectively. Then, $\|(x_n,y_n)-(x,y)\|^2 = \|x_n-x\|^2+\|y_n-y\|^2$, but we know the RHS goes to 0 as $n\to\infty$, so indeed $(x_n,y_n)$ converges to $(x,y)\in\mathscr{H}\times\mathscr{H}$, and indeed $\mathscr{H}\times\mathscr{H}$ is complete.

Now for $\mathscr{M} := \{(x,Ax) : x\in\mathscr{H}\}$ being a closed subspace. It is a subspace because we have that $a(x,Ax)+b(x',Ax') = (ax+bx', a(Ax)+b(Ax')) = (ax+bx', A(ax)+A(bx')) = (ax+bx', A(ax+bx'))$, where we used that $A\in L(\mathscr{H},\mathscr{H})$. For closedness, let's start by supposing that we have a sequence $\{(x_n,Ax_n)\}_{n=1}^\infty \subseteq \mathscr{M}$ that converges to $(x,y)$, and try to prove that $y=Ax$. Well by convergence, we must have that $\|(x_n,Ax_n)-(x,y)\|^2\to 0$ as $n\to\infty$. But $\|(x_n,Ax_n)-(x,y)\|^2 = \|x_n-x\|^2+\|Ax_n-y\|^2$, and the only way that the LHS goes to 0 as $n\to\infty$ is that each term in the RHS goes to 0 as $n\to\infty$ (since both are $\geq 0$ since they are squares). Then, for any $n\in\mathbb{N}$, $\|Ax-y\| \leq \|Ax-Ax_n\|+\|Ax_n-y\| \leq \|A\|\|x-x_n\|+\|Ax_n-y\|$, but because $\|A\|$ is bounded (since $A\in L(\mathscr{H},\mathscr{H})$), and both $\|x-x_n\|, \|Ax_n-y\|$ go to 0 as $n\to\infty$, we have that $\|Ax-y\|$ is bounded above by any arbitrarily small positive quantity, meaning it must be 0, and so $Ax=y$ as desired.

We now determine $\mathscr{M}^\perp = \{(z,y)\in\mathscr{H}\times\mathscr{H} : \langle x,z\rangle + \langle Ax,y\rangle = 0 \text{ for all } x\in\mathscr{H}\}$. By the definition of adjoint, we have $\langle x,z\rangle + \langle Ax,y\rangle = \langle x,z\rangle + \langle x,A^*y\rangle = \langle x, z+A^*y\rangle = 0$ for all $x\in\mathscr{H}$. But this happens if and only if $z+A^*y=0$, which happens if and only if $z=A^*(-y)$, i.e. we have $\mathscr{M}^\perp = \{(A^*(-y),y) : y\in\mathscr{H}\} = \{(A^*y,-y) : y\in\mathscr{H}\}$.

By Theorem 5.24, which we can apply because $\mathscr{H}\times\mathscr{H}$ is a Hilbert space with $\mathscr{M}$ a closed subspace of it, we know that any $(a,b)\in\mathscr{H}\in\mathscr{H}$ there is unique $(x,Ax)\in\mathscr{M}$ and $(A^*y,-y)\in\mathscr{M}^\perp$ such that $(a,b) = (x,Ax)+(A^*y,-y)$. Unique $(x,Ax)\in\mathscr{M}$ means unique $x\in\mathscr{H}$, because if $x\neq x'$, then $(x,Ax)\neq(x',Ax')$; and similarly unique $(A^*y,-y)$ means unique $y\in\mathscr{H}$. That is to say, we have unique $x,y\in\mathscr{H}$ s.t. $(a,b) = (x,Ax)+(A^*y,-y) \iff (1)a = x+A^*y, (2)Ax-y=b$, and we have finished the problem.

## Problem 3

Let $P$ be the orthogonal projection associated with the closed subspace $\mathscr{S}$ of a Hilbert space $\mathscr{H}$; i.e. $P$ is a linear map that satisfies $P(f)=f$ if $f\in\mathscr{S}$ and $P(f)=0$ if $f\in\mathscr{S}^\perp$. Since $\mathscr{S}$ is a closed subspace, we know that any $x\in\mathscr{H}$ can be expressed uniquely as $y+z$, where $y\in\mathscr{S}$ and $z\in\mathscr{S}^\perp$ (Theorem 5.24 in Folland). Thus, we have that $P$ is bounded (because $P(x)=P(y)+P(z)=y$, so $\|Px\|^2 = \|y\|^2 \leq \|y\|^2+\|z\|^2 = \|x\|^2 \iff \|Px\|\leq\|x\|$, where we used the Pythagorean theorem), so $P\in L(\mathscr{H},\mathscr{H})$.

(a) We want to show that $P^2 := P \circ P = P$ and $P^* = P$. Using Theorem 5.24, we have that any $x \in \mathscr{H}$ can be written as $y + z$, where $y \in \mathscr{S}, z \in \mathscr{S}^\perp$. Thus, $P(x) = P(y) + P(z) = y + 0 = y$, and clearly $[P \circ P](x) = P(P(x)) = P(y) = y$, meaning that $P \circ P$ and $P$ agree for all $x \in \mathscr{H}$, so they are equal.

Recall from Problem 2(a) of last week that the adjoint $P^*$ is unique; thus to prove $P^* = P$, it suffices to show that $\langle Px, y\rangle = \langle x, Py\rangle$ for all $x, y \in \mathscr{H}$, or equivalently $\langle Px - x, y - Py\rangle = 0$ for all $x, y \in \mathscr{H}$. Again using Theorem 5.24, we can write $x, y \in \mathscr{H}$ as $a + b, c + d$ respectively, for $a, c \in \mathscr{S}$ and $b, d \in \mathscr{S}^\perp$. Then, $\langle Px - x, y - Py\rangle = \langle (a + 0) - (a + c), (c + d) - (c + 0)\rangle = \langle -c, d\rangle$. But because $c \in \mathscr{S} \iff -c \in \mathscr{S}$, and $d \in \mathscr{S}^\perp$, this inner product is 0, so we have indeed shown that $\langle Px - x, y - Py\rangle = 0$ for all $x, y \in \mathscr{H}$, as desired.

(b) Conversely, we also show that if $P \in L(\mathscr{H}, \mathscr{H})$ satisfies $P \circ P = P$ and $P^* = P$, then $P$ is the orthogonal projection associated with some closed subspace $\mathscr{H}$. By Ex. 5.15 in Folland (Problem 3 of 525 Homework 3), we know that $\mathscr{N}(P) = \{x \in \mathscr{H} : Px = 0\}$ is a closed subspace of $\mathscr{H}$ (we can apply Ex. 5.15 because Hilbert spaces are normed vector spaces). Let $\mathscr{S} = (\mathscr{N}(P))^\perp$. Note that $\mathscr{S}$ is a closed subspace of $\mathscr{H}$ (pg. 173 of Folland says $E^\perp$ is closed subspace for any subset $E \subseteq \mathscr{H}$). Also, by Problem 1, we have that $\mathscr{S}^\perp$ is the smallest closed subspace containing $\mathscr{N}(P)$, but because $\mathscr{N}(P)$ is itself a closed subspace, it is *also* the smallest closed subspace containing $\mathscr{N}(P)$; that is to say, $\mathscr{N}(P)$ must equal $S^\perp$.

We now have defined $\mathscr{S}$, a closed subspace of $\mathscr{H}$ s.t. for any $x \in \mathscr{S}^\perp$, $Px = 0$. Thus, to conclude that $P$ is the orthogonal projection associated with $\mathscr{S}$, we just need to show that for any $x \in \mathscr{S}$, $Px = x$. Let us fix $x \in \mathscr{S}$. We can show that $Px = x$, by showing that $\|Px - x\|^2 = \langle Px - x, Px - x\rangle = 0$. Note that $\langle Px - x, Px - x\rangle = \langle Px - x, Px\rangle - \langle Px - x, x\rangle$. But because $P = P^*$ and $P \circ P = P$, we see that $\langle Px - x, Px\rangle = \langle P(Px - x), x\rangle = \langle Px - Px, x\rangle = \langle 0, x\rangle = 0$. Note also that we just proved that $P(Px - x) = 0$, so $Px - x \in \mathscr{N}(P) = \mathscr{S}^\perp$. But because $x \in \mathscr{S}$, $\langle Px - x, x\rangle = 0$, and so indeed $\langle Px - x, Px - x\rangle = 0$. Because $x \in \mathscr{S}$ was arbitrary, we have that $Px = x$ for all $x \in \mathscr{S}$, and we are done.

## Problem 4

Let $\mathscr{X}, \mathscr{Y}$ be Banach spaces with $T \in L(\mathscr{X}, \mathscr{Y})$ a bounded linear injective map whose range $\mathrm{im}(T) = T(\mathscr{X})$ is closed in $\mathscr{Y}$. We want to show that for each $\phi \in \mathscr{X}^*$ (bounded linear functional on $\mathscr{X}$) there is $\psi \in \mathscr{Y}^*$ s.t. $\phi = \psi \circ T$, and there is a constant $C$ independent of $\phi$ s.t. $\psi$ can be chosen to satisfy $\|\psi\| \leq C\|\phi\|$ (where the norms are the operator norms on $\mathscr{X}^*$ and $\mathscr{Y}^*$ respectively).

Note that $\mathrm{im}(T)$ is a subspace (since $y, y' \in \mathrm{im}(T)$ means there are $x, x' \in \mathscr{X}$ s.t. $Tx = y, Tx' = y'$, so for any $a, b \in \mathbb{C}$, $ay + by' = aTx + bTx' = T(ax + bx') \in \mathrm{im}(T)$), so closedness of $\mathrm{im}(T)$ gives that $\mathrm{im}(T)$ is a Banach space (this fact is important, since in the next paragraph we will be using results that depend on $\mathscr{X}$ and $\mathrm{im}(T)$ being Banach spaces).

Let $S$ be $T$ except thought of as in $L(\mathscr{X}, \text{im}(T))$ instead. Then, $S$ is still bounded, linear and injective of course, and it's now surjective by definition of $\text{im}(T)$. Thus, $S$ is a bijection and by Corollary 5.11, we know that $S^{-1} \in L(\text{im}(T), \mathscr{X})$. Also, note that because for all $x \in \mathscr{X}$, $Sx = Tx$ (by definition), we have $S^{-1}(Tx) = S^{-1}(Sx) = x$ for all $x \in \mathscr{X}$. Let us now fix arbitrary $\phi \in \mathscr{X}^*$. Thus, defining $\tilde{\psi}$ on $\text{im}(T)$ to be $\tilde{\psi}(y) = \phi(S^{-1}(y))$, we have that for every $x \in \mathscr{X}$, $\tilde{\psi}(Tx) = \phi(S^{-1}(Tx)) = \phi(x)$, i.e. $\phi = \tilde{\psi} \circ T$ on $\mathscr{X}$.

Now note that by the "product rule for norms" (pg. 154 of Folland), $|\tilde{\psi}y| \leq \|\phi\|\|S^{-1}\|\|y\|$ for all $y \in \text{im}(T)$, i.e. defining $p(y) = \|\phi\|\|S^{-1}\|\|y\|$ on $\mathscr{Y}$, we have that $p$ is a seminorm on $\mathscr{Y}$ with $|\tilde{\psi}(y)| \leq p(y)$ for all $y \in \text{im}(T)$. Because $\tilde{\psi}(y) = \phi(S^{-1}(y))$, a composition of linear maps that outputs a complex number, $\tilde{\psi}$ is a complex linear functional on $\text{im}(T)$. As we said in the second paragraph that $\text{im}(T)$ is a subspace, we can apply the complex Hahn-Banach theorem (Thm. 5.7 in Folland), which tells us that $\tilde{\psi}$ can be extended to a complex linear function $\psi$ on all of $\mathscr{Y}$ s.t. $|\psi(y)| \leq p(y)$ for all $y \in \mathscr{Y}$, and satisfies $\psi|_{\text{im}(T)} = \tilde{\psi}$.

Thus, we still have that $\psi$ satisfies $\phi(x) = \psi(T(x))$ for all $x \in \mathscr{X}$, i.e. $\phi = \psi \circ T$, and also satisfies $|\psi(y)| \leq \|\phi\|\|S^{-1}\|\|y\| \iff \|\psi\| \leq \|\phi\|\|S^{-1}\|$. That is to say, letting $C = \|S^{-1}\|$ (which is independent of the chosen $\phi \in \mathscr{X}^*$ since it only depends on $T$), for any arbitrary $\phi \in \mathscr{X}^*$, we have found $\psi \in \mathscr{Y}^*$ such that $\phi = \psi \circ T$ and $\|\psi\| \leq C\|\phi\|$, exactly as desired.

## Problem 5

Suppose $\mathscr{H}$ is a Hilbert space and $y_n \in \mathscr{H}$ are such that for every $x \in \mathscr{H}$, $\langle x, y_n \rangle$ converges (as $n \to \infty$) to something in $\mathbb{C}$. We want to show that there exists $y \in \mathscr{H}$ s.t. for every $x \in \mathscr{H}$, $\lim_{n\to\infty} \langle x, y_n \rangle = \langle x, y \rangle$. Let us define $f : \mathscr{H} \to \mathbb{C}$ by $f(x)$ is whatever $\langle x, y_n \rangle$ converges to. Then, if we show that $f$ is bounded and linear, i.e. $f \in \mathscr{H}^*$, then by Riesz representation (Theorem 5.25 in Folland), there is unique $y \in \mathscr{H}$ s.t. $f(x) = \langle x, y \rangle$ for all $x \in \mathscr{H}$. That is to say, if we should that $f$ is linear and bounded, then we will be done.

Linearity: because a sum of limits is the limit of the sum (provided all the limits exist), and same for scalar multiples, we have that $f(ax + bx') = \lim_{n\to\infty} \langle ax + bx', y_n \rangle = \lim_{n\to\infty} (a\langle x, y_n \rangle + b\langle x', y_n \rangle) = a \lim_{n\to\infty} \langle x, y_n \rangle + b \lim_{n\to\infty} \langle x', y_n \rangle = af(x) + bf(x')$.

Boundedness: for each $n \in \mathbb{N}$, $f_n : \mathscr{H} \to \mathbb{C}$ defined as $f_n(x) = \langle x, y_n \rangle$ is a bounded linear operator. As $\mathscr{H}$ is a Hilbert space and hence Banach space, and for all $x \in \mathscr{H}$, $\sup_{n\in\mathbb{N}} |f_n(x)| < \infty$ (we know this because for every $x \in \mathscr{H}$, the limit $\lim_{n\to\infty} |f_n(x)| = |\lim_{n\to\infty} \langle x, y_n \rangle| = |f(x)|$ exists, and convergent sequences are bounded), the uniform boundedness principle (Theorem 5.13 in Folland) tells us that $\sup_{n\in\mathbb{N}} \|f_n\| =: C < \infty$. Thus, for any $x \in \mathscr{H}$, $|f(x)| = \lim_{n\to\infty} |\langle x, y_n \rangle| \leq \sup_{n\in\mathbb{N}} |f_n(x)| \leq \sup_{n\in\mathbb{N}} (\|f_n\|\|x\|) \leq C\|x\|$, implying that indeed $f$ is a bounded linear functional, so we are done.

# 525 Homework 5

Daniel Rui - 2/10/20

## Problem 1 - 5.3.38

Let $\mathscr{X}$ and $\mathscr{Y}$ be Banach spaces and let $\{T_n\}_{n=1}^{\infty}$ be a sequence in $L(\mathscr{X},\mathscr{Y})$. s.t. $\lim_{n\to\infty} T_n(x)$ exists for every $x \in \mathscr{X}$. We can therefore define $T(x) = \lim_{n\to\infty} T_n(x)$. We want to show that $T \in L(\mathscr{X},\mathscr{Y})$. Regarding linearity, we have that $T(ax+by) = \lim_{n\to\infty}(T_n(ax+by)) = \lim_{n\to\infty}(aT_n(x)+bT_n(y))$, but a limit of a sum is the sum of the limits (provided they exist), and a limit of a scalar multiple is just a scalar multiple of the limit, so this is equal to $= a\lim_{n\to\infty} T_n(x) + b\lim_{n\to\infty} T_n(y) = aT(x)+bT(y)$. That proved linearity, so we just need to show boundedness. First observe that

$$\|Tx\| = \lim_{n\to\infty}\|T_nx\| \le \lim_{n\to\infty}\|T_n\|\|x\| \le \left(\sup_{n\in\mathbb{N}}\|T_n\|\right)\|x\|,$$

where the first equality is due to the fact that the norm $\|\cdot\|$ is continuous and hence can move limits inside and outside. The reason we introduced the "sup" is because Theorem 5.13 (the uniform boundedness principle) concerns itself with bounds on supremums over sets of the norms of bounded linear maps. Indeed, this principle "of almost magical power" as Folland puts it is the key to the remainder of the problem: recall that for any $x \in \mathscr{X}$, $\|Tx\| = \lim_{n\to\infty}\|T_nx\|$, i.e. we have a sequence of real numbers converging to some real number, implying that $\sup_{n\in\mathbb{N}}\|T_nx\|$ must be bounded; thus, we have that for any $x \in \mathscr{X}$, $\sup_{n\in\mathbb{N}}\|T_nx\| < \infty$, so by the uniform bounded principle, $\sup_{n\in\mathbb{N}}\|T_n\| < \infty$. Letting $C = \sup_{n\in\mathbb{N}}\|T_n\|$, the big equation above says $\|Tx\| \le C\|x\|$ for all $x \in \mathscr{X}$, so $\|T\| \le C < \infty$ and we have shown boundedness, and we are done.

## Problem 2 - 5.5.57

Suppose that $\mathscr{H}$ is a Hilbert space and $T \in L(\mathscr{H},\mathscr{H})$.

(a) We want to show that there is a unique $T^* \in L(\mathscr{H},\mathscr{H})$ (called the adjoint of $T$) s.t. $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x, y \in \mathscr{H}$. Recall from Ex. 5.22 of Folland (i.e. Problem 1 last week) that for normed vector spaces $\mathscr{X},\mathscr{Y}$ and $T \in L(\mathscr{X},\mathscr{Y})$, there is the adjoint $T^\dagger \in L(\mathscr{Y}^*,\mathscr{X}^*)$ s.t. $T^\dagger(f) = f\circ T$ and $\|T^\dagger\| = \|T\|$. Folland's hint suggests that $T^* = V^{-1}T^\dagger V$, where $V : \mathscr{H} \to \mathscr{H}^*$ is defined by $[V(y)](x) = \langle x, y\rangle$, so let us verify this hint. The Riesz representation theorem (Theorem 5.25 in Folland) tells us that $V$ is surjective and injective, since its says that all $f \in \mathscr{H}^*$ are just $\langle x, y\rangle$ for some unique $y \in \mathscr{H}$. Thus, $V^{-1}$ makes sense.

Let us now check the adjoint identity:

$$\langle x, T^*y\rangle = \left\langle x, (V^{-1}T^\dagger V)y\right\rangle = [V((V^{-1}T^\dagger V)y)](x) = [T^\dagger(Vy)](x) = [V(y)\circ T](x) = \langle Tx, y\rangle.$$

With this, we can know check that $T^*$ is linear and bounded.

Linearity: note that for any $x \in \mathscr{H}$, $\langle x, T^*(ay+bz)\rangle = \langle Tx, ay+bz\rangle = \bar{a}\langle Tx, y\rangle + \bar{b}\langle Tx, z\rangle = \bar{a}\langle x, T^*y\rangle + \bar{b}\langle x, T^*z\rangle = \langle x, aT^*(y) + bT^*(z)\rangle$, meaning that in particular for $x = aT^*(y)+bT^*(z)-T^*(ay+bz)$,

$$\langle x, aT^*(y) + bT^*(z) - T^*(ay+bz)\rangle = \|aT^*(y) + bT^*(z) - T^*(ay+bz)\| = 0,$$

implying that indeed $aT^*(y) + bT^*(z) = T^*(ay+bz)$ ($y, z$ arbitrary in $\mathscr{H}$). We will call this the "SUBTRACT, CHOOSE $x$ GET NORM 0 TRICK" in the future.

Boundedness: since

$$\|T^*x\|^2 = \langle T^*x, T^*x\rangle = \langle T(T^*x), x\rangle \leq \|x\|\|T(T^*x)\| \leq \|x\|\|T\|\|T^*x\|$$

(where the first inequality is from Cauchy-Schwarz), we have $\|T^*x\| \leq \|T\|\|x\|$ for all $x \in \mathscr{H}$ (where we divided by $\|T^*x\|$ if it wasn't 0, and if it was 0, the inequality is trivial — we will call this "DIVISION JUSTIFICATION" in the future). Thus, in particular $\|T^*\| \leq \|T\|$, so $T^*$ is bounded telling us that finally $T^* \in L(\mathscr{H}, \mathscr{H})$.

For uniqueness: if $S^* \in L(\mathscr{H}, \mathscr{H})$ also satisfies the adjoint identity, then fixing $y \in \mathscr{H}$, we have that for all $x \in \mathscr{H}$, $\langle x, S^*y\rangle = \langle Tx, y\rangle = \langle x, T^*y\rangle$. Using the "SUBTRACT, CHOOSE $x$ GET NORM 0 TRICK", we see that $S^*(y) = T^*(y)$ for all $y \in \mathscr{H}$, so $S^* = T^*$ and uniqueness is proven.

(b) We want to show that (1) $\|T^*\| = \|T\|$, (2) $\|T^*T\| = \|T\|^2$, (3) $(aS+bT)^* = \bar{a}S^* + \bar{b}T^*$, (4) $(ST)^* = T^*S^*$, and (5) $T^{**} = T$.

(1): at the end of part (a) we proved that $\|T^*\| \leq \|T\|$. We can repeat the calculation:

$$\|Tx\|^2 = \langle Tx, Tx\rangle = \langle x, T^*(Tx)\rangle \leq \|x\|\|T^*(Tx)\| \leq \|x\|\|T^*\|\|Tx\|,$$

which yields $\|Tx\| \leq \|T^*\|\|x\| \implies \|T\| \leq \|T^*\|$ by "DIVISION JUSTIFICATION", and so (1) is done.

(2): by the "product rule for norms" (see pg. 154 of Folland), we have that $\|T^*T\| \leq \|T^*\|\|T\| = \|T\|^2$, so we just need the other direction. Looking at the above calculation, we can get a different conclusion:

$$\|Tx\|^2 = \langle Tx, Tx\rangle = \langle x, T^*(Tx)\rangle \leq \|x\|\|T^*(Tx)\| \leq \|x\|\|T^*T\|\|x\|,$$

implying that $\|Tx\| \leq \sqrt{\|T^*T\|}\|x\|$ (recalling that norms spit out non-negative real numbers, hence we can take the square root) for all $x \in \mathscr{H}$. This gives that $\|T\| \leq \sqrt{\|T^*T\|} \implies \|T\|^2\|T^*T\|$, which is the desired other direction so we are done with (2).

(3): fixing arbitrary $y \in \mathscr{H}$, we have that for all $x \in \mathscr{H}$,

$$\begin{aligned}\langle x, (aS+bT)^* y\rangle &= \langle (aS+bT)x, y\rangle = a\langle Sx, y\rangle + b\langle Tx, y\rangle \\ &= a\langle x, S^*y\rangle + b\langle x, T^*y\rangle = \big\langle x, \bar{a}S^*(y) + \bar{b}T^*(y)\big\rangle = \big\langle x, (\bar{a}S^* + \bar{b}T^*)y\big\rangle.\end{aligned}$$

Using the "SUBTRACT, CHOOSE $x$ GET NORM 0 TRICK", we get that $\bar{a}S^* + \bar{b}T^*$ and $(aS+bT)^*$ agree for all $y \in \mathscr{H}$, so (3) is proven. (4) is proven in much the same way; using the adjoint identity and using the "SUBTRACT, CHOOSE $x$ GET NORM 0 TRICK":

$$\langle x, (ST)^* y\rangle = \langle STx, y\rangle = \langle Tx, S^*y\rangle = \langle x, T^*(S^*y)\rangle = \langle x, (T^*S^*)y\rangle.$$

(5): for any $x, y \in \mathscr{H}$, we have that $\langle T^*x, y\rangle = \overline{\langle y, T^*x\rangle} = \overline{\langle Ty, x\rangle} = \langle x, Ty\rangle$, so by the uniqueness of adjoint from part (a) we know that $T^{**} = T$. We are finally done with part (b).

(c) Let $\mathscr{R}$ and $\mathscr{N}$ denote the range and nullspace. We want to show that $\mathscr{R}(T)^\perp = \mathscr{N}(T^*)$ and $\mathscr{N}(T)^\perp = \overline{\mathscr{R}(T^*)}$. Observe:

$$\begin{aligned}\mathscr{R}(T)^\perp &:= \{x \in \mathscr{H} : \langle x, z\rangle = 0 \text{ for all } z \in \mathscr{R}(T)\} = \{x \in \mathscr{H} : \langle z, x\rangle = 0 \text{ for all } z \in \mathrm{im}(T)\} \\ &= \{x \in \mathscr{H} : \langle Ty, x\rangle = 0 \text{ for all } y \in \mathscr{H}\} = \{x \in \mathscr{H} : \langle y, T^*x\rangle = 0 \text{ for all } y \in \mathscr{H}\}.\end{aligned}$$

But the inner product being 0 for all $y \in \mathscr{H}$ happens if and only if $T^*x = 0$, so we have that $\mathscr{R}(T)^\perp = \{x \in \mathscr{H} : T^*x = 0\} = \mathscr{N}(T^*)$. With this result we see that $\mathscr{R}(T^*)^\perp = \mathscr{N}(T^{**}) = \mathscr{N}(T)$, and so $\mathscr{N}(T)^\perp = (\mathscr{R}(T^*)^\perp)^\perp$. We now show that for any subspace $E \subseteq \mathscr{H}$, $\overline{E} = (E^\perp)^\perp$. For the $\subseteq$ direction, we clearly have $E \subseteq (E^\perp)^\perp$, and we know from pg. 173 of Folland that $E^\perp$ is a closed subspace for any $E \subseteq \mathscr{H}$, so we can take the closure of both sides.

For the other $\supseteq$ direction, suppose on the contrary we have $x_0 \in (E^\perp)^\perp$ but not in $\mathscr{M} = \overline{E}$. By Theorem 5.8(a), we know there exists $f \in \mathscr{H}^*$ s.t. $f(x_0) \neq 0$, but $f|_{\mathscr{M}} = 0$. By Riesz representation (Theorem 5.25), there is a unique $y_0 \in \mathscr{H}$ s.t. $f(x) = \langle x, y_0\rangle$ for all $x \in \mathscr{H}$. Thus in particular, $\langle x, y_0\rangle = 0$ for all $x \in E \subseteq \overline{E}$, meaning that $y_0 \in E^\perp$. But $\langle x_0, y_0\rangle = f(x_0) \neq 0$, so $x_0$ can't be in $(E^\perp)^\perp$; contradiction.

Because $\mathscr{R}(T)$ is a subspace of $\mathscr{H}$, the lemma we just proved indicates that $\mathscr{N}(T)^\perp = (\mathscr{R}(T^*)^\perp)^\perp = \overline{\mathscr{R}(T^*)}$. This concludes part (c).

(d) We want to show that $T$ is unitary iff $T$ is invertible and $T^{-1} = T^*$. ($\implies$): Folland defines a unitary map $\mathscr{H}_1 \to \mathscr{H}_2$ to be an invertible map that preserves inner products, meaning that by definition $T$ unitary implies $T$ invertible. Using the "preserves inner products" part of the definition, we see that $\big\langle x, T^{-1}y\big\rangle = \big\langle Tx, TT^{-1}y\big\rangle = \langle Tx, y\rangle$, so by part (a) (specifically uniqueness) we know that $T^{-1}$ must be $T^*$.

($\impliedby$): if $T$ is invertible and $T^{-1}$ satisfies $T^{-1} = T^*$, then to prove that $T$ is unitary all we need to do is check that it preserves inner products: for any $x, y \in \mathscr{H}$, we have $\langle Tx, Ty\rangle = \langle x, T^*Ty\rangle = \big\langle x, T^{-1}Ty\big\rangle = \langle x, y\rangle$, so we are done.

## Problem 3 - 5.5.59

We want to show that every closed convex set $K$ in a Hilbert space $\mathscr{H}$ has a unique element of minimal norm. As Folland says, if $0 \in K$, then obviously the unique element of minimal norm is just 0, and if not, we can adapt (the first half of) his proof of Theorem 5.24. We shall reproduce (almost word-for-word) his proof with the necessary changes with some elaboration: let us take $x = 0 \in \mathscr{H}$, $\delta = \inf\{\|x - y\| : y \in K\} = \inf\{\|y\| : y \in K\}$, and $\{y_n\}_{n=1}^{\infty}$ a sequence in $K$ s.t. $\|x - y_n\| = \|y_n\| \to \delta$ (which we can do by the definition of infimum). By the parallelogram law,

$$2\big(\|y_n - x\|^2 + \|y_m - x\|^2\big) = \|y_n - y_m\|^2 + \|y_n + y_m - 2x\|^2,$$

so since $\frac{1}{2}(y_n + y_m) \in K$ (by convexity of $K$)

$$\begin{aligned}\|y_n - y_m\|^2 &= 2\|y_n - x\|^2 + 2\|y_m - x\|^2 - 4\|\tfrac{1}{2}(y_n + y_m) - x\|^2 \\ &\leq 2\|y_n - x\|^2 + 2\|y_m - x\|^2 - 4\delta^2\end{aligned}$$

As $m, n \to \infty$ this last quantity tends to zero (goes to $2\delta^2 + 2\delta^2 - 4\delta^2 = 0$), so $\{y_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Let $y = \lim_{n\to\infty} y_n$ (which exists because Cauchy sequence); then $y \in K$ since $K$ is closed, and $\|x - y\| = \|y\| = \delta$ (since by continuity of $\|\cdot\|$ we have $\|y\| = \lim_{n\to\infty} \|y_n\| = \delta$). As for uniqueness, the above big inequality involving $y_n, y_m$ holds true for any $y, y' \in K$, so if we have $y, y' \in K$ s.t. $\|y\| = \|y'\| = \delta$, then the above big inequality tells us that

$$\|y - y'\|^2 \leq 2\|y\|^2 + 2\|y'\|^2 - 4\delta^2 = 0 \implies y = y',$$

and we are done.

## Problem 4 - 5.5.60

Let $(X, \mathscr{M}, \mu)$ be a measure space. For $E \in \mathscr{M}$, we identify $L^2(E, \mu)$ with the subspace of the Hilbert space $L^2(X, \mu)$ consisting of functions that vanish outside $E$. If $\{E_n\}$ is a disjoint sequence in $\mathscr{M}$ with $X = \bigcup_{n=1}^{\infty} E_n$, then we want to show that $\{L^2(E_n, \mu)\}_{n=1}^{\infty}$ is a sequence of mutually orthogonal subspaces of $L^2(X, \mu)$, and moreover every $f \in L^2(X, \mu)$ can be written uniquely as $f = \sum_{n=1}^{\infty} f_n$ (series converging in norm) for $f_n \in L^2(E_n, \mu)$. Lastly, if $L^2(E_n, \mu)$ is separable for every $n$, so is $L^2(X, \mu)$.

Mutually orthogonal: for any $f_n \in L^2(E_n, \mu)$ and $f_m \in L^2(E_m, \mu)$ (arbitrary $m, n \in \mathbb{N}$), we have by disjointness of the $\{E_n\}_{n=1}^{\infty}$ that $f_n$ is 0 on $E_m$ and $f_m$ is 0 on $E_n$, meaning that $\langle f_n, f_m \rangle = \int_X f_n \overline{f_m}\, d\mu = \int_X 0\, d\mu = 0$. As $m, n \in \mathbb{N}$ were arbitrary, and $f_n \in L^2(E_n, \mu)$ and $f_m \in L^2(E_m, \mu)$ were arbitrary, this shows that all the $\{L^2(E_n, \mu)\}_{n=1}^{\infty}$ are mutually orthogonal.

Decomposition: for any $f \in L^2(X, \mu)$, we can define $f_n := f \cdot 1_{E_n}$. Then for any $n \in \mathbb{N}$, $f_n$ is indeed in $L^2(E_n, \mu)$, since $\int_E |f_n|^2\, d\mu = \int_E |f|^2\, d\mu \leq \int_X |f|^2\, d\mu < \infty$, and obviously $f = \sum_{n=1}^{\infty} f_n$ (an equality

of real/complex numbers for every $x \in X$). Also the series converges to $f$ in the norm since

$$\left\| f - \sum_{n=1}^{N} f_n \right\|_{L^2(X,\mu)} \leq \sum_{n=N+1}^{\infty} \|f_n\|_{L^2(X,\mu)} = \sum_{n=N+1}^{\infty} \|f_n\|_{L^2(E_n,\mu)} = \sum_{n=N+1}^{\infty} \left| \int_{E_n} f_n \overline{f_n}\, d\mu \right|$$
$$\leq \sum_{n=N+1}^{\infty} \int_{E_n} |f|^2\, d\mu = \int_{\bigcup_{n=N+1}^{\infty} E_n} |f|^2\, d\mu$$

As $|f|^2 \in L^1(X,\mu)$, we can define a measure $\nu(E) = \int_E |f|^2\, d\mu$ (Ex. 2.14 in Folland, or 524 HW 5 Problem 2). Because $D_N = \bigcup_{n=N+1}^{\infty} E_n$ is a decreasing sequence of sets (as $N \to \infty$), decreasing to $\varnothing$ (since for any $x \in X$, there is unique $k \in \mathbb{N}$ s.t. $x \in E_k$ since $X = \bigcup_{n=1}^{\infty} E_n$ and the $E_n$ are disjoint, so for $N > k$, $D_N$ will not contain $x$), and $\nu(X) = \int_X |f|^2\, d\mu < \infty$, we have by upper continuity of measure that $\nu(D_N) \to 0$ as $N \to \infty$. This proves that the series converges to $f$ in norm.

Uniqueness: if $\sum_{n=1}^{\infty} f_n = f = \sum_{n=1}^{\infty} g_n$ and $f_n, g_n \in L^2(E_n,\mu)$, then multiplying by $1_{E_n}$ we have that $1_{E_n} \cdot f_n = 1_{E_n} \cdot f = 1_{E_n} \cdot g_n$ (since $f_m, g_m$ are 0 on $E_n$, since $E_n$ is outside of $E_m$ by disjointness of the $\{E_i\}_{i=1}^{\infty}$). But $f_n, g_n$ are 0 outside of $E_n$, so this equality is $f_n = 1_{E_n} f = g_n$. This shows that this decomposition is unique.

Separability: by Prop. 5.29 in Folland, we know that there are orthonormal bases for each $L^2(E_n,\mu)$, say $\{u_{n,i}\}_{i=1}^{\infty}$. As we know that the $\{L^2(E_n,\mu)\}_{n=1}^{\infty}$ are mutually orthogonal, the union of all such $\{u_{n,i}\}_{i=1}^{\infty}$ is still an orthonormal set, and moreover a countable one (countable union of countable sets is countable). That means we can write $V = \{v_i\}_{i=1}^{\infty} = \bigcup_{n=1}^{\infty} \{u_{n,i}\}_{i=1}^{\infty}$. We now just need to show that $V$ is an orthonormal basis, or namely that $V$ satisfies property (a) in Theorem 5.27. Well suppose that $\langle f, v_i \rangle = 0$ for all $i \in \mathbb{N}$. Then, in particular for any $n \in \mathbb{N}$, $\langle f, u_{n,i} \rangle = 0$ for all $i \in \mathbb{N}$. Because the $\{u_{n,i}\}_{i=1}^{\infty}$ were an orthonormal basis of $L^2(E_n,\mu)$, property (a) tells us that $f$ must be 0 on $E_n$ a.e., i.e. 0 on $E_n \setminus S_n$ for some null set $S_n$. But this is for every $n \in \mathbb{N}$, and $X = \bigcup_{n=1}^{\infty} E_n$, so $f$ is 0 on all of $X$ except maybe on $\bigcup_{n=1}^{\infty} S_n$, a null set (since it's a countable union of null sets). But in $L^2$ we identify functions that are a.e. equivalent, so $f = 0 \in L^2(X,\mu)$, and so property (a) holds for the orthonormal set $V$ so $V$ is an orthonormal basis of $L^2(X,\mu)$, so again by Prop. 5.29 $L^2(X,\mu)$ is separable.

## Problem 5 - 5.5.61

Let $(X, \mathscr{M}, \mu)$ and $(Y, \mathscr{N}, \nu)$ be $\sigma$-finite measure spaces s.t. $L^2(\mu)$ and $L^2(\nu)$ are separable. If $\{f_m\}$ and $\{g_n\}$ are orthonormal bases for $L^2(\mu)$ and $L^2(\nu)$ and $h_{m,n}(x,y) = f_m(x) g_n(y)$, then we want to show that $\{h_{m,n}\}_{m,n=1}^{\infty}$ is an orthonormal basis for $L^2(\mu \times \nu)$. First we show that this set is an orthonormal set: using Fubini, which tells us that the integral over a product measure can split into

an iterated integral,

$$\begin{aligned}\langle h_{m,n}, h_{i,j}\rangle &= \int_{X\times Y} h_{m,n}\overline{h_{i,j}}\, d[\mu\times\nu] = \int_X\int_Y f_m(x)\overline{f_i(x)}g_n(y)\overline{g_j(y)}\, d\nu(y)\, d\mu(x)\\ &= \int_X f_m(x)\overline{f_i(x)}\left(\int_Y g_n(y)\overline{g_j(y)}\, d\nu(y)\right) d\mu(x) = \int_X f_m(x)\overline{f_i(x)}\, d\mu(x)\cdot\langle g_n, g_j\rangle\\ &= \langle f_m, f_j\rangle\langle g_n, g_j\rangle,\end{aligned}$$

which is 1 if $n = m, i = j$ and 0 otherwise. Now like the above problem (Problem 4), we use property (a) from Theorem 5.27 to check that $\{h_{m,n}\}_{m,n=1}^{\infty}$ is an orthonormal basis of $L^2(\mu\times\nu)$. We now suppose that for all $m, n \in \mathbb{N}$, $\langle h, h_{m,n}\rangle = 0$. Notice that for any $h \in L^2(\mu\times\nu)$ (again using Fubini like before),

$$\begin{aligned}0 = \langle h, h_{m,n}\rangle &= \int_X\int_Y h(x,y)\overline{f_m(x)g_n(y)}\, d\nu(y)\, d\mu(x) = \int_X \overline{f_m(x)}\left(\int_Y h(x,y)\overline{g_n(y)}\, d\nu(y)\right) d\mu(x)\\ &= \left\langle\left(\int_Y h(x,y)\overline{g_n(y)}\, d\nu(y)\right), f_m(x)\right\rangle.\end{aligned}$$

For any fixed $n \in \mathbb{N}$, the above inner product is 0 for all $m \in \mathbb{N}$, so because the $\{f_m\}$ are an orthonormal basis of $L^2(\mu)$, we have that the following function $\phi_n$ of $x$, $\left(\int_Y h(x,y)\overline{g_n(y)}\, d\nu(y)\right)$, is $\mu$-a.e. 0 on $X$, i.e. it's 0 on $X\setminus S_n$ for some null set $S_n$ (and we can do this for every $n \in \mathbb{N}$). Letting $S = \bigcup_{n=1}^{\infty} S_n$, we see that it is also a null set, and moreover for every $n \in \mathbb{N}$, $\phi_n(x)$ is 0 on $X\setminus S$. Fixing any $x_0 \in X\setminus S$, we have that

$$\left(\int_Y h(x_0,y)\overline{g_n(y)}\, d\nu(y)\right) = \langle h(x_0,y), g_n(y)\rangle = 0$$

for all $n \in \mathbb{N}$, so because the $\{g_n\}$ are an orthonormal basis of $L^2(\nu)$, we have that $h(x_0, y)$ is 0 $\nu$-a.e., i.e. it's 0 on $Y\setminus T_{x_0}$ for some null set $T_{x_0}$. Unfortunately the "number" of $x_0 \in X$ may be uncountable, so we can not the "countable union" trick we did before, but we can do a trick with integrals instead: since for any $x_0 \in X\setminus S$, we know that $h(x_0, y)$ is 0 for $\nu$-a.e. $y \in Y$, we know that $\int_Y |h(x_0,y)|\, d\nu(y) = 0$. Then, observe that (again using Fubini like before, and using that integrals remain the same after modification of the function on a null set):

$$\int_{X\times Y}|h(x,y)|\, d[\mu\times\nu] = \int_X\left(\int_Y |h(x,y)|\, d\nu(y)\right) d\mu(x) = \int_X 1_{X\setminus S}(x)\left(\int_Y |h(x,y)|\, d\nu(y)\right) d\mu(x) = 0.$$

This means that $h$ is 0 $[\mu\times\nu]$-a.e. on $X\times Y$, but we identify functions that are equal a.e., so indeed $h = 0 \in L^2(\mu\times\nu)$, and property (a) is verified, implying that the $\{h_{m,n}\}$ are indeed an orthonormal basis.

## Problem 6 - 5.5.66

Let $\mathscr{M}$ be a closed subspace of $L^2([0,1], m)$ that is contained in $C([0,1])$.

(a) We want to show that there is some $C > 0$ s.t. $\|f\|_{\mathrm{u}} \le C\|f\|_{L^2}$ for all $f \in \mathscr{M}$. Folland suggests

to use the closed graph theorem; to use that theorem, we first need Banach spaces $\mathscr{X}, \mathscr{Y}$ and a closed linear map $T : \mathscr{X} \to \mathscr{Y}$. Once we have those, the CGT tells us that $T$ is bounded. This can help us, because if we let $\mathscr{X} = \mathscr{M}$ (Folland "takes it for granted" on pg. 173 that $L^2([0,1])$ is a Hilbert space, so in particular because $\mathscr{M}$ is closed, it is a Banach space w.r.t. the $\|\cdot\|_{L^2}$ norm), and $\mathscr{Y} = C([0,1])$ (indeed $C([0,1])$ is a vector space that is complete under the $\|\cdot\|_{\mathrm{u}}$ norm), then boundedness gives $\|Tf\|_{\mathrm{u}} \leq C\|f\|_{L^2}$ for some $C > 0$. Then, the result follows if $Tf = f$ (i.e. it sends a function $f \in \mathscr{M}$ to that same function but now thought of as an element in $C([0,1])$). We formalize this below.

Letting $T : \mathscr{M} \to C([0,1])$ ($\mathscr{X}$ and $\mathscr{Y}$ respectively) be defined as $T(f) = f$ (in the sense above), we see that it is obviously linear (as $\mathscr{X}, \mathscr{Y}$ are both vector spaces and $T$ just sends a function in $\mathscr{X}$ to its "version" in $\mathscr{Y}$). We now just need to show that $T$ is a closed linear map in order to be done. To verify closedness, we assume that we have some sequence $f_n \to f$ (in the $\mathscr{X}$ norm), and $T(f_n) \to g$ (in the $\mathscr{Y}$ norm), and check that $T(f) = g$.

Since $T(f) = f$, it suffices (actually equivalent) to show that $\|f - g\|_{L^2} = 0$ (since $\|\cdot\|_{L^2}$ is a norm and norms are 0 iff the element is 0). Well, by the triangle inequality for norms, we have that for any $n \in \mathbb{N}$,

$$\|f - g\|_{L^2} = \|f - f_n + T(f_n) - g\|_{L^2} \leq \|f - f_n\|_{L^2} + \|T(f_n) - g\|_{L^2}.$$

Recall that $f_n \to f$ in the $\mathscr{X}$ i.e. $L^2$ norm, so fixing some $\epsilon > 0$, there is indeed some $N \in \mathbb{N}$ s.t. $n \geq N \implies \|f - f_n\|_{L^2} < \epsilon/2$. As for the second term, we do not have an immediate bound, since we only know that $T(f_n) \to g$ in the $\mathscr{Y}$ norm, i.e. $\|T(f_n) - g\|_{\mathrm{u}} \to 0$, but fortunately we can recover an $L^2$ bound from the uniform bound: $\|T(f_n) - g\|_{L^2} = (\int_0^1 |T(f_n) - g|^2 \, dm)^{1/2} \leq (\int_0^1 \|T(f_n) - g\|_{\mathrm{u}}^2 \, dm)^{1/2} = \|T(f_n) - g\|_{\mathrm{u}}$. Therefore, there is indeed some $M \in \mathbb{N}$ s.t. $n \geq M \implies \|T(f_n) - g\|_{L^2} < \epsilon/2$. Combining these results, we see that for any $n \geq \max\{N, M\}$, the big inequality above tells us that $\|f - g\|_{L^2} < \epsilon$. As $\epsilon > 0$ is arbitrary, it must be that $\|f - g\|_{L^2} = 0$, and we are done.

(b) We want to show that for each fixed $x \in [0,1]$, there exists $g_x \in \mathscr{M}$ s.t. $f(x) = \langle f, g_x \rangle$ for all $f \in \mathscr{M}$, and $\|g_x\|_{L^2} \leq C$. Define $F : \mathscr{M} \to \mathbb{C}$ by $F(f) = f(x)$ (the fixed $x \in [0,1]$ chosen above). This is linear because $F(af + bg) = [af + bg](x) = af(x) + bg(x) = aF(f) + bF(g)$, and it's bounded (in fact $\|F\|_{\mathscr{M}^*} \leq C$, the $C$ in (a)) because $|F(f)| = |f(x)| \leq \|f\|_{\mathrm{u}} \leq C\|f\|_{L^2}$ (from (a)). Thus, $F \in \mathscr{M}^*$, and we can use the Riesz representation theorem (Thm. 5.25 in Folland), which tells us that there is a unique $g_x \in \mathscr{M}$ s.t. $F(f) = \langle f, g_x \rangle$ for all $f \in \mathscr{M}$.

As for the upper bound of $C$, observe that $\|g_x\|_{L^2}^2 = \langle g_x, g_x \rangle = F(g_x) = g_x(x) \leq \|g_x\|_{\mathrm{u}} \leq C\|g_x\|_{L^2}$, implying that $\|g_x\|_{L^2} \leq C$ (since if it's non-zero, we can divide, and if it is 0, then obviously it's $\leq C$).

(c) We want to show that the dimension of $\mathscr{M}$ is at most $C^2$, i.e. that if $\{f_i\}_{i \in I}$ is some subset of $\mathscr{M}$ that is linearly independent, then $|I| \leq C^2$. We only need to concern ourselves with finite $I$ and

orthonormal $\{f_i\}_{i\in I}$ since any linearly independent set in $\mathscr{M}$ can be turned into an orthonormal set of the same size by the Gram-Schmidt process (pg. 175 in Folland).

We first prove Folland's hint that given an orthonormal set $\{f_i\}_{i\in I} \subseteq \mathscr{M}$, we have that for every $x \in [0,1]$, $\sum_{i\in I} |f_i(x)|^2 \leq C^2$. Well fixing arbitrary $x \in [0,1]$, we know from (b) that there is some $g_x \in \mathscr{M}$ s.t. for every $f \in \mathscr{M}$, $f(x) = \langle f, g_x\rangle$. Thus, we have that $\sum_{i\in I} |f_i(x)|^2 \leq C^2 = \sum_{i\in I} |\langle f_i, g_x\rangle|^2$. Bessel's inequality (5.26 in Folland) says that since $\{f_i\}_{i\in I}$ is an orthonormal set in $\mathscr{M}$, then for any $g \in \mathscr{M}$, $\sum_{i\in I} |\langle g, f_i\rangle|^2 \leq \|g\|_{L^2}^2$. Because $|\langle f_i, g_x\rangle| = |\langle g_x, f_i\rangle|$, we can combine these facts to get that $\sum_{i\in I} |f_i(x)|^2 \leq \|g_x\|_{L^2} \leq C^2$, as desired.

Now note that

$$|I| = \sum_{i\in I} 1 = \sum_{i\in I} \|f_i\|_{L^2}^2 = \sum_{i\in I} \int_0^1 |f_i|^2 \, dm = \int_0^1 \sum_{i\in I} |f_i|^2 \, dm \leq \int_0^1 C^2 \, dm = C^2,$$

where we can interchange sum and integral because we assumed $I$ is finite. Since $I$ finite implies $|I| \leq C^2$, $I$ can't possible be infinite, and so indeed the dimension of $\mathscr{M}$ is at most $C^2$.

# 525 Homework 4

Daniel Rui - 2/3/20

## Problem 1 - 5.2.22

Suppose that $\mathscr{X}$ and $\mathscr{Y}$ are normed vector spaces and $T \in L(\mathscr{X}, \mathscr{Y})$.

(a) We define $T^\dagger : \mathscr{Y}^* \to \mathscr{X}^*$ by $T^\dagger(f) = f \circ T$ (this is called the adjoint or transpose of $T$). We want to show that $T^\dagger \in L(\mathscr{Y}^*, \mathscr{X}^*)$, and that $\|T^\dagger\| = \|T\|$. Well, note first that $T^\dagger f = f \circ T$ is an element of $\mathscr{X}^*$, since the paragraph following Prop. 5.4 in Folland tells us that $S \in L(\mathscr{Y}, \mathscr{Z}), T \in L(\mathscr{X}, \mathscr{Y}) \implies ST \in L(\mathscr{X}, \mathscr{Z})$ with $\|ST\| \leq \|S\|\|T\|$. To prove $T^\dagger \in L(\mathscr{Y}^*, \mathscr{X}^*)$, we just show that it's linear: $T^\dagger(af + bg) = (af + bf) \circ T = a \cdot (f \circ T) + b \cdot (g \circ T) = aT^\dagger(f) + bT^\dagger(g)$, and that it's bounded:

$$\|T^\dagger f\| \leq \|f\|\|T\| \implies \|T^\dagger\| := \sup\{\|T^\dagger f\| : f \in \mathscr{Y}^*, \|f\| = 1\} \leq 1 \cdot \|T\| = \|T\|.$$

All that's left is to show $\|T\| \leq \|T^\dagger\|$. We know (paragraph above Prop 5.4 in Folland) that $\|T\|$ can be defined as $\inf\{C : \|Tx\| \leq C\|x\| \text{ for all } x\}$, so we just need to show that $\|Tx\| \leq \|T^\dagger\|\|x\|$ for all $x \in \mathscr{X}$. If $Tx \neq 0$, we can use Theorem 5.8(b) which tells us that there is some $f \in \mathscr{Y}^*$ with $\|f\| = 1$ and $f(Tx) = \|Tx\|$. Because $f(Tx)$ also equals $[T^\dagger(f)](x)$, so we get that

$$\|Tx\| = \|[T^\dagger(f)](x)\| \leq \|T^\dagger(f)\|\|x\| \leq \|T^\dagger\|\|x\|,$$

as desired (last inequality uses that $\|f\| = 1$ and that $\|T^\dagger\| = \sup\{\|T^\dagger f\| : f \in \mathscr{Y}^*, \|f\| = 1\}$). If $Tx = 0$, the inequality is trivial, so the result indeed holds for all $x \in \mathscr{X}$, giving us $\|T\| \leq \|T^\dagger\|$ as promised.

(b) Applying the construction in (a) twice, we have an operator $T^{\dagger\dagger} \in L(\mathscr{X}^{**}, \mathscr{Y}^{**})$. By identifying $\mathscr{X}$ and $\mathscr{Y}$ with their natural images $\widehat{\mathscr{X}}$ and $\widehat{\mathscr{Y}}$ (i.e. $x \in \mathscr{X}$ gets mapped to $\widehat{x} \in \widehat{\mathscr{X}} \subseteq \mathscr{X}^{**}$ defined to be $\widehat{x}(f) = f(x)$ for all $f \in \mathscr{X}^*$; and similarly for $\mathscr{Y}$), we want to show that $T^{\dagger\dagger}|_{\mathscr{X}} = T$.

Observe that $T^{\dagger\dagger}(\widehat{x}) \in \mathscr{Y}^{**}$, i.e. something that takes in $f \in \mathscr{Y}^*$ and spits out something in the underlying field, also satisfying $T^{\dagger\dagger}(\widehat{x}) = \widehat{x} \circ T^\dagger$ for some $T^\dagger \in L(\mathscr{Y}^*, \mathscr{X}^*)$ satisfying $T^\dagger(f) = f \circ T$ (by above definition of adjoint). So letting $f \in \mathscr{Y}^*$, we see that

$$[T^{\dagger\dagger}(\widehat{x})](f) = [\widehat{x} \circ T^\dagger](f) = \widehat{x}(T^\dagger(f)) = \widehat{x}(f \circ T) = [f \circ T](x) = f(T(x)).$$

But also $f(T(x)) = \widehat{T(x)}(f)$, and because $f \in \mathscr{Y}^*$ was arbitrary, we see that $T^{\dagger\dagger}(\widehat{x}) = \widehat{T(x)}$. So we do in fact get the desired assertion that $T^{\dagger\dagger}(x) = T(x)$ for all $x \in \mathscr{X}$ if we identify $x \in \mathscr{X}$ with $\widehat{x} \in \widehat{\mathscr{X}}$ and $y \in \mathscr{Y}$ with $\widehat{y} \in \widehat{\mathscr{Y}}$.

(c) We want to show that $T^\dagger$ is injective iff the range of $T$ is dense in $\mathscr{Y}$. ($\implies$): we use the same strategy from paragraphs 2 and 3 of Problem 2 below (I did that one before this one).

We have that $\text{im}(T)$ is a subspace because $y_1, y_2 \in \text{im}(T) \implies$ there are $x_1, x_2 \in \mathscr{X}$ s.t. $T(x_i) = y_i$, so $ay_1 + by_2 = T(ax_1 + bx_2) \in \text{im}(T)$. Then $\mathscr{M} := \overline{\text{im}(T)}$ is a closed subspace of $\mathscr{Y}$ (again like paragraph 2 below, closedness is obvious, and we can prove subspace by taking limits: $y_n \to y, y_n' \to y' \implies ay_n + by_n' \to ay + by'$). If we assume that $\mathscr{M}$ is not equal to $\mathscr{Y}$, then there is some $y \in \mathscr{Y}$ s.t. we can apply Theorem 5.8(a), which tells us that there is some $f \in \mathscr{Y}^*$ s.t. $f(y) \neq 0$ and $f_{\mathscr{M}} = 0$. This means in particular that $f \circ T$ is identically 0, but $T^\dagger(f) = f \circ T$, meaning that $T^\dagger$ has non-trivial kernel (since it contains $f$, and $f(y) \neq 0$); contradiction.

For ( $\impliedby$ ): we also do a proof by contradiction. We suppose that $T^\dagger$ is not injective, meaning that there is some non-trivial $f \in \mathscr{Y}^*$ in the kernel, i.e. $f \circ T = T^\dagger(f) = 0$ identically, but $f(y) \neq 0$ for some $y \in \mathscr{Y}$. But $f \in \mathscr{Y}^*$, the space of bounded linear functionals, implying that $f$ is continuous (we know from Prop. 5.2 that bounded is equivalent to continuous for linear maps between normed vector spaces). By continuity, there is a non-empty open set $U \subseteq \mathscr{Y}$ s.t. $f(U) \neq 0$ (e.g. $U = |f|^{-1}(\frac{1}{2}|f(y)|, 2|f(y)|)$). But since $f \circ T = 0 \iff f(\text{im}\, T) = 0$, it must be that $\text{im}\, T \cap U = \varnothing$, contradicting that $\text{im}\, T$ is dense.

(d) We want to show that if the range of $T^\dagger$ is dense in $\mathscr{X}^*$, then $T$ is injective. We will also show that the converse is true if $\mathscr{X}$ is reflexive ($\widehat{\mathscr{X}} = \mathscr{X}^{**}$).

For the FIRST assertion, we proceed again by contradiction. Suppose that $T$ is not injective; then there is some non-trivial $x \in \mathscr{X}$ in the kernel, i.e. $Tx = 0$ for some $x \neq 0$. Then, obviously we have that for any $f \in \mathscr{Y}^*$, $f(T(x)) = 0$. But recall from the big chain of equalities from part (b) that $f(T(x)) = \widehat{x}(T^\dagger(f))$. So we've just shown that $\widehat{x}(T^\dagger(f)) = 0$ for all $y \in \mathscr{Y}^*$, or in other words, $\widehat{x}(\text{im}\, T^\dagger) = 0$. As $\widehat{x}$ is in $\mathscr{X}^{**}$ (see above part (b)), it is bounded, or equivalently continuous. Combining the facts that $\widehat{x}(\text{im}\, T^\dagger) = 0$, the closure of $\text{im}\, T^\dagger$ equals $\mathscr{X}^*$ (remember we assumed the range of $T^\dagger$ is dense in $\mathscr{X}^*$), and $\widehat{x}$ is continuous, we see that $\widehat{x}$ is identically 0 on all of $\mathscr{X}^*$. The only way that can happen is if $x = 0$; contradiction. (Yes I realize that we didn't need contradiction here, but by the time I realized that I had already wrote it up).

For the SECOND assertion, we proceed again by contradiction, and assume on the contrary that the range of $T^\dagger$ is not dense in $\mathscr{X}^*$. We use the same strategy as in the forward direction of part (c) above: we have a proper closed subspace $\mathscr{M} := \overline{\text{im}(T^\dagger)}$ of $\mathscr{X}^*$ (proper by contrary assumption, closed and subspace by above arguments in part (c)). Hence, we can use Theorem 5.8(a), which tells us that there is some non-trivial $g \in \mathscr{X}^{**}$ s.t. $g|_{\mathscr{M}} = 0$.

But by reflexivity, $g$ must be $\widehat{x}$ for some $x \in \mathscr{X}$ (note that $x \neq 0$ because we specified $g$ was non-trivial). So we have that $\widehat{x}(\mathscr{M}) = 0$, implying in particular that $\widehat{x}(T^\dagger(f)) = 0$ for all $f \in \mathscr{Y}^*$. Again recall from the big chain of equalities in part (b) that $\widehat{x}(T^\dagger(f)) = f(T(x))$, so we have just shown that $f(T(x)) = 0$ for every $f \in \mathscr{Y}^*$. This forces $T(x) = 0$, contradicting injectivity of $T$ since we specified $x \neq 0$.

## Problem 2 - 5.2.25

Let $\mathscr{X}$ be a Banach space. We wand to show that $\mathscr{X}^*$ separable $\implies$ $\mathscr{X}$ separable. Well, by separability of $\mathscr{X}^*$, we have $\{f_n\}_{n=1}^\infty$ a countable dense subset in $\mathscr{X}^*$. We follow Folland's hint, and choose for each $n \in \mathbb{N}$ an $x_n \in \mathscr{X}$ satisfying $\|x_n\| = 1$ and $|f_n(x_n)| \geq \frac{1}{2}\|f_n\|$ (which we can do because $\|f_n\| = \sup\{|f_n(x)| : \|x\| = 1\}$). We show that the (finite rational) linear combinations $\mathscr{S}$ of $\{x_n\}_{n=1}^\infty$ are dense in $\mathscr{X}$, i.e. $X = \overline{\mathscr{S}}$; this suffices since the set of such combinations is indeed countable (the number of rational linear combinations of a given length $\ell$ is countable — it's just a finite product of countable sets $(\mathbb{Q} \times \mathbb{N})^\ell$ — and a countable union (over all $\ell \in \mathbb{N}$) of countable sets is still countable). We proceed by contradiction: assume that $\mathscr{X}$ strictly contains $\overline{\mathscr{S}}$, i.e. there is some $x' \in \mathscr{X} \setminus \overline{\mathscr{S}}$.

It is clear that $\overline{\mathscr{S}}$ is closed; it is furthermore a subspace of $\mathscr{X}$ because if $s, t \in \overline{\mathscr{S}}$ and $a, b \in \mathbb{R}$ (could also be $\mathbb{C}$), we know that there must be some sequences $\{s_n\}_{n=1}^\infty, \{t_n\}_{n=1}^\infty \subseteq \mathscr{X}$ and $\{a_n\}_{n=1}^\infty, \{b_n\}_{n=1}^\infty \subseteq \mathbb{Q}$ (could also be $\mathbb{Q} + \mathbb{Q}i$) s.t. $s_n \to s, t_n \to t$ (w.r.t. the $\mathscr{X}$ norm) and $a_n \to a, b_n \to b$ (just convergence of real/complex numbers), meaning that $as+bt \in \overline{\mathscr{S}}$ because $a_n s_n + b_n t_n \to as+bt$ (w.r.t. the $\mathscr{X}$ norm).

With that in place, we can use Theorem 5.8(a) in Folland, which tells us that since $\mathscr{M} := \overline{\mathscr{S}}$ is a closed subspace of $\mathscr{X}$ and $x' \in \mathscr{X} \setminus \mathscr{M}$, we can find some $f \in \mathscr{X}^*$ s.t. $f(x') \neq 0$ and $f|_{\mathscr{M}} = 0$. Because $\{f_n\}_{n=1}^\infty$ is dense in $\mathscr{X}^*$, for any $\epsilon > 0$, we can find some $N \in \mathbb{N}$ s.t. $\|f - f_N\| < \frac{\epsilon}{3}$. But now (motivated by the fact that we have bounds involving $\|f - f_N\|$ and $\|f_N\|$, by $\frac{\epsilon}{3}$ and $2|f_N(x_N)|$ respectively) notice that

$$\begin{aligned}\|f\| = \|f - f_N + f_N\| &\leq \|f - f_N\| + \|f_N\| < \tfrac{\epsilon}{3} + 2|f_N(x_N)| \\ &\leq \tfrac{\epsilon}{3} + 2|f_N(x_N) - f(x_N)| \leq \tfrac{\epsilon}{3} + 2\|f_N - f\|\|x_N\| < \tfrac{\epsilon}{3} + 2 \cdot \tfrac{\epsilon}{3} \cdot 1 = \epsilon,\end{aligned}$$

where we used the fact that $f(x_N) = 0$ since $x_N \in \mathscr{S} \subseteq \mathscr{M}$. Because $\epsilon > 0$ was arbitrary, $\|f\|$ must in fact be 0. But this is impossible, since $\|f\| = 0 \implies f = 0$ by the properties of norms, contradicting what we said earlier about $f(x') \neq 0$. Thus, our initial assumption was wrong; $\mathscr{X}$ must equal $\overline{\mathscr{S}}$.

## Problem 3 - 5.3.35

Let $\mathscr{X}$ and $\mathscr{Y}$ be Banach spaces, and $T \in L(\mathscr{X}, \mathscr{Y})$. Like in Problem 3 of last week's homework (Folland problem 5.15(b)), we let $\mathscr{N}(T) = \{x \in \mathscr{X} : Tx = 0\}$. we also define $\mathscr{M} = \text{range}(T)$. We want to show that $\mathscr{X}/\mathscr{N}(T)$ is isomorphic to $\mathscr{M}$ if and only if $\mathscr{M}$ is closed.

($\implies$): $\mathscr{X}/\mathscr{N}(T)$ and $\mathscr{M}$ being isomorphic means that there is isomorphism (i.e. bounded bijective linear map with bounded inverse) $S \in L(\mathscr{X}/\mathscr{N}(T), \mathscr{M})$. Suppose we have a sequence $\{y_n\}_{n=1}^\infty \subseteq \mathscr{M}$ that converges to some $y \in \mathscr{Y}$. Then $x_n + \mathscr{N}(T) := S^{-1}(y_n)$ is a sequence in $\mathscr{X}/\mathscr{N}(T)$ that converges to some $x + \mathscr{N}(T)$ because $S^{-1}$ is continuous (bounded $\iff$ continuous for linear maps by Prop. 5.2) and $\mathscr{X}/\mathscr{N}(T)$ is closed ($\mathscr{N}(T)$ is closed by Problem 3(a) of last week (Folland 5.15), so we can apply Problem 3(d) from two weeks ago (Folland 5.12) which tells us that $\mathscr{X}/\mathscr{N}(T)$ is complete because $\mathscr{X}$ is). But $S(x_n + \mathscr{N}(T)) = y_n$, so by continuity of $S$, $S(x + \mathscr{N}(T)) = S(\lim_{n\to\infty} x_n + \mathscr{N}(T)) = \lim_{n\to\infty} y_n = y$. Thus, $y \in \text{range}(S) \subseteq \mathscr{M}$ (remark: "$\subseteq$" is actually "$=$" by surjectivity of $S$), and so $\mathscr{M}$ is closed.

($\Longleftarrow$): with $T \in L(\mathscr{X}, \mathscr{Y})$ and $\mathscr{M} = \text{range}(T)$, we can easily make the map $T' \in L(\mathscr{X}, \mathscr{M})$ defined as literally just $T'(x) = T(x)$, since we are guaranteed all outputs $T(x) \in \mathscr{M}$. $T'$ retains linearity and boundedness from $T$. Also by definition of range, $T'$ is surjective. Now note that $\mathscr{M}$ is a Banach space: first, it's a subspace because if $y_1, y_2 \in \mathscr{M}$, $ay_1 + by_2 = aT(x_1) + bT(x_2) = T(ax_1 + bx_2) \in \mathscr{M}$, and it's normed because we can just have it take the norm of $\mathscr{Y}$, and finally it's complete because we assumed that $\mathscr{M}$ was closed. Note also that $\mathscr{X}/\mathscr{N}(T)$ is also a Banach space ($\mathscr{N}(T)$ is closed by Problem 3(a) of last week (Folland 5.15), so we can apply Problem 3(a),(d) from two weeks ago (Folland 5.12) which gives us what we want).

Like in the "general case" of Problem 6 below (I did that one before this one), we can construct a bijective map $S \in T(\mathscr{X}/\mathscr{N}(T), \mathscr{M})$ (to summarize what we already did below, we took the unique $S$ guaranteed in Problem 3(b) of last week's homework satisfying $T' = S \circ \pi$ where $\pi : \mathscr{X} \to \mathscr{X}/\mathscr{N}(T)$ is the projection map; $S$ is obviously surjective since $T'$ is, and injectivity is also easy to show: $S(x_1 + \mathscr{N}(T)) = S(x_2 + \mathscr{N}(T)) \implies T(x_1) = T'(x_1) = T'(x_2) = T(x_2) \implies x_1 - x_2 \in \mathscr{N}(T) \implies x_1 + \mathscr{N}(T) = x_2 + \mathscr{N}(T)$). Corollary 5.11 then says that because $\mathscr{X}/\mathscr{N}(T)$ and $\mathscr{M}$ are Banach spaces and $S$ is a bounded bijective linear map, $S$ is an isomorphism, so $\mathscr{X}/\mathscr{N}(T)$ and $\mathscr{M}$ are isomorphic.

## Problem 4 - 5.3.36

Let $\mathscr{X}$ be a separable Banach space and let $\mu$ be counting measure on $\mathbb{N}$. Let $\{x_n\}_{n=1}^{\infty}$ be a countable dense subset of the unit ball $\mathscr{X}$, and define $T : L^1(\mathbb{N}, 2^{\mathbb{N}}, \mu) \to \mathscr{X}$ by $Tf = \sum_{n=1}^{\infty} f(n)x_n$. $T$ is linear, because $T(af + bg) = \sum_{n=1}^{\infty} af(n)x_n + bg(n)x_n = a\sum_{n=1}^{\infty} f(n)x_n + b\sum_{n=1}^{\infty} g(n)x_n = aT(f) + bT(g)$.

(a) We show that $T$ is bounded: using infinite triangle inequality for norms (proven by noting for any $N \in \mathbb{N}$, $\|\sum_{i=1}^{N} x_i\| \le \sum_{i=1}^{N} \|x_i\| \le \sum_{i=1}^{\infty} \|x_i\|$, and taking $N \to \infty$), we have that

$$\|T(f)\|_{\mathscr{X}} \le \sum_{n=1}^{\infty} \|f(n)x_n\|_{\mathscr{X}} = \sum_{n=1}^{\infty} |f(n)|\|x_n\|_{\mathscr{X}} \le \sum_{n=1}^{\infty} |f(n)| = \|f\|_{L^1(\mathbb{N}, 2^{\mathbb{N}}, \mu)}.$$

By one of the definitions, $\|T\| = \inf\{C : \|T(f)\| \le C\|f\|\}$, so $\|T\| \le 1$, meaning that $T$ is definitely bounded. This and the aforementioned linearity of $T$ gives that $T \in (L^1(\mathbb{N}, 2^{\mathbb{N}}, \mu), \mathscr{X})$, where $L^1(\mathbb{N}, 2^{\mathbb{N}}, \mu)$ is a Banach space (basically proved in class sometime in 524).

(b) To prove that $T$ is surjective, it suffices to show that there is some open ball centered at 0 contained in $\operatorname{im} T$ (since we can extend to all of $\mathscr{X}$ by linearity, i.e. multiplying by scalars). First, let us review the last paragraph of Folland's proof of the open mapping theorem: in that paragraph, Folland essentially proves that if we have Banach spaces $\mathscr{X}, \mathscr{Y}$ with $T \in L(\mathscr{X}, \mathscr{Y})$ and some $r > 0$ s.t. $B_{\mathscr{Y}}(r) = \{y \in \mathscr{Y} : \|y\|_{\mathscr{Y}} < r\} \subseteq \overline{T(B_{\mathscr{X}}(r_0))}$, then $B_{\mathscr{Y}}(r/2) \subseteq T(B_{\mathscr{X}}(r_0))$.

We use this as follows: consider $B_{L^1}(2) = \{f \in L^1(\mu) : \sum_{n=1}^{\infty} |f(n)| < 2\}$. Clearly, for all $n \in \mathbb{N}$, $f_n(x) := 1_{\{n\}}(x) \subseteq B_{L^1}(2)$, implying that $\{x_n\}_{n=1}^{\infty} \subseteq T(B_{L^1}(2))$. Taking closures of each side, we get that (since $\{x_n\}_{n=1}^{\infty}$ is dense in the unit ball of $\mathscr{X}$) $B_{\mathscr{X}}(1) \subseteq \overline{\{x_n\}_{n=1}^{\infty}} \subseteq \overline{T(B_{L^1}(2))}$. Folland's fact now tells us that $B_{\mathscr{X}}(\frac{1}{2}) \subseteq T(B_{L^1}(2)) \subseteq \operatorname{im} T$, so we have found an open ball in $\operatorname{im} T$, which as we said suffices to show that $T$ is surjective.

(c) Finally, because $\operatorname{im} T = \mathscr{X}$, and $\mathscr{X}$ is complete (hence closed), we have by Folland 5.35 (Problem 3 above) that $\mathscr{X}$ is indeed isomorphic to a quotient of $L^1(\mu)$ (more explicitly $\mathscr{X} \sim L^1(\mathbb{N}, 2^{\mathbb{N}}, \mu)/\mathscr{N}(T)$ where $\mathscr{N}(T) = \{f \in L^1(\mu) : T(f) = 0\}$).

## Problem 5 - 5.3.42

Let $E_n$ be the set of all $f \in C([0,1])$ for which there exists $x_0 \in [0,1]$ (depending on $f$) s.t. $|f(x) - f(x_0)| \leq n|x - x_0|$ for all $x \in [0,1]$.

(a) We want to show that each $E_n$ is nowhere dense in $C([0,1])$. First, let's show that $E_n$ is equal to its closure (i.e. it's closed). Suppose we have a sequence $\{f_k\}_{k=1}^{\infty} \subseteq E_n$ that converges to some $f \in C([0,1])$ (in the uniform norm $\|\cdot\|_{\mathrm{u}}$!). Let us fix an arbitrary $\epsilon > 0$. Because the $f_k$ are in $E_n$, we have that there is $\{x_k\}_{k=1}^{\infty} \subseteq [0,1]$ s.t. for all $x \in [0,1]$ and each $k \in \mathbb{N}$, $|f_k(x) - f_k(x_k)| \leq n|x - x_0|$. Because $[0,1]$ is a compact set, the sequence $\{x_k\}_{k=1}^{\infty}$ has a convergent subsequence converging to say $x_0 \in [0,1]$. Now note that for any $N \in \mathbb{N}$ and $x \in [0,1]$,

$$\begin{aligned}
|f_N(x) - f_N(x_0)| &\leq |f_N(x) - f_N(x_N)| + |f_N(x_N) - f_N(x_0)| \\
&\leq n|x - x_N| + |f_N(x_N) - f(x_N)| + |f(x_N) - f(x_0)| + |f(x_0) - f_N(x_0)| \\
&\leq n|x - x_0| + n|x_0 - x_N| + 2\|f - f_N\|_{\mathrm{u}} + |f(x_N) - f(x_0)|
\end{aligned}$$

Since $f$ is continuous on a compact set, we know that we can find $\delta > 0$ s.t. $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \frac{\epsilon}{6}$. By the definition of limit, there is some $N \in \mathbb{N}$ s.t. $|x_0 - x_N| < \min\{\frac{\epsilon}{6n}, \delta\}$ and $\|f - f_N\|_{\mathrm{u}} < \frac{\epsilon}{6}$, which yield the bound $|f_N(x) - f_N(x_0)| \leq n|x - x_0| + \frac{4\epsilon}{6}$. With this in hand, we have that for any $x \in [0,1]$,

$$\begin{aligned}
|f(x) - f(x_0)| &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)| \\
&\leq 2\|f - f_N\|_{\mathrm{u}} + n|x - x_0| + \frac{4\epsilon}{6} = n|x - x_0| + \epsilon.
\end{aligned}$$

As $\epsilon > 0$ was picked arbitrarily, we do indeed have that $|f(x) - f(x_0)| \leq n|x - x_0|$, proving that $E_n = \overline{E_n}$.

We will now show that for any $f \in C([0,1])$, $\epsilon > 0$, and $n \in \mathbb{N}$, there is some $h \in C([0,1])$ satisfying $\|f - h\|_{\mathrm{u}} < \epsilon$ that is piecewise linear, with finitely many pieces, with each piece having slope in absolute value $\geq 2n$.

These properties guarantee that $h \notin E_n$ because for any $x_0 \in [0,1]$, $\frac{|h(x) - h(x_0)|}{|x - x_0|}$ for $x$ close enough to $x_0$ is just the absolute value of the slopes of the 2 line segments on either side of $x_0$ (there are finitely many line segments, so their widths are bounded below by some $w > 0$), which we said were $\geq 2n$, not $\leq n$. This suffices to prove that $E_n$ is nowhere dense, because it tells us that for any $\epsilon > 0$ and $f \in E_n$, we do NOT have that $B_{\|\cdot\|_{\mathrm{u}}}(f, \epsilon) \subseteq E_n = \overline{E_n}$ (recall that a definition of nowhere dense is that the closure does not contain any non-empty open set).

Let us fix $n \in \mathbb{N}$, $\epsilon > 0$, and $f \in C([0,1])$. Then, the function $g = f - 2n$ is also in $C([0,1])$. As $[0,1]$ is a compact set, $g$ is uniformly continuous, so there is some $\delta$ s.t. for any $x, x' \in [0,1]$, $|x - x'| < \delta \implies |g(x) - g(x')| < \frac{\epsilon}{2}$. Let $N \in \mathbb{N}$ be s.t. $\frac{1}{N} < \delta$; then we have $N$ intervals $I_1 = [0, \frac{1}{N}], I_2 = [\frac{1}{N}, \frac{2}{N}], \ldots, I_N = [1 - \frac{1}{N}, 1]$ s.t. on $I_j$, $g(x) \in (g(\frac{j-1}{N}) - \frac{\epsilon}{2}, g(\frac{j-1}{N}) + \frac{\epsilon}{2})$. We now define $h'$ to be piecewise constant function (NOT continuous!) on $[0,1]$ s.t. $h'(x) = g(\frac{\lfloor Nx \rfloor}{N})$, so $h'$ is constant on $I_j$ except maybe at the right endpoint. The previous sentence can now be rephrased as: on each $I_j$, $g(x) \in (h'(x) - \frac{\epsilon}{2}, h'(x) + \frac{\epsilon}{2})$. This in particular means that $\|g - h'\|_{\mathrm{u}} < \frac{\epsilon}{2}$.

We can now make $h'$ into a continuous function $h''$, by "erasing" the values of $h'$ for $x$ on the left of but "super close" to the right endpoint of $I_j$ (which is $\frac{j}{N}$) and instead drawing a line to connect it to $h'(\frac{j}{N})$. Because the interpolation is just a line going from $h'(\frac{j-1}{N})$ to $h'(\frac{j}{N})$, those values lie between $h'(\frac{j-1}{N})$ to $h'(\frac{j}{N})$. Because $\frac{j}{N} \in I_j$, we have that $h'(\frac{j}{N}) = g(\frac{j}{N}) \in (h'(\frac{j-1}{N}) - \frac{\epsilon}{2}, h'(\frac{j-1}{N}) + \frac{\epsilon}{2})$, and so the values of the interpolation line also lie in this interval. That means that overall, $\|g - h''\|_{\mathrm{u}} \leq \|g - h'\|_{\mathrm{u}} + \|h' - h''\|_{\mathrm{u}} < \frac{\epsilon}{2} + \frac{\epsilon}{2} + \epsilon$. By erasing close enough, we can make the slope of the interpolation lines as high (in absolute value) as we want (same amount of change in $y$ over smaller change in $x$ means arbitrarily large (in abs val) slope). So make it so that all the interpolation lines (there are only finitely many) have slope in absolute value $> 4n$. Note that in total we only have $2N$ (plus or minus one or two) lines.

Now, we just add back the $2n$ we originally subtracted, i.e. we define $h := h'' + 2n$; all the pieces that were constant now have slope $+2n$, and all the interpolated lines still have slope in absolute value $\geq 2n$, and of course $\|f - h\|_{\mathrm{u}} = \|g - h''\|_{\mathrm{u}} < \epsilon$. Thus, we conclude the proof of the lemma we used, and so we are done with the whole proof that each $E_n$ is nowhere dense.

(b) We want to show that the set of functions in $C([0,1])$ that are nowhere differentiable is residual, i.e. the complement of a meager set, where a meager set is a countable union of nowhere dense sets. Obviously, this is equivalent to show that the set of functions in $C([0,1])$ that are somewhere differentiable is meager.

Let's consider a function $f \in C([0,1])$ that is differentiable at say $x_0 \in [0,1]$. Then, by the definition of differentiability, we have some $D \in \mathbb{R}$ such that for every $\epsilon > 0$, there is $\delta > 0$ s.t. for $x \in [0,1]$ satisfying $|x - x_0| < \delta$, $|\frac{f(x) - f(x_0)}{x - x_0} - D| < \epsilon$. This in particular gives that $|x - x_0| < \delta \implies \frac{|f(x) - f(x_0)|}{|x - x_0|} \leq |D| + \epsilon$. Thus, we can pick say $\epsilon = 1$, get the corresponding $\delta$, and define $B = |D| + 1$ to see that for $x \in [0,1]$ satisfying $|x - x_0| < \delta$, $\frac{|f(x) - f(x_0)|}{|x - x_0|} \leq B$.

For $x \in [0,1]$ s.t. $|x - x_0| \geq \delta$, we have the basic bound of $\frac{|f(x) - f(x_0)|}{|x - x_0|} \leq \frac{|f(x)| + |f(x_0)|}{\delta} \leq \frac{2M}{\delta}$, where $M = \sup_{x \in [0,1]} |f(x)|$ (which we know to be finite because $f$ is a continuous function on a compact set). Thus, for any $n \geq \max\{B, \frac{2M}{\delta}\}$, we have that $f \in E_n$. This tells us that the complement of the set of $C([0,1])$ nowhere differentiable functions is meager, since it is contained in a countable union of nowhere dense sets (any subset of a meager set is meager, since any subset $A$ of a nowhere dense set $B$ is nowhere dense, since $\mathrm{int}(\overline{A}) \subset \mathrm{int}(\overline{B}) = \varnothing$). We have thus proved the desired claim that the set of $C([0,1])$ nowhere differentiable functions is residual.

## Problem 6

We state the open mapping theorem and closed graph theorem for Banach spaces, and then derive the open mapping theorem from the closed graph theorem.

The **open mapping theorem** states that for Banach spaces $\mathscr{X}$ and $\mathscr{Y}$, if $T \in L(\mathscr{X}, \mathscr{Y})$ is surjective, then $T$ is open. The **closed graph theorem** states that for Banach spaces $\mathscr{X}$ and $\mathscr{Y}$, if $T : \mathscr{X} \to \mathscr{Y}$ is a closed linear map (i.e. the graph of $T$, $\Gamma(T) := \{(x, y) \in \mathscr{X} \times \mathscr{Y} : y = T(x)\}$ is a closed subspace of $\mathscr{X} \times \mathscr{Y}$), then $T$ is bounded (which is equivalent to $T$ being continuous, by Prop. 5.2); note that the converse, $T$ bounded (equiv. continuous) $\implies$ $T$ (equiv. $\Gamma(T)$) closed is true for any normed vector spaces $\mathscr{X}, \mathscr{Y}$ (fairly obvious consequence of limit interchange property of continuity).

**Bijective case:** suppose we have Banach spaces $\mathscr{X}$ and $\mathscr{Y}$ and a surjective bounded linear map $T \in L(\mathscr{X}, \mathscr{Y})$. If we suppose further that $T$ is bijective, then we can define $T^{-1}$, the inverse of $T$ (i.e. $T^{-1}$ satisfies $T \circ T^{-1} = \mathrm{id} = T^{-1} \circ T$), which is a linear map $\mathscr{Y} \to \mathscr{X}$ since $T^{-1}(ax + by) = aT^{-1}(x) + bT^{-1}(y)$ (because $T$ is injective and $T(T^{-1}(ax + by)) = ax + by$ and also $T(aT^{-1}(x) + bT^{-1}(y)) = aT(T^{-1}(x)) + bT(T^{-1}(y)) = ax + by$). As Folland notes in the proof of Corollary 5.11, the continuity (equiv. boundedness) of $T^{-1}$ is equivalent to the openness of $T$, and so we only need to show that $T^{-1}$ is bounded in order to show the open mapping theorem in the case of bijective $T$.

We do this using the closed graph theorem: note that $\Gamma(T^{-1}) = \{(y, x) \in \mathscr{Y} \times \mathscr{X} : x = T^{-1}(y)\}$, which is closed because $\Gamma(T) = \{(x, y) \in \mathscr{X} \times \mathscr{Y} : y = T(x)\}$ is closed. More explicitly, if $(y_n, x_n) \to (y, x)$ in the norm of $\mathscr{Y} \times \mathscr{X}$ (which is $\|(y, x)\| = \max\{\|y\|, \|x\|\}$), we know that that $y_n \to y$ (in the sense of convergence in $\mathscr{Y}$ w.r.t. the $\mathscr{Y}$ norm) and similarly $x_n \to x$, and so $(x_n, y_n) \to (x, y)$ (in the norm of $\mathscr{X} \times \mathscr{Y}$). Then by the closedness of $\Gamma(T)$, we know $(x, y) \in \Gamma(T)$, which happens iff $(y, x) \in \Gamma(T^{-1})$. This shows that $\Gamma(T^{-1})$ is closed, so by the closed graph theorem, $T^{-1}$ is bounded, and as the above paragraph promised, we have shown the open mapping theorem in the case of bijective $T$.

**General case:** we now tackle the case when $T$ is not necessarily bijective. Let $\mathscr{M}$ denote $\mathscr{N}(T) = \{x \in \mathscr{X} : T(x) = 0\}$. In Problem 3(b) of last week's homework (Folland problem 5.15(b)), we showed that there is a unique $S \in L(\mathscr{X}/\mathscr{M}, \mathscr{Y})$ s.t. $T = S \circ \pi$ where $\pi : \mathscr{X} \to \mathscr{X}/\mathscr{M}$ is the projection map. Given that $T$ is surjective, we show that $S$ is in fact bijective. Surjectivity is obvious given surjectivity of $T$, so for injectivity, suppose we have $x_1 + \mathscr{M}, x_2 + \mathscr{M} \in \mathscr{X}/\mathscr{M}$ s.t. $S(x_1 + \mathscr{M}) = S(x_2 + \mathscr{M})$. By $T = S \circ \pi$, this means that $T(x_1) = T(x_2)$, and so by linearity, $T(x_1 - x_2) = 0 \implies x_1 - x_2 = m$ for some $m \in \mathscr{M}$. This implies that $x_1 = x^2 + m$, i.e. $x_1 + \mathscr{M} = x_2 + \mathscr{M}$, so injectivity is proven.

Thus, by the previous bijective case, $S$ is an open map. To conclude that $T$ is an open map, we just need to show that the projection map $\pi$ is also an open map (since $T = S \circ \pi$, if $\pi$ and $S$ are both open maps, then $T(\text{open set}) = S(\pi(\text{open set})) = S(\text{open set}) = \text{open set}$). Suppose $U$ is an open set in $\mathscr{X}$. We want to show that $\pi(U) = U + \mathscr{M}$ is open in $\mathscr{X}/\mathscr{M}$. We will show (i.e. it will suffice to show) that if $x_0 \in U$ and $\delta > 0$ are s.t. $B(x_0, \delta) \subseteq U$ (a ball w.r.t. the $\mathscr{X}$ norm/metric), then $B(x_0 + \mathscr{M}, \delta) \subseteq U + \mathscr{M}$ (a ball w.r.t. the $\mathscr{X}/\mathscr{M}$ norm/metric).

By definition, $x \in B(x_0 + \mathscr{M}, \delta) \iff \|x - x_0 + \mathscr{M}\| < \delta$, which is equivalent to there existing some $m \in \mathscr{M}$ s.t. $\|x - x_0 + m\| < \delta \iff \|(x + m) - x_0\| < \delta$. But this last proposition means that $x + m \in U$, or in other words $x \in U + \mathscr{M}$! This is exactly what we wanted to show, so as advertised, we have shown that $\pi$ is an open map, so indeed $T$ is an open map.

# 525 Homework 3

Daniel Rui - 1/27/20

## Problem 1 - 5.1.7

Let $\mathscr{X}$ be a Banach space.

(a) let $T \in L(\mathscr{X}, \mathscr{X})$ satisfying $\|I - T\| < 1$, where $I$ is the identity operator. We show that $T$ has inverse $U := \sum_{n=0}^{\infty}(I-T)^n$. First of all, since $\mathscr{X}$ is a Banach space, we have by Proposition 5.4 in Folland (and the paragraphs preceding it) that $L(\mathscr{X}, \mathscr{X})$ is also a Banach space. Thus, since we have that $\sum_{n=0}^{\infty}\|(I-T)^n\| \leq \sum_{n=0}^{\infty}\|I-T\|^n < \infty$ because a geometric series with ratio $\|I-T\| < 1$ converges (note that the inequality between the sums comes from the "product rule" for the operator norm: $\|ST\| \leq \|S\|\|T\|$, found in the paragraph below Prop 5.4), we have by Theorem 5.1 that $\sum_{n=0}^{\infty}(I-T)^n$ does in fact converge to some $U \in L(\mathscr{X}, \mathscr{X})$.

We now verify that it is in fact the inverse to $T$: since $\|U - \sum_{n=0}^{N}(I-T)^n\| = \|\sum_{n=N+1}^{\infty}(I-T)^n\| \leq \sum_{n=N+1}^{\infty}\|I-T\|^n$ goes to 0 as $N \to \infty$, we have that for any $\epsilon > 0$, there is some $N \in \mathbb{N}$ s.t. for any $N' \geq N$, $\|U - \sum_{n=0}^{N'}(I-T)^n\| < \frac{\epsilon}{2}$, and so

$$\begin{aligned}
\|I - TU\| = \|I + (I-T)U - U\| &= \left\| I + (I-T)U - \sum_{n=0}^{N+1}(I-T)^n + \sum_{n=0}^{N+1}(I-T)^n - U \right\| \\
&\leq \left\| I + (I-T)U - (I-T)^0 - \sum_{n=1}^{N+1}(I-T)^n \right\| + \left\| \sum_{n=0}^{N+1}(I-T)^n - U \right\| \\
&\leq \left\| (I-I) + (I-T)\left( U - \sum_{n=0}^{N}(I-T)^n \right) \right\| + \frac{\epsilon}{2} \\
&\leq \|I-T\| \left\| U - \sum_{n=0}^{N}(I-T)^n \right\| + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

As $\epsilon > 0$ was arbitrary, it must be that $0 \leq \|I - TU\| \leq 0$, and as the operator norm on $L(\mathscr{X}, \mathscr{X})$ is actually a norm, it must be that $I - TU = 0 \iff I = TU$. For $I = UT$, it is much the same — split up $I - UT$ into $I + U(I-T) - U = [I + U(I-T) - \Sigma] + [\Sigma - U]$ and the bounds follow exactly the same way. Thus, we have indeed verified that $U$ is the inverse to $T$ in $L(\mathscr{X}, \mathscr{X})$.

(b) We show that if $T \in L(\mathscr{X}, \mathscr{X})$ is invertible and $\|S - T\| < \|T^{-1}\|^{-1}$, then $S$ is invertible. (Remark: this shows that the set of invertible operators in $L(\mathscr{X}, \mathscr{X})$ is open, because for any invertible $T \in L(\mathscr{X}, \mathscr{X})$, the "ball" of radius $\|T^{-1}\|^{-1}$ is contained in the subspace of invertible operators). First notice that $\|T^{-1}(S-T)\| \leq \|T^{-1}\|\|S-T\| < \|T^{-1}\|\|T^{-1}\|^{-1} = 1$, and $T^{-1}(S-T) = T^{-1}S - I$, so we can apply part (a) to get that $T^{-1}S$ is invertible, say with inverse $U$. Then, $U(T^{-1}S) = (T^{-1}S)U = I$. Observing since $U$ is the inverse of some operator, it is itself invertible, so we have $S = TT^{-1}SUU^{-1} = TU^{-1}$. Thus, we have that $(UT^{-1})S = I = S(UT^{-1})$, so $S$ is indeed invertible, as desired.

## Problem 2 - 5.1.8

Let $(X, \mathscr{M})$ be a measurable space and let $M(X)$ be the space of finite signed measures on $(X, \mathscr{M})$. Defining $\|\mu\| = |\mu|(X)$, we show that this is a norm on $M(X)$, and in fact one that makes $M(X)$ into a Banach space. First, norm business:

- Norm 0 implies 0: et $\mu^+$ and $\mu^-$ be the unique positive measures given by Jordan's decomposition theorem (Theorem 3.4 in Folland); i.e. they satisfy $\mu = \mu^+ - \mu^-$, $\mu^+ \perp \mu^-$, and $|\mu| = \mu^+ + \mu^-$. Well, $|\mu|(X) = 0$ means that $\mu^+(X)$ and $\mu^-(X)$ would both have to be 0 (since both are $\geq 0$ and sum to $|\mu|(X) = 0$), and by monotonicity for positive measures, we would have for any measurable $E \subseteq X$ that $\mu^+(E) = \mu^-(E) = 0$, forcing $\mu(E) = 0$ for all measurable $E \subseteq X$. That is to say, $|\mu|(X) = 0$ implies that $\mu$ is the 0 measure.
- Scalar multiples: we have the Jordan decomposition of a measure $\mu = \mu^+ - \mu^-$, then $\lambda\mu = \lambda\mu^+ - \lambda\mu^-$. If $\lambda = 0$, then $\lambda\mu$ will be the 0 measure, so indeed $\|\lambda\mu\| = 0 = \lambda\|\mu\|$. We will now assume that $\lambda > 0$; otherwise just slap absolute values on $\lambda$ and switch the places of $\mu^+$ and $\mu^-$. Anyways, because the Jordan decomposition theorem additionally tells us that the decomposition is unique, and because $\mu^+ \perp \mu^- \implies \lambda\mu^+ \perp \lambda\mu^-$, we see that $\lambda\mu^+$ and $\lambda\mu^-$ are indeed the Jordan decomposition of $\lambda\mu$, meaning that $|\lambda\mu| = \lambda\mu^+ + \lambda\mu^-$, yielding that $\|\lambda\mu\| = \lambda\mu^+(X) + \lambda\mu^-(X) = \lambda(\mu^+(X) + \mu^-(X)) = \lambda\|\mu\|$, as desired.
- Triangle inequality: let $P, N$ be a Hahn decomposition of the measure $\mu + \nu$ (Theorem 3.3 in Folland), and let $P_\mu, N_\mu$ and $P_\nu, N_\nu$ be the Hahn decompositions of $\mu$ and $\nu$ respectively. Then, by the Jordan decomposition theorem, $[\mu+\nu]^+(E) = [\mu+\nu](E \cap P)$ and $[\mu+\nu]^-(E) = -[\mu+\nu](E \cap N)$, and similarly for $\mu$ and $\nu$. Thus, as desired,

$$\begin{aligned}
\|\mu+\nu\| &= [\mu+\nu](X\cap P) - [\mu+\nu](X\cap N) = \mu(P) + \nu(P) - \mu(N) - \nu(N) \\
&= \mu(P\cap P_\mu) + \underbrace{\mu(P\cap N_\mu)}_{\leq 0} + \nu(P\cap P_\nu) + \underbrace{\nu(P\cap N_\nu)}_{\leq 0} \\
&\quad - \underbrace{\mu(N\cap P_\mu)}_{\geq 0} - \mu(N\cap N_\mu) - \underbrace{\nu(N\cap P_\nu)}_{\geq 0} - \nu(N\cap N_\nu) \\
&\leq \mu(P\cap P_\mu) - \mu(P\cap N_\mu) + \nu(P\cap P_\nu) - \nu(P\cap N_\nu) \\
&\quad + \mu(N\cap P_\mu) - \mu(N\cap N_\mu) + \nu(N\cap P_\nu) - \nu(N\cap N_\nu) \\
&= \mu((P\cup N)\cap P_\mu) - \mu((P\cup N)\cap N_\mu) + \nu((P\cup N)\cap P_\nu) - \nu((P\cup N)\cap N_\nu) \\
&= \mu(P_\mu) - \mu(N_\mu) + \nu(P_\nu) - \nu(N_\nu) = |\mu|(X) + |\nu|(X) = \|\mu\| + \|\nu\|.
\end{aligned}$$

We now show that under this norm, $M(X)$ is complete. By Theorem 5.1, it suffices to show that every absolutely convergent series in $M(X)$ converges. Suppose we have that $\sum_{n=1}^\infty \|\mu_n\| < \infty$. Then, for any $A \in \mathscr{M}$ we have that $\sum_{n=1}^\infty |\mu_n(A)| \leq \sum_{n=1}^\infty |\mu_n|(A) \leq \sum_{n=1}^\infty |\mu_n|(X) = \sum_{n=1}^\infty \|\mu_n\| < \infty$, where $|\mu_n(A)| \leq |\mu_n|(A)$ can be proven from Ex. 3.3(b) (Problem 1 of 525 HW 1) by setting $f = 1_A$, and $|\mu_n|(A) \leq |\mu_n|(X)$ comes from monotonicity of positive measures. Then because a series of real numbers converging absolutely implies it converges, we have that indeed $\sum_{n=1}^\infty \mu_n(A)$ converges to some value which we will denote $\mu(A)$.

We now prove that $\mu$ is a measure. First note that because all the convergent sums above are $< \infty$, i.e. finite, $\mu(A)$ for any $A \in \mathscr{M}$ is finite. Also, $\mu(\varnothing) = \sum_{n=1}^{\infty} \mu_n(\varnothing) = \sum_{n=1}^{\infty} 0 = 0$, and so we only need to prove countably disjoint additivity — first we prove the following sum converges absolutely; that then will allow us to use Fubini's theorem (applied to counting measures) to interchange sums, since Fubini requires "$f \in L^1(\mu \times \nu)$", which translates in our case to "$\sum \sum |\bullet| < \infty$", which we can compute as an iterated sum in either order by Tonelli (since $|\bullet|$ is $\geq 0$):

$$\sum_{j=1}^{\infty} |\mu(E_j)| = \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} |\mu_n(E_j)| \leq \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} |\mu_n|(E_j) = \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} |\mu_n|(E_j) = \sum_{n=1}^{\infty} |\mu_n|(E) \leq \sum_{n=1}^{\infty} \|\mu_n\| < \infty$$

where $E = \bigcup_{j=1}^{\infty} E_j$. Thus, as stated above, since this absolute sum is convergent, Fubini allows us to interchange sums in the following computation:

$$\sum_{j=1}^{\infty} \mu_n(E_j) = \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} \mu_n(E_j) = \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \mu_n(E_j) = \sum_{n=1}^{\infty} \mu_n(E) =: \mu(E).$$

We are now done with showing that $\mu \in M(X)$. Finally, we show that $\mu$ and $\sum_{n=1}^{\infty} \mu_n$ converge in the norm of $M(X)$:

$$\left\| \mu - \sum_{n=1}^{N} \mu_n \right\| = \left\| \sum_{n=N+1}^{\infty} \mu_n \right\| \leq \sum_{n=N+1}^{\infty} \|\mu_n\| \to 0 \text{ as } N \to \infty.$$

To justify the use of "infinite triangle inequality", we give the same argument as above but with infinitely many measures instead we denote $\Sigma = \sum_{n=1}^{\infty} \mu_n$ with Hahn decomposition $P, N$; measure $\mu_i$ will have Hahn decomposition $P_i, N_i$ (note that all the sums below are absolutely convergent, bound by $\sum_{n=1}^{\infty} \|\mu_n\| < \infty$, and so the terms can be rearranged within summation allowing us to do things like split up sums and combine two sums):

$$\begin{aligned}
\|\Sigma\| &= [\Sigma](X \cap P) - [\Sigma](X \cap N) = \sum_{n=1}^{\infty} \mu_n(P) - \sum_{n=1}^{\infty} \mu_n(N) \\
&= \sum_{n=1}^{\infty} \mu_n(P \cap P_n) + \underbrace{\sum_{n=1}^{\infty} \mu_n(P \cap N_n)}_{\leq 0} \\
&\quad - \underbrace{\sum_{n=1}^{\infty} \mu_n(N \cap P_n)}_{\geq 0} - \sum_{n=1}^{\infty} \mu_n(N \cap N_n) \\
&\leq \sum_{n=1}^{\infty} \mu_n(P \cap P_n) - \sum_{n=1}^{\infty} \mu_n(P \cap N_n) \\
&\quad + \sum_{n=1}^{\infty} \mu_n(N \cap P_n) - \sum_{n=1}^{\infty} \mu_n(N \cap N_n) \\
&= \sum_{n=1}^{\infty} \mu_n(P_n) - \sum_{n=1}^{\infty} \mu_n(N_n) = \sum_{n=1}^{\infty} |\mu_n|(X) = \sum_{n=1}^{\infty} \|\mu_n\| < \infty
\end{aligned}$$

## Problem 3 - 5.1.15

Suppose that $\mathscr{X}$ and $\mathscr{Y}$ are normed vector spaces and $T \in L(\mathscr{X}, \mathscr{Y})$ Let $\mathscr{N}(T) = \{x \in \mathscr{X} : Tx = 0\}$.

(a) We want to show that $\mathscr{N}(T)$ is a closed subspace. First, it is a subspace because if $x, y \in \mathscr{N}(T)$ and $a, b$ are scalars, $T(ax + by) = aT(x) + bT(y) = 0$. Now for closed-ness: from Prop 5.2 in Folland, we know that for linear maps $T : \mathscr{X} \to \mathscr{Y}$ between normed vector spaces, boundedness, continuity at 0, and continuity are all equivalent. Thus, if $x_n$ is some sequence in $\mathscr{N}(T)$ that converges to some $x \in \mathscr{X}$, then we have by continuity of $T$ that $T(x) = T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} T(x_n) = 0$, and so $x \in \mathscr{N}(T)$.

(b) We want to show that there is a unique $S \in L(\mathscr{X}/\mathscr{N}(T), \mathscr{Y})$ s.t. $T = S \circ \pi$, where $\pi : \mathscr{X} \to \mathscr{X}/\mathscr{N}(T)$ is the projection, and moreover that $\|T\| = \|S\|$.

If $x_1, x_2 \in \mathscr{X}$ satisfy $x_1 + \mathscr{N}(T) = x_2 + \mathscr{N}(T)$, or equivalently there is some $m \in \mathscr{N}(T)$ s.t. $x_1 + m = x_2$, then we have that $T(x_1) = T(x_1) + 0 = T(x_1 + m) = T(x_2)$. Thus, the map $S$ that sends $x + \mathscr{N}(T)$ to $T(x)$ is well defined. We just need to show that it is a bounded linear operator. It's linear because $S(x + \mathscr{N}(T) + y + \mathscr{N}(T)) = T(x + y) = T(x) + T(y) = S(x + \mathscr{N}(T)) + S(y + \mathscr{N}(T))$ and $S(\lambda x + \mathscr{N}(T)) = T(\lambda x) = \lambda T(x) = \lambda S(x + \mathscr{N}(T))$.

To show that it's bounded, note that for any $x \in \mathscr{X}, m \in \mathscr{N}(T)$, $\|S(x + \mathscr{N}(T))\| = \|T(x) + 0\| = \|T(x + m)\| \leq \|T\|\|x + m\|$, implying that for any $x \in \mathscr{X}$, $\|S(x + \mathscr{N}(T))\| \leq \|T\| \inf_{m\in\mathscr{N}(T)} \|x + m\| = \|T\|\|x + \mathscr{N}(T)\|$. But $\|S\| := \sup_{x\neq 0} \frac{\|S(x+\mathscr{N}(T))\|}{\|x+\mathscr{N}(T)\|}$ (definition 2 in Folland, found right about Prop. 5.4), so the previous sentences show that $\|S\| \leq \|T\|$, where $\|T\|$ is bounded. By the "product rule" for the operator norm, $\|AB\| \leq \|A\|\|B\|$, found right after Prop. 5.4 in Folland, and by Ex. 12(c) where we showed that $\|\pi\| = 1$, we have that $\|T\| \leq \|S\|\|\pi\| = \|S\|$, and so indeed $\|T\| = \|S\|$.

Finally, to show that it is unique, let's say we had $S'$ s.t. $S \circ \pi = T = S' \circ \pi$. Then, for all $x \in \mathscr{X}$, we have $S(x + \mathscr{N}(T)) = T(x) = S'(x + \mathscr{N}(T))$, meaning that $S$ and $S'$ agree on all elements of $\mathscr{X}/\mathscr{N}(T)$. Thus proves uniqueness, which finishes the problem.

## Problem 4 - 5.2.17

We want to show that a linear functional $f$ on a normed vector space $\mathscr{X}$ is bounded if and only if $f^{-1}(\{0\})$ is closed.

( $\implies$ ) from Prop 5.2 in Folland, we know that for linear maps $T : \mathscr{X} \to \mathscr{Y}$ between normed vector spaces, boundedness, continuity at 0, and continuity are all equivalent. Thus, if $x_n$ is some sequence in $f^{-1}(\{0\})$ that converges to some $x \in \mathscr{X}$, then we have by continuity of $f$ that $f(x) = f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n) = 0$, and so $x \in f^{-1}(\{0\})$.

( $\impliedby$ ) Exercise 12(b) says that if we have a normed vector space $\mathscr{X}$ and proper closed subspace of $\mathscr{X}$, then for every $\epsilon > 0$ there is some $x \in \mathscr{X}$ s.t. $\|x\| = 1$ and $\|x + \mathscr{M}\| := \inf_{y\in\mathscr{M}} \|x + y\| \geq 1 - \epsilon$. In

our case, if $f^{-1}(\{0\})$ is not a proper subspace, $f$ is literally just the 0 functional, and hence trivially bounded. So, suppose $f^{-1}(\{0\})$ is a proper subspace; then by Exercise 12(b) quoted above, there is some $\|x_0 + f^{-1}(\{0\})\| \geq 1 - \epsilon$, which is in particular $> 0$, which means that $x_0$ is not in $f^{-1}(\{0\})$, or in other words $f(x_0) \neq 0$. If we assume that $f$ is not bounded; i.e. we assume there exists $\{x_n\}_{n=1}^{\infty}$ s.t. $\|x_n\| = 1$ but $f(x_n) \geq n$, and we somehow construct a sequence of numbers in $f^{-1}(\{0\})$ converging to $x_0$, then we will reach a contradiction (because by closed-ness all convergent sequences in $f^{-1}(\{0\})$ would have to converge to something in $f^{-1}(\{0\})$).

Well, first note that by the scalar multiple rule for norms, $\|\frac{x_n}{f(x_n)}\| = \frac{\|x_n\|}{f(x_n)} = \frac{1}{f(x_n)} \leq \frac{1}{n}$. Thus, something like $x_0 - x_n/f(x_n)$ would converge to $x_0$. However, $f(x_0 - x_n/f(x_n)) = f(x_0) - 1$ is not necessarily equal to 0. But, no worries; a quick tweak fixes the issue: $f(x_0 - \frac{f(x_0)}{f(x_n)}x_n) = f(x_0) - f(x_0) = 0$. We have in fact found a sequence in $f^{-1}(\{0\})$ that converges to $x_0$ (explicitly, we have $\|x_0 - (x_0 - \frac{f(x_0)}{f(x_n)}x_n)\| \leq f(x_0)\frac{1}{n}$, which goes to 0 as $n \to \infty$), which contradicts that $f(x_0) \neq 0$, implying that our assumption was wrong and so indeed $f$ has to be bounded.

## Problem 5 - 5.2.18

Let $\mathscr{X}$ be a normed vector space.

(a) If $\mathscr{M}$ is a closed subspace and $x \in \mathscr{X} \setminus \mathscr{M}$, we want to show that $\mathscr{M} + \mathbb{C}x$ is closed. As Folland's hint suggests we use Theorem 5.8(a), which says that if $\mathscr{M}$ is a closed and $x \in \mathscr{X} \setminus \mathscr{M}$, there exists $f \in \mathscr{X}^* := L(\mathscr{X}, K)$ (where $K$ is the underlying field, which in our case is $\mathbb{C}$) such that $f(x) \neq 0$ and $f|_{\mathscr{M}} = 0$. Suppose we have a sequence $\{m_n + \lambda_n x\}_{n=1}^{\infty} \subseteq \mathscr{M} + \mathbb{C}x$ converging to some $y \in \mathscr{X}$. We want to prove that $y \in \mathscr{M} + \mathbb{C}x$ (this suffices to prove the claim since our convergent sequence was arbitrary).

Well, note that $f(m_n + \lambda_n x) = f(m_n) + \lambda_n f(x) = \lambda_n f(x)$. As $f \in L(\mathscr{X}, \mathbb{C})$, the space of bounded linear functionals, we have by Prop 5.2 in Folland that $f$ is continuous, and so we can interchange limits to get $f(y) = f(\lim_{n\to\infty} m_n + \lambda_n x) = \lim_{n\to\infty} \lambda_n f(x) = f(x) \lim_{n\to\infty} \lambda_n$. As it was specified by Theorem 5.8(a), $f(x) \neq 0$, so we can divide yielding $\lim_{n\to\infty} \lambda_n = \frac{f(y)}{f(x)}$. Then, $m_n$ converge to $y - \frac{f(y)}{f(x)}x$:

$$\begin{aligned}\left\| m_n - \left( y - \frac{f(y)}{f(x)}x \right) \right\| &\leq \|m_n + \lambda_n x - y\| + \left\| \lambda_n x - \frac{f(y)}{f(x)}x \right\| \\ &= \|(m_n + \lambda_n x) - y\| + \left| \lambda_n - \frac{f(y)}{f(x)} \right| \|x\| \to 0 \text{ as } n \to \infty.\end{aligned}$$

But the $m_n$ are in $\mathscr{M}$, and $\mathscr{M}$ is closed! That means that $y - \frac{f(y)}{f(x)}x$ must also be in $\mathscr{M}$, meaning that $y \in \mathscr{M} + \mathbb{C}x$, as desired.

(b) We want to show that every finite-dimensional subspace of $\mathscr{X}$ is closed. Well, every finite-dimensional subspace is of the form $\mathbb{C}x_1 + \ldots, +\mathbb{C}x_n$ for some $n \in \mathbb{N}$ and basis elements $x_1, \ldots, x_n \in \mathscr{X}$. Thus, applying part (a) iteratively, starting with $\mathscr{M} = \{0\}$, we show that $\{0\} + \mathbb{C}x_1 = \mathbb{C}x_1$ is closed, then $\mathbb{C}x_1 + \mathbb{C}x_2$ and so on, until at the $n$th iteration we show that

$\mathbb{C}x_1 + \ldots, +\mathbb{C}x_n$ is closed.

## Problem 6 - 5.2.19

Let $\mathscr{X}$ be infinite dimensional.

(a) We want to construct a sequence $\{x_j\}_{j=1}^{\infty} \subseteq \mathscr{X}$ s.t. $\|x_j\| = 1$ for all $j \in \mathbb{N}$, and $\|x_j - x_k\| \geq \frac{1}{2}$ for $j \neq k$ (or equivalently all $j < k$, since $\|x_j - x_k\| = |-1|\|x_k - x_j\|$). As Folland's hint suggests, we use Exercises 12(b) and 18(b) to do this construction. Recall that Exercise 12(b) says that if we have a normed vector space $\mathscr{X}$ and proper closed subspace of $\mathscr{X}$, then for every $\epsilon > 0$ there is some $x \in \mathscr{X}$ s.t. $\|x\| = 1$ and $\|x + \mathscr{M}\| := \inf_{y \in \mathscr{M}} \|x + y\| \geq 1 - \epsilon$.

Alright. We start with some $x_1 \in \mathscr{X}$ with $\|x_1\| = 1$ (such an $x_1$ exists because choosing any $x \in \mathscr{X} \setminus \{0\}$, we can e.g. set $x_1 = x/\|x\|$, where the norm is non-zero because the norm is 0 iff $x = 0$). Denote $\mathscr{M}_1 := Kx_1$ (where $K$ is the underlying field). By Ex. 18(b), $\mathscr{M}_1$ is closed, so using Ex. 12(b), we know that there is some $x_2 \in \mathscr{X}$ s.t. $\|x_2\| = 1$, and $\|x_2 + \mathscr{M}_1\| \geq \frac{1}{2}$. Because $\|x_2 + \mathscr{M}_1\| = \inf_{y \in \mathscr{M}_1} \|x_2 + y\|$, we have in particular that $\|x_2 - x_1\| \geq \frac{1}{2}$. We can now define $\mathscr{M}_2 = \mathscr{M}_1 + Kx_2$, and continue like this.

More explicitly, given $\mathscr{M}_k = \mathscr{M}_{k-1} + Kx_k$, Ex. 18(b) says it's closed, and Ex. 12(b) gives us $x_k$ s.t. $\|x_k\| = 1$ and $\|x_k + \mathscr{M}_k\| \geq \frac{1}{2}$, implying that for all $j < k$, $x_j \in M_k$ and so $\|x_k - x_j\| \geq \frac{1}{2}$. The fact that $\mathscr{X}$ is infinite dimensional allows us to continue to get an infinite sequence (all our $\mathscr{M}_k$ will be finite dimensional, and hence proper closed subspaces of $\mathscr{X}$, allowing us to keep using Ex. 12(b)). Thus, we have constructed a sequence with the desired properties.

(b) We want to show that $\mathscr{X}$ is not locally compact. A locally compact space is one s.t. every point has a compact neighborhood. Since $\mathscr{X}$ is normed, it is a metric space with metric $d(x, y) = \|x - y\|$. In metric spaces, compact subsets are those s.t. every sequence has a convergent subsequence. We show that $0 \in \mathscr{X}$ has no compact neighborhood: any neighborhood $N$ of $0 \in \mathscr{X}$ contains a "sphere" $S_r := \{x \in \mathscr{X} : \|x\| = r\}$ for some $r > 0$; but from part (a), we know there exist $\{x_j\}_{j=1}^{\infty} \subseteq \mathscr{X}$ s.t. $\|x_j\| = 1$ for all $j \in \mathbb{N}$, and $\|x_j - x_k\| \geq \frac{1}{2}$ for $j \neq k$, meaning that defining $y_1 = rx_1, y_2 = rx_2, \ldots$, we see that $\|y_j\| = r\|x_j\| = r$ and $\|y_j - y_k\| = r\|x_j - x_k\| \geq \frac{r}{2}$ for $j \neq k$.

Thus, we've found subsequence $\{y_j\}_{j=1}^{\infty} \subseteq S_r \subseteq N$ in the neighborhood $N$ fixed above that has no convergent subsequence, meaning that the neighborhood is not compact. But the neighborhood was arbitrary, and so there are NO compact neighborhoods of 0, which proves that $\mathscr{X}$ is not locally compact.

# 525 Homework 2

Daniel Rui - 1/20/21

## Problem 1 - 3.2.17

Let $(X, \mathscr{M}, \mu)$ be a finite measure space, $\mathcal{N}$ a sub-$\sigma$-algebra of $\mathscr{M}$, and $\nu$ be the restriction of $\mu$ on $\mathcal{N}$. If $f \in L^1(\mu)$, we want to show that there is some $g \in L^1(\nu)$ s.t. $\int_E f\, d\mu = \int_E g\, d\nu$ for all $E \in \mathcal{N}$, and furthermore that $g$ is unique $\nu$-a.e. (i.e. it is unique up to null-sets of $\nu$).

Let us first discuss Theorem 3.8 in Folland, the so-called Lebesgue-Radon-Nikodym theorem. As Folland says in the paragraph following the proof of Theorem 3.8, his result actually is two results bundled together: the Lebesgue decomposition of some measure w.r.t. some other measure, and the Radon-Nikodym theorem (applying when $\rho \ll \nu$, for $\rho, \nu$ $\sigma$-finite resp. signed/positive measures on the same measurable space) which shows the existence and uniqueness $\nu$-a.e. of the $\nu$-integrable (i.e. $L^1(\nu)$) Radon-Nikodym derivative $\frac{d\rho}{d\nu}$ satisfying $d\rho = \frac{d\rho}{d\nu}\, d\nu$ (i.e. $\rho(E) = \int_E \frac{d\rho}{d\nu}\, d\nu$ for all measurable sets $E$ in the measurable space). We will in fact be using the Radon-Nikodym theorem, which is why I described it in detail (and made some notation changes differing from the book).

Having laid out the problem statement and main theorem in this way, it should be fairly clear what we are going to do: we shall define $\rho$ on the measurable space $(X, \mathcal{N})$ to be $\rho(E) = \int_E f\, d\mu$ for all $E \in \mathcal{N}$. If we show that $\rho \ll \nu$, then (because we said that $(X, \mathscr{M}, \mu)$ was a finite measure space, $\rho$ and $\nu$ are finite resp. signed/positive measures on the same measurable space, namely $(X, \mathcal{N})$) as I described above, the Radon-Nikodym theorem gives the existence and uniqueness $\nu$-a.e. of the Radon-Nikodym derivative $g := \frac{d\rho}{d\nu} \in L^1(\nu)$ satisfying $\rho(E) = \int_E f\, d\mu = \int_E g\, d\nu$ for all $E \in \mathcal{N}$. This is exactly what we wanted to show, so showing $\rho \ll \nu$ will finish the problem.

We want to show that $\nu(E) = 0 \implies \rho(E) = 0$, where the sets $E$ are taken to be in $\mathcal{N}$. Well, since $\nu$ is the restriction of $\mu$ on $\mathcal{N}$, and we are only looking at sets from $\mathcal{N}$, we see that $\nu(E) = 0 \iff \mu(E) = 0$. Since $\rho(E)$ is defined as the integral $\rho(E) = \int_E f\, d\mu$, having $\mu(E) = 0$ clearly implies that $\int_E f\, d\mu$ must be 0 as well. Thus, we've have that $\nu(E) = 0 \iff \mu(E) = 0 \implies \rho(E) = 0$, so we are done.

## Problem 2 - 5.1.11

If $0 < \alpha \leq 1$, let $\Lambda_\alpha([0,1])$ be the space of Hölder continuous functions of exponent $\alpha$ on [0,1] . That is, $f \in \Lambda_\alpha([0,1])$ iff $\|f\|_{\Lambda_\alpha} < \infty$, where

$$\|f\|_{\Lambda_\alpha} = |f(0)| + \sup_{x,y \in [0,1], x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}$$

(a) We want to show that $\|\cdot\|_{\Lambda_\alpha}$ is a norm that makes $\Lambda_\alpha([0,1])$ into a Banach space. First, $\|\cdot\|_{\Lambda_\alpha}$ is indeed a norm:

- non-negative and 0-norm implies 0: obviously, $\|\cdot\|_{\Lambda_\alpha}$ is $\geq 0$ because of all the absolute values.

Moreover, if $\|f\|_{\Lambda_\alpha} = 0$, it must be that $f$ is identically 0: first, $|f(0)|$ must be 0, and second $|f(x) - f(y)|$ must be 0 for all $x, y \in [0,1]$; the latter assertion says that $f$ must be constant, and $f(0) = 0$ forces $f$ to be constantly 0.

- scalar multiplication: for any fixed $\lambda \in \mathbb{R}$,

$$\|\lambda f\|_{\Lambda_\alpha} = |\lambda| \cdot |f(0)| + |\lambda| \cdot \sup_{x,y\in[0,1],x\neq y} |\lambda| \cdot \frac{|f(x) - f(y)|}{|x-y|^\alpha} = |\lambda| \cdot \|f\|_{\Lambda_\alpha}$$

- and triangle inequality:

$$\begin{aligned}
\|f+g\|_{\Lambda_\alpha} &= |f(0)+g(0)| + \sup_{x,y\in[0,1],x\neq y} \frac{|f(x)+g(x)-f(y)-g(y)|}{|x-y|^\alpha} \\
&\leq |f(0)| + |g(0)| + \sup_{x,y\in[0,1],x\neq y} \left( \frac{|f(x)-f(y)|}{|x-y|^\alpha} + \frac{|g(x)-g(y)|}{|x-y|^\alpha} \right) \\
&\leq |f(0)| + |g(0)| + \sup_{x,y\in[0,1],x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha} + \sup_{x,y\in[0,1],x\neq y} \frac{|g(x)-g(y)|}{|x-y|^\alpha} \\
&= \|f\|_{\Lambda_\alpha} + \|g\|_{\Lambda_\alpha}.
\end{aligned}$$

We now use Theorem 5.1 in Folland to show that $(\Lambda_\alpha([0,1]), \|\cdot\|_{\Lambda_\alpha})$ is complete. Suppose we have a sequence of functions $\{f_n\}$ s.t. $\sum_{n=1}^\infty \|f_n\|_{\Lambda_\alpha} < \infty$; we want to show that $\sum_{n=1}^N f_n \to f$ (w.r.t. the norm $\|\cdot\|_{\Lambda_\alpha}$) as $N \to \infty$ for some $f \in \Lambda_\alpha([0,1])$. We can show that $\lim_{N\to\infty} \sum_{n=1}^N f_n(x)$ exists for all $x \in [0,1]$ if we show that $\sum_{n=1}^\infty |f_n(x)| < \infty$ for all $x \in [0,1]$ (it's a intro calculus/real-analysis fact that an absolutely convergence of a series of real numbers implies convergence (and furthermore convergence to the same value irrespective of rearrangement)); this latter proposition is in fact true: for $x = 0$, we have that $\sum_{n=1}^\infty |f_n(0)| \leq \sum_{n=1}^\infty \|f_n\|_{\Lambda_\alpha} < \infty$, and for $x \in (0,1]$, we have that

$$\sum_{n=1}^\infty |f_n(x)| \leq \sum_{n=1}^\infty |f_n(x) - f_n(0)| + |f_n(0)| \leq \sum_{n=1}^\infty \frac{|f_n(x) - f_n(0)|}{|x-0|^\alpha} + |f_n(0)| \leq \sum_{n=1}^\infty \|f_n\|_{\Lambda_\alpha} < \infty$$

where we used the fact that for $x \in (0,1]$ and $\alpha \in (0,1]$, $x^\alpha \leq 1 \iff \frac{1}{x^\alpha} \geq 1$. Again, as I said before, this shows that we can define the function $f(x) = \sum_{n=1}^\infty f_n(x)$ on $[0,1]$. We now show that $\sum_{n=1}^N f_n \to f$ w.r.t. the norm $\|\cdot\|_{\Lambda_\alpha}$:

$$\left\| f - \sum_{n=1}^N f_n \right\|_{\Lambda_\alpha} = \left\| \sum_{n=N+1}^\infty f_n \right\|_{\Lambda_\alpha} \leq \sum_{n=N+1}^\infty \|f_n\|_{\Lambda_\alpha}$$

indeed goes to 0 as $N \to \infty$ (since $\sum_{n=1}^\infty \|f_n\|_{\Lambda_\alpha} < \infty$). The last thing we have to show is that

the $f$ we found is in $\Lambda_\alpha([0,1])$: for any $x, y \in [0,1]$, $x \neq y$,

$$|f(0)| + \frac{|f(x)-f(y)|}{|x-y|^\alpha} = \left|\sum_{n=1}^{\infty} f_n(0)\right| + \frac{|\sum_{n=1}^{\infty}(f_n(x)-f_n(y))|}{|x-y|^\alpha}$$
$$\leq \sum_{n=1}^{\infty} |f_n(0)| + \frac{|f_n(x)-f_n(y)|}{|x-y|^\alpha} \leq \sum_{n=1}^{\infty} \|f_n\|_{\Lambda_\alpha} < \infty,$$

where in the 1st equality we used that we can rearrange terms with no bad consequences due to absolute convergence (mentioned above). The above bound holds for any $x, y \in [0,1]$, $x \neq y$, so it holds for the supremum too, meaning that indeed $\|f\|_{\Lambda_\alpha} \leq \sum_{n=1}^{\infty} \|f_n\|_{\Lambda_\alpha} < \infty$.

(b) Let $\lambda_\alpha([0,1])$ be the set of all $f \in \Lambda_\alpha([0,1])$ such that $\frac{|f(x)-f(y)|}{|x-y|^\alpha} \to 0$ as $x \to y$, for all $y \in [0,1]$. If $\alpha < 1$, we want to show that $\lambda_\alpha([0,1])$ is an infinite-dimensional closed subspace of $\Lambda_\alpha([0,1])$, and furthermore that if $\alpha = 1$, $\lambda_\alpha([0,1])$ contains only constant functions.

We'll do the $\alpha = 1$ case first. We see that $f \in \lambda_1([0,1])$ if and only if for all $x_0 \in [0,1]$, $\lim_{x \to x_0} \frac{|f(x)-f(x_0)|}{|x-x_0|} = 0$. This is exactly the definition of the derivative of $f$ existing and being 0 everywhere on $[0,1]$, and this is if and only if $f$ is constant.

Now for $\alpha < 1$. We'll tackle the assertions in the order (1) subspace, (2) closed, and (3) infinite-dimensional. To show (1), i.e. that $\lambda_\alpha([0,1])$ is a subspace, we just need to show that if $f, g \in \lambda_\alpha([0,1])$, then $af + bg \in \lambda_\alpha([0,1])$ for any scalars $a, b$, and that 0 and 1 are in $\lambda_\alpha([0,1])$. It is clear that the constant functions 0 and 1 are indeed in $\lambda_\alpha([0,1])$, and $af + bg$ indeed satisfies

$$\frac{|af(x)+bg(x)-af(y)-bg(y)|}{|x-y|^\alpha} \leq |a| \cdot \frac{|f(x)-f(y)|}{|x-y|^\alpha} + |b| \cdot \frac{|f(x)-f(y)|}{|x-y|^\alpha} \to 0$$

as $x \to y$ for all $y \in [0,1]$.

To show it's closed, we consider a sequence $\{f_n\}_{n=1}^{\infty} \subseteq \lambda_\alpha([0,1])$ converging to some $f \in \Lambda_\alpha([0,1])$.

## Problem 3 - 5.1.12 ¬(e)

Let $\mathscr{X}$ be a normed vector space and $\mathscr{M}$ a proper closed subspace of $\mathscr{X}$.

(a) $\|x + \mathscr{M}\| = \inf\{\|x + y\| : y \in \mathscr{M}\}$ is a norm on $\mathscr{X}/\mathscr{M}$.

- non-negative and 0-norm implies 0: obviously, $\|x+\mathscr{M}\|$ is $\geq 0$ because it's taking the infimum of values returned by the norm of $\mathscr{X}$, which is $\geq 0$. Moreover, if $\|x + \mathscr{M}\| = 0$, that must mean that for every $n \in \mathbb{N}$, there is some $y_n \in \mathscr{M}$ s.t. $\|x + y_n\| < \frac{1}{n}$. This, by definition of convergence in the normed vector space $\mathscr{X}$, means that the sequence $-y_n$ converges to $x$. But $\mathscr{M}$ is closed, and all the points $-y_n$ are in $\mathscr{M}$, meaning that because $x$ is a limit point of points in $\mathscr{M}$, $x$ must be in $\mathscr{M}$ too. In other words, we've shown that $\|x+\mathscr{M}\| = 0 \implies x+\mathscr{M} = 0+\mathscr{M}$.

- scalar multiplication: for any fixed $\lambda$ in the scalar field,

$$\|\lambda(x+\mathscr{M})\| = \|\lambda x+\mathscr{M}\| = \inf_{y\in\mathscr{M}} \|\lambda x+y\| = \inf_{y\in\mathscr{M}} |\lambda|\cdot\|x+\tfrac{1}{\lambda}y\| = |\lambda|\cdot \inf_{y\in\mathscr{M}} \|x+y\| = |\lambda|\cdot\|x+\mathscr{M}\|$$

- and triangle inequality:

$$\begin{aligned}\|x+\mathscr{M}+x'+\mathscr{M}\| &= \|(x+x')+\mathscr{M}\| = \inf_{y\in\mathscr{M}} \|(x+x')+y\| = \inf_{y,y'\in\mathscr{M}} \|(x+x')+(y+y')\| \\ &\le \inf_{y,y'\in\mathscr{M}} \Big(\|x+y\|+\|x'+y'\|\Big) = \inf_{y\in\mathscr{M}} \|x+y\| + \inf_{y'\in\mathscr{M}} \|x'+y'\| = \|x+\mathscr{M}\|+\|x'+\mathscr{M}\|\end{aligned}$$

(b) For any $\epsilon > 0$ there exists $x \in \mathscr{X}$ such that $\|x\| = 1$ and $\|x+\mathscr{M}\| \ge 1-\epsilon$. First, some intuition: tweaking the formula a little bit $\|x+\mathscr{M}\| = \inf_{y\in\mathscr{M}} \|x-y\|$, we can think of minimizing $\|x-y\|$ over $y \in \mathscr{M}$ as finding the "closest approximation" to $x$ we can in the subspace $\mathscr{M}$ (i.e. finding the "projection" of $x$ onto $\mathscr{M}$). Then, we can think of $x+\mathscr{M}$ as the vector "$x - \text{proj}_{\mathscr{M}}(x)$" (i.e. the vector "perpendicular" to $\mathscr{M}$ pointing to $x$), so that it's "length" is indeed the "minimum" $\|x-y\|$ over all $y \in \mathscr{M}$. Having this picture in mind, this question can be thought of as asking if we can find a norm-1 vector that is "basically already perpendicular" to $\mathscr{M}$.

This intuition helps us construct a solution: for any $x \in \mathscr{X}$ with norm 1, if we find a "close approximation to the projection" $y \in \mathscr{M}$, then $x-y$ is "almost perpendicular" to $\mathscr{M}$, so scaling it to be norm 1, $\frac{x-y}{\|x-y\|}$ may be our desired vector. We now formalize this idea: suppose $\epsilon \in (0,1)$ (if it were $\ge 1$, the problem is trivial). Then, $\frac{1}{1-\epsilon}$ is a bit bigger than 1, so we can find $y \in \mathscr{M}$ s.t. $\|x-y\| < \frac{1}{1-\epsilon}\cdot\|x+\mathscr{M}\|$ since $\|x+\mathscr{M}\| = \inf_{y\in\mathscr{M}} \|x-y\| < \frac{1}{1-\epsilon}\cdot\|x+\mathscr{M}\|$ (so although we can not necessarily find $y$ "optimal" in the sense that $\|x-y\| = \|x+\mathscr{M}\|$, where again $x+\mathscr{M}$ can be thought of as the "projection of $x$ onto $\mathscr{M}$", the $\frac{1}{1-\epsilon}$ gives a bit of wiggle room so that we can find a "close enough approximation to the projection"). Then, taking $x' = \frac{x-y}{\|x-y\|}$, we see that indeed

$$\|x'+\mathscr{M}\| = \left\|\frac{x-y}{\|x-y\|}+\mathscr{M}\right\| = \frac{\|x-y+\mathscr{M}\|}{\|x-y\|} = \frac{\|x+\mathscr{M}\|}{\|x-y\|} > 1-\epsilon.$$

(c) The projection map $\pi(x) = x+\mathscr{M}$ from $\mathscr{X}$ to $\mathscr{X}/\mathscr{M}$ has norm 1. Recall that the norm of an linear map is defined to be $\|T\| = \sup\{\|T(x)\| : \|x\| = 1\}$. Applying this to our scenario, we have that $\|\pi\| = \sup\{\|x+\mathscr{M}\| : \|x\| = 1\}$. For the upper bound, by the definition of infimum and because $\mathscr{M}$ is a subspace implying that $0 \in \mathscr{M}$, we see that $\|x+\mathscr{M}\| \le \|x+0\| = \|x\| = 1$ for any $x \in \mathscr{X}$. For the lower bound, part (b) above gives that $\|x+\mathscr{M}\|$ gets arbitrarily close to 1 for some particular $x$'s, and so by the definition of supremum, it must be that $\|\pi\| \ge 1$. Combining both bounds, we get that $\|\pi\| = 1$ exactly.

(d) If $\mathscr{X}$ is complete, so is $\mathscr{X}/\mathscr{M}$. We use Theorem 5.1: it suffices to show that if $\sum_{n=1}^{\infty} \|x_n+\mathscr{M}\| < \infty$, then $\sum_{n=1}^{N}(x_n+\mathscr{M}) = (\sum_{n=1}^{N} x_n)+\mathscr{M}$ converges to some $x+\mathscr{M}$ as $N \to \infty$. If we show that $\sum_{n=1}^{N}(x_n+y_n) \to x$ for some $x \in \mathscr{X}$ and $y_n \in \mathscr{M}$ (w.r.t. to the norm of $\mathscr{X}$), then $(\sum_{n=1}^{N} x_n)+\mathscr{M}$

indeed converges to some $x+\mathscr{M}$ (w.r.t. the norm of $\mathscr{X}/\mathscr{M}$), since

$$\left\|\left(\sum_{n=1}^{N} x_n\right)+\mathscr{M}-(x+\mathscr{M})\right\| = \left\|\left(x-\sum_{n=1}^{N}(x_n+y_n)\right)+\mathscr{M}\right\| \leq \left\|\left(x-\sum_{n=1}^{N}(x_n+y_n)\right)+0\right\|.$$

So we only need to show that $\sum_{n=1}^{N}(x_n+y_n)\to x$ for some $x\in\mathscr{X}$ and $y_n\in\mathscr{M}$. Well, first we know we can find $y_n\in\mathscr{M}$ s.t. for each $n\in\mathbb{N}$, $\|x_n+y_n\|\leq\|x_n+\mathscr{M}\|+\frac{1}{2^n}$, and so we have that

$$\sum_{n=1}^{\infty}\|x_n+y_n\|\leq\sum_{n=1}^{\infty}\|x_n+\mathscr{M}\|+\frac{1}{2^n}=1+\sum_{n=1}^{\infty}\|x_n+\mathscr{M}\|<\infty,$$

and second, since we assume that $\mathscr{X}$ is complete, we can use Theorem 5.1 again (but in the other direction) which tells us that indeed $\sum_{n=1}^{N}(x_n+y_n)\to x$ for some $x\in\mathscr{X}$, as desired.

## Problem 4

For $f\in L^1_{\text{loc}}(\mathbb{R}^n)$, and $0<s<n$ define

$$\Lambda_s=\left\{x\in\mathbb{R}^n : \liminf_{r\to 0}\frac{1}{r^s}\int_{B(x,r)}|f|(y)\,dy>0\right\}$$

We want to show that $m(\Lambda_s)=0$, for all $s\in(0,n)$.

By the Lebesgue differentiation theorem (or more precisely a slightly weaker version of the corollary) which we covered in class on 12-02-20, we have for any $f\in L^1_{\text{loc}}(\mathbb{R}^n)$ that

$$\lim_{r\to 0}\frac{1}{m(B(x,r))}\int_{B(x,r)}|f|(y)\,dy=|f|(x)$$

for $m$-a.e. $x$. But recall that for $n$-dimensional Lebesgue measure, $m(B(x,r))=r^n m(B(x,1))$ (I don't think we proved this fact, but we used it repeatedly throughout last quarter; e.g. in the Lecture of 11-23-20. It's also in Folland). Thus, for any fixed $s\in(0,n)$ and $\epsilon>0$, we can find $\delta>0$ s.t. $r\in(0,\delta)\implies r^{n-s}<\epsilon$, implying that

$$\frac{1}{r^s}\int_{B(x,r)}|f|(y)\,dy=\frac{r^{n-s}}{r^n}\int_{B(x,r)}|f|(y)\,dy\leq\frac{\epsilon}{r^n}\int_{B(x,r)}|f|(y)\,dy.$$

Therefore,

$$\liminf_{r\to 0}\frac{1}{r^s}\int_{B(x,r)}|f|(y)\,dy\leq\liminf_{r\to 0}\frac{\epsilon}{r^n}\int_{B(x,r)}|f|(y)\,dy=\epsilon|f|(x)m(B(x,1))$$

for $m$-a.e. $x$, (recalling that the liminf and lim are the same if the limit exists). Because $f\in L^1_{\text{loc}}(\mathbb{R}^n)$, it is bounded $m$-a.e., and obviously $m(B(x,1))$ is bounded, and so because $\epsilon>0$ can be taken arbitrarily small, we have that the RHS can become arbitrarily small for $m$-a.e. $x$. Hence overall, we have that for $m$-a.e. $x$, the liminf expression is 0 (since the inside absolute value forces the integral

to be $\geq 0$, and the RHS bound we just discussed forces it to be $\leq 0$), showing that indeed for any $s \in (0, n)$, $m(\Lambda_s) = 0$.

## Problem 5

Let $(X, d)$ be a metric space, $(Y, \rho)$ a complete metric space, and $\{f_n\}_n$ a sequence of functions from $X$ to $Y$ such that $\sup_{x \in X} \rho(f_n(x), f_m(x)) \to 0$ as $m, n \to \infty$ (the uniform Cauchy assumption). We want to show that there is a unique function $f : X \to Y$ such that $\sup_{x \in X} \rho(f_n(x), f(x)) \to 0$, and moreover that if each $f_n$ is continuous, so is $f$.

Due to the uniform Cauchy assumption, we have for all $x \in X$, the sequence of elements of $Y$ $\{f_n(x)\}$ is a Cauchy sequence. By completeness of $(Y, \rho)$, each of these Cauchy sequences converge to some element $f(x)$ in $Y$. In other words, we've defined for each $x \in X$ $f(x) = \lim_{n \to \infty} f_n(x)$, where the limit exists because $\{f_n(x)\}$ is a Cauchy sequence and $(Y, \rho)$ is complete. Now let us fix an arbitrary $\epsilon > 0$. Then, we know (by the uniform Cauchy assumption) that there is some $N \in \mathbb{N}$ s.t. $n, m \geq N \implies \sup_{x \in X} \rho(f_n(x), f_m(x)) < \epsilon/2$. Let us take an arbitrary $x \in X$, and consider $\rho(f_n(x), f(x))$ for any $n \geq N$. By the triangle inequality, we have that for any $m \in \mathbb{N}$,

$$\rho(f_n(x), f(x)) \leq \rho(f_n(x), f_m(x)) + \rho(f_m(x), f(x)).$$

Because this is true for all $m \in \mathbb{N}$, we can choose $m$ large enough (in particular we choose it to be at least $\geq N$) s.t. $\rho(f_m(x), f(x)) < \epsilon/2$ (which we can do because $f(x) = \lim_{m \to \infty} f_m(x)$). Then, using this and the uniform Cauchy bound, we get that $\rho(f_n(x), f(x)) < \epsilon$. But recall that $x$ was chosen arbitrarily, so indeed this is a uniform bound: $n \geq N \implies \sup_{x \in X} \rho(f_n(x), f(x)) < \epsilon$. Since $\epsilon > 0$ was arbitrary, this proves the desired result $\sup_{x \in X} \rho(f_n(x), f(x)) \to 0$.

If we further assume that each $f_n$ is continuous, we see that $f$ must be too. First we write out a consequence of the triangle inequality: for all $x, x_0 \in X$ and $n \in \mathbb{N}$,

$$\rho(f(x), f(x_0)) \leq \rho(f(x), f_n(x)) + \rho(f_n(x), f_n(x_0)) + \rho(f_n(x_0), f(x_0)).$$

Let us now fix arbitrary $\epsilon > 0$. Because $\sup_{x \in X} \rho(f_n(x), f(x)) \to 0$, we know there must be some $N \in \mathbb{N}$ s.t. $n \geq N \implies \sup_{x \in X} \rho(f_n(x), f(x)) < \epsilon/3$ (this takes care of the first and third terms on the RHS of the inequality above). Moreover, since $f_n$ is continuous for all $n \in \mathbb{N}$, choosing $n = N$, we get that $f_N$ is continuous implying that there is some $\delta > 0$ s.t. $d(x, x_0) < \delta \implies \rho(f_N(x), f_N(x_0)) < \epsilon/3$. Putting all these bounds together in the above inequality (taking $n = N$), we get that for $x, x_0 \in X$ s.t. $d(x, x_0) < \delta$

$$\rho(f(x), f(x_0)) \leq \rho(f(x), f_N(x)) + \rho(f_N(x), f_N(x_0)) + \rho(f_N(x_0), f(x_0)) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon,$$

which indeed shows that $f$ is continuous.

# 525 Homework 1

Daniel Rui - 1/13/20

## Problem 1 - 3.1.3

Let $\nu$ be a signed measure on $(X, \mathscr{M})$. We want to show the following:

(a) $L^1(\nu) = L^1(|\nu|)$. Looking at Folland's proof of Theorem 3.4 (the Jordan Decomposition Theorem), we see that letting $X = X^+ \cup X^-$ be the Hahn decomposition for $\nu$ and defining $\nu^+(E) = \nu(E \cap X^+)$ and $\nu^-(E) = -\nu(E \cap X^-)$, we get that $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$. Furthermore, Folland defines $|\nu| = \nu^+ + \nu^-$, and $L^1(\nu) = L^1(\nu^+) \cap L^1(\nu^-)$, so we just need to prove that $L^1(|\nu|) = L^1(\nu^+) \cap L^1(\nu^-)$.

Well because $|\nu| = \nu^+ + \nu^-$, $\int f\, d|\nu|$ is just $\int f\, d\nu^+ + \int f\, d\nu^-$ (one could prove this via the "standard machine"; starting with $f$ as indicator functions, then as simple functions, and then non-negative measurable functions by the MCT, and then to general measurable functions by taking the difference of the integrals of $f^+$, $f^-$). Anyways, it is clear that $\int |f|\, d|\nu|$ is finite if and only if $\int |f|\, d\nu^+$ and $\int |f|\, d\nu^-$ are finite (and of course $|f| \in L^1(|\nu|) \iff f \in L^1(|\nu|)$), so indeed the claim is true.

(b) If $f \in L^1(\nu)$, $|\int f\, d\nu| \leq \int |f|\, d|\nu|$. Let $A = [f \geq 0]$ and $B = [f < 0]$, and $X^+$ and $X^-$ be like in (a) (i.e. the Hahn decomposition of $X$). Note that on $A$, $f = |f|$ and on $B$, $f = -|f|$; and on $X^+$, $\phi = |\phi|$, and on $X^-$, $\phi = -|\phi|$. Now observe:

$$\begin{aligned}
\left|\int_X f\, d\phi\right| &= \left|\int_{A\cap X^+} f\, d\phi + \int_{B\cap X^+} f\, d\phi + \int_{A\cap X^-} f\, d\phi + \int_{B\cap X^-} f\, d\phi\right| \\
&= \left|\int_{A\cap X^+} |f|\, d\phi + \int_{B\cap X^+} -|f|\, d\phi + \int_{A\cap X^-} |f|\, d\phi + \int_{B\cap X^-} -|f|\, d\phi\right| \\
&= \left|\int_{A\cap X^+} |f|\, d|\phi| + \int_{B\cap X^+} -|f|\, d|\phi| - \int_{A\cap X^-} |f|\, d|\phi| - \int_{B\cap X^-} -|f|\, d|\phi|\right| \\
&= \left|\int_{A\cap X^+} |f|\, d|\phi| - \int_{B\cap X^+} |f|\, d|\phi| - \int_{A\cap X^-} |f|\, d|\phi| + \int_{B\cap X^-} |f|\, d|\phi|\right| \\
&\leq \left|\int_{A\cap X^+} |f|\, d|\phi|\right| + \left|\int_{B\cap X^+} |f|\, d|\phi|\right| + \left|\int_{A\cap X^-} |f|\, d|\phi|\right| + \left|\int_{B\cap X^-} |f|\, d|\phi|\right| \\
&= \int_{A\cap X^+} |f|\, d|\phi| + \int_{B\cap X^+} |f|\, d|\phi| + \int_{A\cap X^-} |f|\, d|\phi| + \int_{B\cap X^-} |f|\, d|\phi| \\
&= \int_X |f|\, d|\phi|
\end{aligned}$$

(c) If $E \in \mathscr{M}$, $|\nu|(E) = \sup\{|\int_E f\, d\nu| : |f| \leq 1\}$. By definition of $|\nu|$ (see part (a)), $|\nu|(E) = \nu^+(E \cap X^+) + \nu^-(E \cap X^-)$. Taking $f = 1_{X^+} - 1_{X^-}$ (indeed $|f| \leq 1$), by definition of integral w.r.t. $\nu$ (see top of page 88 in Folland) and because $\nu^+ \perp \nu^-$, we get that

$$\int_E f\, d\nu = \int_E f\, d\nu^+ - \int_E f\, d\nu^- = \int_E 1_{X^+}\, d\nu^+ - \int_E -1_{X^-}\nu^- = \nu^+(E\cap X^+) + \nu^-(E\cap X^-) = |\nu|(E).$$

This proves that $|\nu|(E) \leq \sup\{|\int_E f\, d\nu| : |f| \leq 1\}$. For the other direction, we use part (b), which tells us that $|\int_E f\, d\nu| \leq \int_E |f|\, d|\nu| \leq \int_E 1\, d|\nu| = |\nu|(E)$. This concludes this problem.

## Problem 2 - 3.2.9

Suppose $\{v_j\}$ is a sequence of positive measures. We want to show that if $v_j \perp \mu$ for all $j$, then $[\sum_{j=1}^{\infty} v_j] \perp \mu$, and that if $v_j \ll \mu$ for all $j$, then $[\sum_{j=1}^{\infty} v_j] \ll \mu$.

By definition of $\perp$ (page 87 of Folland), there exist measurable sets $S_j$ for each $\nu_j$ s.t. for all measurable sets $S \subseteq S_j$, $\nu_j(S) = 0$, and all measurable $S \subseteq S_j^{\complement}$, $\mu(S) = 0$. Then, for all measurable $S \subseteq \bigcap_{j=1}^{\infty} S_j$, $\nu_j(S) = 0$ for all $j \in \mathbb{N}$, implying that $\sum_{j=1}^{\infty} \nu_j(S) = 0$. Moreover, for all measurable $S \subseteq (\bigcap_{j=1}^{\infty} S_j)^{\complement} = \bigcup_{j=1}^{\infty} S_j^{\complement}$, implying that $\mu(S) \leq \sum_{j=1}^{\infty} \mu(S_j^{\complement}) = \sum_{j=1}^{\infty} 0 = 0$. Thus, we've found that $[\sum_{j=1}^{\infty} v_j]$ is null on $\bigcap_{j=1}^{\infty} S_j$ (measurable because it's an intersection of measurable sets) and $\mu$ is null on $(\bigcap_{j=1}^{\infty} S_j)^{\complement}$, so indeed $[\sum_{j=1}^{\infty} v_j] \perp \mu$.

By definition of $\ll$, we know that for all $j \in \mathbb{N}$, $\mu(S) = 0 \implies \nu_j(S) = 0$. Thus, $\mu(S) = 0 \implies \sum_{j=1}^{\infty} \nu_j(S) = 0$, so indeed $[\sum_{j=1}^{\infty} v_j] \ll \mu$.

## Problem 3 - 3.5.40

Let $F$ denote the Cantor function on $[0,1]$ and set $F(x) = 0$ for $x < 0$ and $F(x) = 1$ for $x > 1$ (to continuously extend $F$ to $\mathbb{R}$). Let $\{[a_n, b_n]\}$ be an enumeration of the closed subintervals of $[0,1]$ with rational endpoints, and let $F_n(x) = F((x - a_n)/(b_n - a_n))$. We want to show that $G = \sum_{n=1}^{\infty} F_n/2^n$ is continuous and strictly increasing on $[0,1]$, and that $G' = 0$ a.e..

Continuity: by the Weierstrass $M$-test, we see that because $F_n/2^n = |F_n/2^n| \leq 1/2^n$ and $\sum_{n=1}^{\infty} 1/2^n = 1 < \infty$, the partial sums $S_N = \sum_{n=1}^{N} F_n/2^n$ converges uniformly on $[0,1]$. As $[0,1]$ is a compact set, we have that a uniformly convergent sequence of continuous functions converges to a continuous function, and because $F_n/2^n$ is continuous for all $n \in \mathbb{N}$ (see page 39 of Folland), we see that $G$ must be continuous on $[0,1]$.

Strictly increasing: for any $x, y \in [0,1]$, $x < y$, we must have two rationals $r_1, r_2$ satisfying $x < r_1 < r_2 < y$ (by denseness of $\mathbb{Q}$). Because $\{[a_n, b_n]\}$ is an enumeration of all such intervals with rational endpoints, there must be some $n \in \mathbb{N}$ s.t. $a_n = r_1$ and $b_n = r_2$. Then, because $x < a_n$, $F_n(x) = 0$, and because $y > b_n$, $F_n(y) = 1$, so indeed $G(x) < G(x) + \frac{1}{2^n} \leq G(y)$; i.e. $G$ is strictly increasing on $[0,1]$.

$G' = 0$ a.e.: we know from Lec 01-04 or from Corollary 3.7 that increasing implies differentiability a.e. with derivative $\geq 0$ a.e., so we have that $G' \geq 0$ on $[0,1]$. Recall from Proposition 2.16 in Folland that a function $\geq 0$ is 0 a.e. on some set if and only if its integral on that set is 0. In our case, that says that $G' = 0$ a.e. $\iff \int_0^1 G'(x)\, dx = 0$. We know that for any finite $N \in \mathbb{N}$, $(\sum_{n=1}^{N} F_n/2^n)' = \sum_{n=1}^{N} F_n'/2^n = 0$, so we need only consider the remainder sums, $R_N = \sum_{n=N+1}^{\infty} F_n/2^n$, and show

that $\int_0^1 R_N'(x)\,dx \to 0$ as $N \to \infty$. Again using Corollary 3.7, we see that

$$\int_0^1 R_N'(x)\,dx \le R_N(1) - R_N(0) = \sum_{n=N+1}^{\infty} (F_n(1) - F_n(0)) \le \sum_{n=N+1}^{\infty} \frac{1}{2^n} = \frac{1}{2^N},$$

which indeed goes to 0 as $N \to \infty$. And so we are done.

## Problem 4

Show that if $F$ is of bounded variation in $[a,b]$, then $\int_a^b |F'(x)|\,dx \le T_a^b(F)$, and furthermore that $\int_a^b |F'(x)|\,dx = T_a^b(F)$ if and only if $F$ is absolutely continuous in $[a,b]$.

Solely assuming that $F$ is of bounded variation on $[a,b]$, we can use the properties discussed in Lec 12-04-20, namely that $f(x)-f(a) = P_a^x(f) - N_a^x(f)$ and $T_a^x(f) = P_a^x(f) + N_a^x(f)$ for $P_a^x(f)$ and $N_a^x(f)$ (resp. the positive and negative variation of $f$ on $[a,x]$, also defined in that lecture) increasing. From Lec 01-04, we know that increasing functions are differentiable a.e. and have derivative $\ge 0$ a.e., and so we have that $F'(x) = P_a^{x\prime}(F) - N_a^{x\prime}(F)$ a.e.. Furthermore, we see that

$$|F'(x)| \le |P_a^{x\prime}(F)| + |N_a^{x\prime}(F)| = P_a^{x\prime}(F) + N_a^{x\prime}(F).$$

Using monotonicity of the integral and the inequality from Lec 01-04, we get

$$\begin{aligned}\int_a^b |F'(x)|\,dx &\le \int_a^b P_a^{x\prime}(F)\,dx + \int_a^b N_a^{x\prime}(F)\,dx \\ &\le P_a^b(F) - P_a^a(F) + N_a^b(F) - N_a^a(F) = P_a^b(F) + N_a^b(F) = T_a^b(F).\end{aligned}$$

If we further assume that $F$ is absolutely continuous, the inequality we just proved continues to hold (since absolutely continuous $\implies$ BV by Remark 2 from Lec 01-04), but furthermore we get equality (i.e. we get the other direction of inequality). For any partition $\mathscr{P} = \{t_i\}_0^n$ of $[a,b]$, we see that (using the fact that for the absolutely continuous function $F$, $F(x) - F(a) = \int_a^x F'(y)\,dy$):

$$\sum_{i=1}^n |F(t_i) - F(t_{i-1})| = \sum_{i=1}^n \left| \int_{t_{i-1}}^{t_i} F'(x)\,dx \right| \le \sum_{i=1}^n \int_{t_{i-1}}^{t_i} |F'(x)|\,dx = \int_a^b |F'(x)|\,dx.$$

As this is true for any partition, it is remains true after we take the supremum over all partitions, yielding $T_a^b(F) \le \int_a^b |F'(x)|\,dx$.

The converse is also true — if $F$ is BV and $T_a^b(F) = \int_a^b |F'(x)|\,dx$, then $F$ must be absolutely continuous. First, from $\int_a^b |F'(x)|\,dx = T_a^b(F)$, we can conclude that $\int_a^x |F'(x)|\,dx = T_a^x(F)$, since we have that

$$T_a^b(F) - T_x^b(F) = T_a^x(F) \ge \int_a^x |F'(y)|\,dy = \int_a^b |F'(y)|\,dy - \int_x^b |F'(y)|\,dy \ge T_a^b(F) - T_x^b(F)$$

where the first "$\ge$" is what we proved about BV functions a couple paragraphs up. To rephrase, we

have shown that $T_a^x(F)$ is equal to the integral from $a$ to $x$ of a function in $L^1[a,b]$, so by Remark 4 from Lec 01-04, $T_a^x(F)$ is absolutely continuous. By the definition of total variation, we have that for any $a \le a_i < b_i \le b$, $|F(b_i) - F(a_i)| \le T_x^y(F) = T_a^y(F) - T_a^x(F)$. Absolute continuity of $T_a^x(F)$ means that for every $\epsilon > 0$, we can find $\delta > 0$ s.t. for disjoint intervals $\{(a_k, b_k)\}_{k=1}^N$ in $[a,b]$ satisfying $\sum_{k=1}^N (b_k - a_k) < \delta$, we have

$$\sum_{k=1}^N |F(b_k) - F(a_k)| \le \sum_{k=1}^N |T_a^{b_k}(F) - T_a^{a_k}(F)| < \epsilon,$$

proving that $F$ is also absolutely continuous on $[a,b]$.

## Problem 5

A bounded function $F : \mathbb{R} \to \mathbb{R}$ is said to be of bounded variation on $\mathbb{R}$ if $F$ is of bounded variation on any finite interval $[a,b]$ and $\sup_{[a,b]} T_a^b(F) < \infty$. We want to prove that

(a) $\int_{\mathbb{R}} |F(x+h) - F(x)|\, dx \le A|h|$, for some constant $A$ and all $h \in \mathbb{R}$.

I suppose we are assuming that $F$ is measurable, in which case we have that $|F(x+h) - F(x)|$ is measurable and hence in $L^+$. Without loss of generality, let us take $h > 0$ (if $h < 0$, then $\int_{\mathbb{R}} |F(x+h) - F(x)|\, dx = \int_{\mathbb{R}} |F(y+|h|+h) - F(y+|h|)|\, dy = \int_{\mathbb{R}} |F(y+|h|) - F(y)|\, dy$, and of course the statement is true for $h = 0$). Let us now split the integral as follows: $\int_{\mathbb{R}} |F(x+h) - F(x)|\, dx = \sum_{n=-\infty}^{\infty} \int_{nh}^{(n+1)h} |F(x+h) - F(x)|\, dx$. Doing a change of variable, and interchanging the sum and integral (which we can do for $L^+$ functions by a corollary of the MCT), we get that

$$\begin{aligned}\int_{\mathbb{R}} |F(x+h) - F(x)|\, dx &= \sum_{n=-\infty}^{\infty} \int_0^h |F(x+(n+1)h) - F(x+nh)|\, dx \\ &= \int_0^h \sum_{n=-\infty}^{\infty} |F(x+(n+1)h) - F(x+nh)|\, dx.\end{aligned}$$

But recall that

$$\sum_{n=-\infty}^{\infty} |F(x+(n+1)h) - F(x+nh)| = \lim_{N\to\infty} \sum_{n=-N}^{N} |F(x+(n+1)h) - F(x+nh)|,$$

where for any fixed $x$, the sum $\sum_{n=-N}^{N} |F(x+(n+1)h) - F(x+nh)|$ is simply the variation of $F$ over the partition $\{x - Nh, \ldots, x + (N+1)h\}$. By the definition of total variation and supremum, we know that this sum is $\le A := \sup_{[a,b]} T_a^b(F) < \infty$, a fact that remains true as we push $N$ to $\infty$. Thus, our integral can now be bounded by

$$\int_{\mathbb{R}} |F(x+h) - F(x)|\, dx \le \int_0^h \sup_{[a,b]} A\, dx = Ah.$$

Thus, because we took $h > 0$ w.l.o.g., we can extend this to all $h \in \mathbb{R}$ to the desired statement: $\int_{\mathbb{R}} |F(x+h) - F(x)|\, dx \le A|h|$, where $A = \sup_{[a,b]} T_a^b(F)$.

(b) $|\int_{\mathbb{R}} F(x)\phi'(x)\,dx| \leq A$, where $\phi$ ranges over all $C^1$ functions with compact support satisfying $\sup_{x\in\mathbb{R}} |\phi(x)| \leq 1$.

Since $\phi'(x)$ exists, we have that $\phi'(x) = \lim_{n\to\infty} \frac{\phi(x+1/n)-\phi(x)}{1/n}$. Substituting into the integral, we get that

$$\int_{\mathbb{R}} F(x)\phi'(x)\,dx = \int_{\mathbb{R}} \lim_{n\to\infty} f_n(x)\,dx := \int_{\mathbb{R}} \lim_{n\to\infty} F(x)\frac{\phi(x+1/n)-\phi(x)}{1/n}\,dx.$$

To move the limit outside the integral, we use the DCT. First we have to bound the $|f_n|$; because $\phi$ is $C^1$, we can use the MVT, allowing us to bound $|f_n| \leq \sup_{x\in\mathbb{R}} |F(x)| \cdot \sup_{x\in\mathbb{R}} |\phi'|$. But $\phi$ has compact support, say $K$, so $\phi'$ will be a continuous function on the compact set $K$, and hence $|\phi'|$ will be bounded. We are also assuming that $F$ is bounded, so we do have that $|f_n|$ is bounded by some $B \cdot 1_K \in \mathbb{R}$, which is in fact in $L^1(\mathbb{R})$. Thus, we can apply the DCT, yielding

$$\int_{\mathbb{R}} F(x)\phi'(x)\,dx = \lim_{n\to\infty} \int_{\mathbb{R}} F(x)\frac{\phi(x+1/n)-\phi(x)}{1/n}\,dx = \lim_{n\to\infty} \frac{1}{1/n}\int_{\mathbb{R}} F(x)(\phi(x+1/n)-\phi(x))\,dx.$$

Now note that (keeping in mind all the integrals are bounded):

$$\begin{aligned}\int_{\mathbb{R}} F(x)(\phi(x+1/n)-\phi(x))\,dx &= \int_{\mathbb{R}} F(x)\phi(x+1/n)\,dx - \int_{\mathbb{R}} F(x)\phi(x)\,dx \\ &= \int_{\mathbb{R}} F(y-1/n)\phi(y)\,dy - \int_{\mathbb{R}} F(x)\phi(x)\,dx \\ &= \int_{\mathbb{R}} (F(x-1/n)-F(x))\phi(x)\,dx.\end{aligned}$$

Substituting and manipulating (e.g. using continuity of absolute value, pulling absolute values inside to bound, using the bound proved in (a) and the fact that $\phi$ is bounded by 1), we see that

$$\begin{aligned}\left|\int_{\mathbb{R}} F(x)\phi'(x)\,dx\right| &= \left|\lim_{n\to\infty} \frac{1}{1/n}\int_{\mathbb{R}} (F(x-1/n)-F(x))\phi(x)\,dx\right| \\ &= \lim_{n\to\infty}\left|\frac{1}{1/n}\int_{\mathbb{R}} (F(x-1/n)-F(x))\phi(x)\,dx\right| \\ &\leq \lim_{n\to\infty} \frac{1}{1/n}\int_{\mathbb{R}} |(F(x-1/n)-F(x))\phi(x)|\,dx \\ &\leq \lim_{n\to\infty} \frac{1}{1/n}\int_{\mathbb{R}} |F(x-1/n)-F(x)|\,dx \\ &\leq \lim_{n\to\infty} \frac{1}{1/n}\cdot A|-1/n| = A,\end{aligned}$$

and we are done.

## Problem 6

Let $F$ be an increasing right-continuous real-valued function on $[0,1]$ with $F(0) = 0$. We want to show that $F$ can be written uniquely as $F = F_A + F_S + F_J$ where the functions $F_A, F_S, F_J$ are all increasing, all evaluate to 0 at $x = 0$, and satisfy

(a) $F_A$ is absolutely continuous

(b) $F_S$ is continuous and singular, i.e. $F_S'(x) = 0$ for a.e. $x \in [0,1]$

(c) and $F_J$ is a right-continuous "jump function", meaning that there is is an empty, finite, or countable subset $S \subseteq (0,1]$ and positive constants $\{c_s\}_{s\in S}$ with $\sum_{s\in S} c_s < \infty$ for which $F_j(x) = \sum_{s\in S: s\leq x} c_s$ for ALL $x \in [0,1]$ (following the convention that an empty sum is 0).

EXISTENCE: Let $F_J$ be exactly $J_F$ mentioned in the section above Lemma 3.13 in Stein & Shakarchi (we will be using their notation in this paragraph). In our case, because we have that $F$ is right-continuous, i.e. at every discontinuity $x_n$ (of which there are only countably many; see Lemma 3.12 or Problem 5(a) from Homework 1 of Math 524), $F(x_n) = F(x_n+) := \lim_{y\to x^+} F(y)$, our $F_J$ will be

$$F_J(x) = \sum_{n=1}^{\infty} \alpha_n 1_{[x\geq x_n]} = \sum_{n\in\mathbb{N}, x_n\leq x} \alpha_n$$

for $\alpha_n$ defined as $\alpha_n = F(x_n+) - F(x_n-)$. Clearly $F_J$ satisfies the conditions set forth in (c). By Lemma 3.13, $F - F_J$ is increasing and continuous. It is clear that $F_J$ is increasing, and also $x = 0$ is not a point of discontinuity of $F$, since $F(0) = F(0+)$ by right continuity, so $F_J(0) = 0$.

From Lec 01-04 or from Corollary 3.7, we know that increasing implies differentiability a.e. with derivative $\geq 0$ a.e., so let $g$ be the derivative of $F - F_J$. Now define $F_A(x) = \int_0^x g(y)\,dy$. Then, by Theorem 3.11 in S&S, or Remark 4 in Lec 01-06, $F_A$ is indeed absolutely continuous. Note that by definition of $\int_0^x g(y)\,dy$ (where $g \geq 0$ a.e.), $F_A(0) = 0$, and $F_A$ is increasing. The inequality given both in Corollary 3.7 or Lec 01-04 furthermore yields that $F_A(x) - F_A(a) = \int_a^x g(y)\,dy \leq (F - F_J)(x) - (F - F_J)(a)$ for all $a, x \in [0,1]$, $a \leq x$.

Finally, define $F_S = (F - F_J) - F_A$. Clearly $F_S(0) = 0$, and from the last line of the the above paragraph, we know that $F_S$ is increasing. $F_S$ is a difference of continuous and differentiable a.e. functions, so it it also continuous and differentiable a.e. with derivative $F_S' = g - g = 0$ (where we used that $F_A' = g$, given again in Theorem 3.11 or page 6 of Lec 01-06).

Onward to UNIQUENESS: First, $F_J$. Say we have two decompositions of $F$, $F_A + F_S + F_J$ and $G_A + G_S + G_J$ (both satisfying the requirements listed in (a), (b), and (c)). Then, we have that $F - F_J$ and $F - G_J$ are continuous, implying that $F_J - G_J$ is continuous. But $F_J - G_J$ is also a jump function, and the only continuous jump functions are constant functions. But $F_J(0) = G_J(0) = 0$, so $F_J - G_J$ must be identically 0.

Now for $F_A$. Notice that all the functions we are dealing with are differentiable a.e. (Theorem 3.14 in S&S give that jump functions are differentiable a.e. and have derivative 0 a.e.), so we can differentiate both sides and get $F_A' + 0 + 0 = G_A' + 0 + 0$. But absolutely continuous functions can be recovered from their derivatives; i.e. by Theorem 3.11, $F_A(x) = \int_0^x F_A'(y)\,dy = \int_0^x G_A'(y)\,dy = G_A(x)$. Finally, $F_S = G_S$ because $F_S = F - F_J - F_A = F - G_J - G_A = G_S$, and we are done.