534 Homework 9

Daniel Rui - 5/30/20

...

Problem 1 - DO NOT GRADE

We want to show that every rational function $f = \frac{g}{h}$ of degree $d \geq 1$ (degree of rational function is max of degrees of numerator and denominator) assumes every value $w \in \hat{\mathbb{C}}$ exactly d times in $\hat{\mathbb{C}}$ and has $2d-2$, COUNTING MULTIPLICITY. More explicitly, "assuming w at z with multiplicity m" means that $f(z) - w$ has a zero of order m at $z \in \hat{\mathbb{C}}$, a critical point is a point where the derivative is 0, and counting multiplicity of critical points is again the order of the zero. Let degree of numerator, denominator be d_n, d_d respectively. Then $f(z) - w = 0 \iff g(z) - w = 0$ for $z \in \mathbb{C}$, so the fundamental theorem of algebra gives d_n values of $z \in \mathbb{C}$ assuming w (counting multiplicity). Order of zero at ∞ is order of 0 at 0 of $f(\frac{1}{z}) = \frac{g(1/z)}{h(1/z)} = \frac{z^d g(1/z)}{z^d h(1/z)}$ where number of "z"s in the numerator (i.e. the order of the zero at 0) is exactly $d - d_1$. Similar analysis on the derivative yields desired claims.

Problem 2 - VI.2.11

Suppose that $f(z) = \sum a_k z^k$ is analytic for $|z| < R$, and suppose that $f(z)$ extends to be meromorphic for $|z| < R + \epsilon$ with one pole z_0 on the circle $|z| = R$. We want to show that $a_k/a_{k+1} \to z_0$ as $k \to \infty$.

Looking at the discussion at the end of page 172 and start of page 173 for Gamelin, let

$$f(z) - \sum_{j=1}^{m} \frac{b_j}{(z - z_0)^j} = \sum c_k z^k$$

be analytic in $|z| < R + \epsilon$ (i.e. we are subtracting the principle part of f). We know that for $|z| < |z_0| = R$,

$$\frac{1}{z - z_0} = -\frac{1/z_0}{1 - \frac{z}{z_0}} = -\frac{1}{z_0} \sum_{n=0}^{\infty} \left(\frac{z}{z_0} \right)^n$$

and so taking derivatives (we can take derivatives of power series, and those power series will have the same radius of convergence) we see that

$$f^{(1)}(z) = (-1)\frac{1}{(z - z_0)^2} = -\frac{1}{z_0} \sum_{n=1}^{\infty} (z_0^{-n}) n z^{n-1} = -\frac{1}{z_0} \sum_{n=0}^{\infty} (z_0^{-n-1})(n+1) z^n$$

and in general:

$$f^{(j-1)}(z) = (-1)(-2)\cdots(-(j-1))\frac{1}{(z - z_0)^j} = -\frac{1}{z_0} \sum_{n=0}^{\infty} (z_0^{-n-(j-1)})(n+1)\cdots(n+(j-1)) z^n$$

or via rearrangement:

$$\frac{1}{(z-z_0)^j} = \frac{(-1)^j}{(j-1)!}\frac{1}{z_0}\sum_{n=0}^{\infty}\frac{(n+1)\cdots(n+(j-1))}{z_0^{n+(j-1)}}z^n$$

and so looking at the coefficients of z^k, we see that

$$a_k = c_k + \sum_{j=1}^{m}\frac{(-1)^j}{(j-1)!z_0}b_j\frac{(k+j-1)\dots(k+1)}{z_0^{k+j-1}}$$
$$a_{k+1} = c_{k+1} + \sum_{j=1}^{m}\frac{(-1)^j}{(j-1)!z_0}b_j\frac{(k+j)\dots(k+2)}{z_0^{k+j}}.$$

Dividing and multiplying top and bottom by z_0^{k+1} we get

$$\frac{a_k}{a_{k+1}} = z_0\frac{c_kz_0^k + \sum_{j=1}^{m}\frac{(-1)^j}{(j-1)!}b_j(k+j-1)\dots(k+1)z_0^{-j}}{c_{k+1}z_0^{k+1} + \sum_{j=1}^{m}\frac{(-1)^j}{(j-1)!}b_j(k+j)\dots(k+2)z_0^{-j}} \qquad (1)$$
$$= z_0\frac{c_kz_0^k + \frac{(-1)^m}{(m-1)!}b_mz_0^{-m}k^{m-1}+\dots}{c_{k+1}z_0^{k+1} + \frac{(-1)^m}{(m-1)!}b_mz_0^{-m}k^{m-1}+\dots}.$$

where the "..." in the numerator/denominator are different (we just expand out the polynomials in k in equation (1), and indicate that they have the same first term). Since $\sum c_kz_0^k$ converges, $c_kz_0^k \to 0$ as $k\to\infty$. Dividing numerator and denominator by k^{m-1} and letting $k\to\infty$, we see that

$$\frac{a_k}{a_{k+1}} \to z_0\frac{0+\frac{(-1)^m}{(m-1)!}b_mz_0^{-m}+0}{0+\frac{(-1)^m}{(m-1)!}b_mz_0^{-m}+0} = z_0.$$

Problem 3

We prove the following assertion from pg. 200 of Gamelin. Let P, Q be polynomials, where Q has no zeroes on $\mathbb{R}$ and $d_Q \geq d_P + 2$ where d_Q, d_P are the degrees of Q, P respectively. We want to show $\int_{-\infty}^{\infty}\frac{P(x)}{Q(x)}\,dx = 2\pi i\sum \mathrm{Res}\left[\frac{P(z)}{Q(z)}, z_j\right]$ summed over the poles (i.e. zeroes of Q) in the UPPER HALF PLANE. As Gamelin suggests, we use the semicircle S_R contour found on pg. 200 (composed of the line from $-R$ to R, and the semicircle Γ_R going counterclockwise). Since there are finitely many zeroes of Q, past some point, there are no pole on S_R, and because $\frac{P}{Q}$ is analytic on $\mathbb{C}$ except for finitely many isolated singularities, we can apply the residue theorem which tells us that

$$\int_{\Gamma_R}\frac{P(z)}{Q(z)}\,dz = \int_{-R}^{R}\frac{P(x)}{Q(x)}\,dx + \int_{\Gamma_R}\frac{P(z)}{Q(z)}\,dz = \sum_{\mathrm{Im}(z_j)>0}\mathrm{Res}\left[\frac{P(z)}{Q(z)}, z_j\right]. \qquad (1)$$

Note that $Q(z) = a_{d_Q} z^{d_Q} + \ldots + a_0$ (and $P(z) = b_{d_P} z^{d_P} + \ldots + b_0$), and by the reverse triangle inequality, $|Q(z)| \geq |a_{d_Q}||z|^{d_Q} - |a_{d_Q-1} z^{d_Q-1} + \ldots + a_0|$, and by the triangle inequality $|a_{d_Q-1} z^{d_Q-1} + \ldots + a_0| \leq |a_{d_Q-1}||z|^{d_Q-1} + \ldots + |a_0|$, so $|Q(z)| \geq |a_{d_Q}||z|^{d_Q} - (|a_{d_Q-1}||z|^{d_Q-1} + \ldots + |a_0|)$. The ML-estimate then yields

$$\left| \int_{\Gamma_R} \frac{P(z)}{Q(z)}\, dz \right| \leq \frac{|b_{d_P}| R^{d_P} + \ldots + |b_0|}{|a_{d_Q}| R^{d_Q} - (|a_{d_Q-1}| R^{d_Q-1} + \ldots + |a_0|)} \cdot \pi R.$$

Since $d_Q \geq d_P + 2$, dividing top and bottom by R^{d_P+2} makes the numerator only have R's in the bottom, while still having at least one term in the denominator with a non-negative power of R. That means that taking $R \to \infty$, the numerator would go to 0, and denominator would either go to the non-zero constant term (if there are no terms with positive power of R in the denominator), or otherwise ∞. Either way, the integral goes to 0 as $R \to \infty$, so taking the limit as $R \to \infty$ of equation (1) above yields the desired claim $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\, dx = 2\pi i \sum_{\mathrm{Im}(z_j)>0} \mathrm{Res}\left[\frac{P(z)}{Q(z)}, z_j\right]$ summed over the poles (i.e. zeroes of Q) in the upper half plane.

We use this fact to find $\int_0^{\infty} \frac{1}{(1+x^2)^2}\, dx$. By symmetry, this is exactly $\frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^2}\, dx$. The only pole of the integrand $f(z)$ in the upper half plane is at i, and it is a double pole (because $g(z) := \frac{1}{(z+i)^2}$ is analytic and non-zero at $z = i$, and $f(z) = g(z)/(z-i)^2$, and so it is a double pole by the first theorem on pg. 173). Rule 2 (pg. 197) gives that $\mathrm{Res}[f(z), i]$ is the derivative of $g(z)$ at $z = i$, which is $-i/4$, and so the fact above tells us $\int_0^{\infty} \frac{1}{(1+x^2)^2}\, dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^2}\, dx = \pi i\, \mathrm{Res}[f(z), i] = \pi i(-i/4) = \frac{\pi}{4}$.

Problem 4

We want to evaluate the integral $\int_0^{\infty} \frac{\log x}{(1+x^2)^2}\, dx$. We will integrate around a half keyhole contour κ (see pg. 206). We use the same notation as the keyhole contour (pg. 206-7), where $\Gamma_R, \gamma_\epsilon$ are just π radian arcs of circles of radius R and ϵ respectively, and where the "integral from R back to ϵ" is done along the line $z = -t \implies dz = -\,dt$ (t going from R to ϵ). We choose a branch of $\log z$ that say has a branch cut along the negative imaginary axis, and agrees with the real-values log function on the positive real axis. In particular we have $\log(-t) = \log t + \pi i$ for $t \in [\epsilon, R]$. Denote $f(z) := \frac{\log z}{(1+z^2)^2}$ again for the branch of $\log z$ we just specified above.

Thus the integral around the half keyhole contour $\oint_\kappa f(z)\, dz$ is (very similar to Gamelin pg. 207)

$$\oint_\kappa f(z)\, dz = \int_\epsilon^R \frac{\log x}{(1+x^2)^2}\, dx + \int_{\Gamma_R} f(z)\, dz + \int_R^\epsilon \frac{\log(t) + \pi i}{(1+t^2)^2}(-1)\, dt + \int_{\gamma_\epsilon} f(z)\, dz.$$

The ML-estimate gives that the Γ_R integral is bounded in absolute value by $\frac{\log|R| + \pi}{(R^2-1)^2} \cdot \pi R$, which goes to 0 as $R \to \infty$. Similarly, the ML-estimate gives that the γ_ϵ integral is bounded in absolute value by $\frac{\log|\epsilon| + \pi}{(1-\epsilon^2)^2} \cdot \pi\epsilon$. Recalling that $\lim_{x \searrow 0} x \log x = \lim_{t \to \infty}(e^{-t})(-t) = 0$ (decaying exponentials decay MUCH faster than linear growth), which one can check rigorously using an easy application of l'Hôpital's rule, the numerator goes to 0 and denominator goes to 1 as $\epsilon \to 0$, so the whole thing goes to 0 as $\epsilon \to 0$.

The residue theorem (pg. 196) on the other hand gives that $\oint_\kappa f(z)\,dz = 2\pi i \operatorname{Res}[f(z), i]$ (since that's the only isolated singularity of f in the half keyhole domain), where the pole is a double pole (because $g(z) := \frac{\log z}{(z+i)^2}$ is analytic and non-zero at $z = i$, and $f(z) = g(z)/(z-i)^2$, and so it is a double pole by the first theorem on pg. 173). Rule 2 (pg. 197) gives that $\operatorname{Res}[f(z), i]$ is the derivative of $g(z)$ at $z = i$, which is $\frac{\pi}{8} + \frac{i}{4}$. Thus, putting the two pieces together we have

$$2\int_0^\infty \frac{\log x}{(1+x^2)^2}\,dx + \pi i \int_0^\infty \frac{1}{(1+t^2)^2}\,dt = \lim_{R\to\infty,\epsilon\to 0} \oint_\kappa f(z)\,dz = 2\pi i\left(\frac{\pi}{8} + \frac{i}{4}\right)$$

so (using the concluding example of Problem 3 above)

$$2\int_0^\infty \frac{\log x}{(1+x^2)^2}\,dx + \pi i \cdot \frac{\pi}{4} = \pi i \cdot \frac{\pi}{4} + \frac{-\pi}{2} \implies \int_0^\infty \frac{\log x}{(1+x^2)^2}\,dx = -\frac{\pi}{4}.$$

Problem 5 - VII.1.6

Consider the integral

$$\int_{\partial D_R} \frac{e^{\pi i (z-1/2)^2}}{1 - e^{-2\pi i z}}\,dz,$$

where D_R is the parallelogram with vertices $\pm\frac{1}{2} \pm (1+i)R$ (so think of the diagonal line $1+i$ stretched to $-R(1+i)$ and $R(1+i)$, and move it left $\frac{1}{2}$ and right $\frac{1}{2}$ to create the parallelogram).

(a) We use the residue theorem to show the integral is $(1+i)/\sqrt{2}$. The singularities of the integrand are exactly where $1 = e^{-2\pi i z}$, i.e. at exactly $z = k$ for $k \in \mathbb{Z}$. Thus, the only singularity in D_R is at $z = 0$ (we shift the line $(-R(1+i), R(1+i))$ left and right by $\frac{1}{2}$, so we do not hit any integers on $\mathbb{R}$ other than 0), and indeed the integrand (which we denote $f(z)$) is analytic everywhere else on $D_R \cup \partial D_R$. The residue theorem (pg. 196) then gives us that $\int_{\partial D_R} f(z)\,dz = 2\pi i \operatorname{Res}[f(z), 0]$.

I claim now that the singularity at $z = 0$ is a simple pole. Defining $g(z) := \frac{e^{\pi i (z-1/2)^2}}{1-e^{-2\pi i z}} \cdot (z-0)$, note that

$$-\lim_{z\to 0} \frac{1 - e^{-2\pi i z}}{z - 0} = \lim_{z\to 0} \frac{e^{-2\pi i z} - e^{-2\pi i \cdot 0}}{z - 0}$$

exists and is exactly the derivative of $e^{-2\pi i z}$ at $z = 0$, i.e. $(-2\pi i \cdot e^{-2\pi i z})|_{z=0} = -2\pi i$. Then observe that the singularities of g are exactly the same as the singularities of f, but because $\lim_{z\to 0} g(z)$ exists (equals $-\frac{e^{\pi i \cdot 1/4}}{-2\pi i} = \frac{e^{\pi i \cdot 1/4}}{2\pi i}$), the Riemann removable singularity theorem (pg. 172) tells us that g can be extended to be analytic at 0 (by defining $g(0) = \frac{e^{\pi i \cdot 1/4}}{2\pi i}$), hence analytic on $D_R \cup \partial D_R$. Then, the first theorem on pg. 173 tells us that because $f(z) = g(z)/(z-0)^1$ where g is analytic and non-zero at 0, the singularity of f at 0 is a simple pole.

Rule 1 (pg. 196) tells us that $\operatorname{Res}[f(z), 0] = \lim_{z\to 0} z f(z) = \lim_{z\to 0} g(z)$ which we found above to be $\frac{e^{\pi i \cdot 1/4}}{2\pi i}$, so indeed $\int_{\partial D_R} f(z)\,dz = e^{\pi i/4} = \cos(\frac{\pi}{4}) + i\sin(\frac{\pi}{4}) = (1+i)/\sqrt{2}$, as desired.

(b) Let γ_1 denote the path $\frac{1}{2} + t(1+i)$, t going from $-R$ to R, γ_2 denote the path $-t + \frac{1}{2} + R(1+i)$, t going from 0 to 1, γ_3 denote the path $-\frac{1}{2} + t(1+i)$, t going from R to $-R$, and γ_4 denote the

path $t - \frac{1}{2} - R(1+i)$, t going from 0 to 1. Note all paths depend on R, but I don't want to deal with long subscripts so just keep that in the back of your mind.

- γ_1: we substitute $z = \frac{1}{2} + t(1+i) \implies dz = (1+i)\,dt$, so

$$\int_{\gamma_1} f(z)\,dz = \int_{-R}^{R} \frac{e^{\pi i(t(1+i))^2}}{1 - e^{-\pi i(1+2t(1+i))}}(1+i)\,dt = (1+i)\int_{-R}^{R} \frac{e^{-2\pi t^2}}{1-(-1)e^{-2\pi i t(1+i)}}\,dt,$$

  which approaches $(1+i)\int_{-\infty}^{\infty} \frac{e^{-2\pi t^2}}{1+e^{-2\pi i t(1+i)}}\,dt$ as $R \to \infty$.

- γ_2: we substitute $z = -t + \frac{1}{2} + R(1+i) \implies dz = (-1)\,dt$, so

$$\begin{aligned}\int_{\gamma_2} f(z)\,dz &= \int_0^1 \frac{e^{\pi i(-t+R(1+i))^2}}{1 - e^{-\pi i(-2t+1+2R(1+i))}}\,dt = \int_0^1 \frac{e^{\pi i(t^2-2Rt)}e^{\pi i(-2Rit+(2i)R^2)}}{1 - e^{\pi i(2t-1-2R)}e^{2\pi R}}\,dt \\ &= \int_0^1 \frac{e^{\pi i(t^2-2Rt)}e^{-2\pi(-Rt+R^2)}}{1 - e^{\pi i(2t-1-2R)}e^{2\pi R}}\,dt\end{aligned}$$

  (note we have separated the exponential into an imaginary exponent and real exponent) and so (using reverse triangle inequality in the second "$\le$")

$$\begin{aligned}\left|\int_{\gamma_2} f(z)\,dz\right| &\le \sup_{0\le t\le 1}\left|\frac{e^{\pi i(t^2-2Rt)}e^{-2\pi(-Rt+R^2)}}{1 - e^{\pi i(2t-1-2R)}e^{2\pi R}}\right| \le \sup_{0\le t\le 1}\frac{e^{-2\pi(-Rt+R^2)}}{|1 - |e^{\pi i(2t-1-2R)}e^{2\pi R}||} \\ &= \sup_{0\le t\le 1}\frac{e^{-2\pi(-Rt+R^2)}}{|1-e^{2\pi R}|} = \frac{e^{-2\pi(-R\cdot 1+R^2)}}{e^{2\pi R}-1}\end{aligned}$$

  which approaches 0 as $R \to \infty$ (numerator approaches 0, denominator approaches ∞, everything ≥ 0).

- γ_3: we substitute $z = -\frac{1}{2} + t(1+i) \implies dz = (1+i)\,dt$, so

$$\begin{aligned}\int_{\gamma_3} f(z)\,dz &= \int_R^{-R} \frac{e^{\pi i(t(1+i)-1)^2}}{1 - e^{-\pi i(-1+2t(1+i))}}(1+i)\,dt = -(1+i)\int_{-R}^{R} \frac{e^{\pi i(2it^2-2t(1+i)+1)}}{1-(-1)e^{-2\pi i t(1+i)}}\,dt \\ &= -(1+i)\int_{-R}^{R} \frac{e^{-2\pi t^2}e^{-2\pi i t(1+i)}(-1)}{1+e^{-2\pi i t(1+i)}}\,dt,\end{aligned}$$

  which approaches $(1+i)\int_{-\infty}^{\infty} \frac{e^{-2\pi t^2}e^{-2\pi i t(1+i)}}{1+e^{-2\pi i t(1+i)}}\,dt$ as $R \to \infty$.

- γ_4: we substitute $z = t - \frac{1}{2} - R(1+i) \implies dz = dt$, so

$$\begin{aligned}\int_{\gamma_4} f(z)\,dz &= \int_0^1 \frac{e^{\pi i((t-1-R(1+i))^2}}{1 - e^{-\pi i(2t-1-2R(1+i))}}\,dt = \int_0^1 \frac{e^{\pi i((t-(1+R))^2+(Ri)^2)}e^{\pi i(-2(t-(1+R))\cdot Ri)}}{1 - e^{-\pi i(2t-1-2R)}e^{-2\pi R}}\,dt \\ &= \int_0^1 \frac{e^{\pi i((t-(1+R))^2-R^2)}e^{2\pi R(t-(1+R))}}{1 - e^{-\pi i(2t-1-2R)}e^{-2\pi R}}\,dt\end{aligned}$$

and so

$$\left|\int_{\gamma_4} f(z)dz\right| \leq \sup_{0\leq t\leq 1}\left|\frac{e^{\pi i((t-(1+R))^2-R^2)}e^{2\pi R(t-(1+R))}}{1-e^{-\pi i(2t-1-2R)}e^{-2\pi R}}\right|$$
$$\leq \sup_{0\leq t\leq 1}\frac{e^{-2\pi(R^2+R-Rt)}}{|1-|e^{-\pi i(2t-1-2R)}e^{-2\pi R}||} \leq \sup_{0\leq t\leq 1}\frac{e^{-2\pi(R^2+R-Rt)}}{|1-e^{-2\pi R}|} = \frac{e^{-2\pi(R^2+R-R\cdot 1)}}{|1-e^{-2\pi R}|}$$

which approaches 0 as $R\to\infty$ (numerator goes to 0, denominator goes to 1).

Thus, we have that

$$\int_{\partial D_R} f(z)\,dz = \sum_{i=1}^{4}\int_{\gamma_i} f(z)\,dz$$

approaches

$$(1+i)\int_{-\infty}^{\infty}\frac{e^{-2\pi t^2}}{1+e^{-2\pi it(1+i)}} + 0 + (1+i)\int_{-\infty}^{\infty}\frac{e^{-2\pi t^2}e^{-2\pi it(1+i)}}{1+e^{-2\pi it(1+i)}} + 0 = (1+i)\int_{-\infty}^{\infty} e^{-2\pi t^2}\cdot 1\,dt$$

as $R\to\infty$.

(c) Using (a) and (b), we see that $(1+i)\int_{-\infty}^{\infty} e^{-2\pi t^2}\,dt = (1+i)/\sqrt{2} \iff \int_{-\infty}^{\infty} e^{-2\pi t^2}\,dt = 1/\sqrt{2}$. Performing the variable substitution $s=\sqrt{2\pi}t \implies s^2 = 2\pi t^2$, $ds = \sqrt{2\pi}\,dt$, we see

$$\int_{-\infty}^{\infty} e^{-2\pi t^2}\,dt = \int_{-\infty}^{\infty} e^{-s^2}\cdot(1/\sqrt{2\pi})\,ds = 1/\sqrt{2} \implies \int_{-\infty}^{\infty} e^{-s^2}\,ds = \sqrt{\pi}.$$

Problem 6(a) - VII.4.1

We are asked to integrate around the keyhole contour κ (pg. 206) in order to show that $\int_0^\infty \frac{x^{-a}}{1+x}\,dx = \frac{\pi}{\sin\pi a}$ for any $a\in(0,1)$ (we will denote the integrand by $f(z)$). We use the notation from pg. 206-7, i.e. "the integral around ∂D breaks into the sum of four integrals, from ϵ to R along the top edge of the slit, around the circular contour Γ_R of radius R in the counterclockwise direction, from R back to ϵ along the bottom edge of the slit, and around the circular contour γ_ϵ of radius ϵ in the clockwise direction". As Gamelin does on pg. 206, we consider the branch z^{-a} defined on the split plane $\mathbb{C}\setminus[0,\infty)$ by $r^{-a}e^{-ia\theta}$ for $z=re^{i\theta}$, $0<\theta<2\pi$, and we regard the slit $[0,\infty)$ as having a top edge and a bottom edge, and we extend the function by continuity to each edge of the slit, so it is defined by $r^{-a}e^{-ia\theta}$ with $\theta=0$ on the top edge, and by the same formula with $\theta=2\pi$ on the bottom edge (i.e. values of the bottom edge are just the values of the top edge times $e^{-2\pi ia}$).

Thus the integral around the keyhole contour $\oint_\kappa f(z)\,dz$ is (very similar to Gamelin pg. 207)

$$\oint_\kappa f(z)\,dz = \int_\epsilon^R \frac{x^{-a}}{1+x}\,dx + \int_{\Gamma_R} f(z)\,dz + e^{-2\pi ia}\int_R^\epsilon \frac{x^{-a}}{1+x}\,dx + \int_{\gamma_\epsilon} f(z)\,dz.$$

The ML-estimate gives that the Γ_R integral is bounded in absolute value by $R^{-a}|e^{-ia\theta}|/(R-1)\cdot 2\pi R = 2\pi\frac{R^{1-a}}{R-1} \leq 2\pi\frac{R^{1-a}}{R/2} = 4\pi R^{-a}$ (the M bound uses the reverse triangle inequality to see that

$|1+z| \geq |1-|z|| = R-1$ for any $|z| = R$). Because $a > 0$, this goes to 0 as $R \to \infty$. Similarly, the ML-estimate gives that the γ_ϵ integral is bounded in absolute value by $\epsilon^{-a}|e^{-ia\theta}|/(1-\epsilon) \cdot 2\pi\epsilon = 2\pi\frac{\epsilon^{1-a}}{1-\epsilon}$. Because $a < 1$, the exponent of ϵ is positive, so as $\epsilon \to 0$, the numerator goes to 0 (and denominator goes to 1), giving us that integral is also 0 as $\epsilon \to 0$.

The residue theorem (pg. 196) on the other hand gives that $\oint_\kappa f(z)\,dz = 2\pi i \operatorname{Res}[f(z), -1]$ (since that's the only isolated singularity of f on the keyhole domain, given our branch cut of z^a), where the pole is simple because z^{-a} (the branch cut we chose) is analytic at $z = -1$, and so the first theorem on pg. 173 tells us that the pole at $z = -1$ is simple. Rule 1 (pg. 196) then gives that $\operatorname{Res}[f(z), -1] = (-1)^{-a} := 1^{-a}e^{-ia\pi}$. Thus, putting the two pieces together we have

$$(1 - e^{-2\pi ia})\int_0^\infty \frac{x^{-a}}{1+x}\,dx = \lim_{R\to\infty,\epsilon\to 0}\oint_\kappa f(z)\,dz = 2\pi i e^{-ia\pi}$$

so

$$\int_0^\infty \frac{x^{-a}}{1+x}\,dx = \frac{2\pi i e^{-ia\pi}}{1-e^{-2\pi ia}} = \frac{2\pi i}{e^{ia\pi} - e^{-ia\pi}} = \frac{\pi}{\sin(\pi a)}.$$

Problem 6(b) - VII.4.2

We are asked to integrate around a pie-slice π domain of aperture $2\pi/b$ in order to show that $\int_0^\infty \frac{1}{1+x^b}\,dx = \frac{\pi}{b\sin(\pi/b)}$ for any $b > 1$. We then check our result by changing variables and comparing with Problem 6(a) above (the preceding Exercise). We use the same notation as the keyhole contour (pg. 206-7), where $\Gamma_R, \gamma_\epsilon$ are just $2\pi/b$ radian arcs of circles, and where the "integral from R back to ϵ" is done along the line $z = te^{(2i\pi/b)} \implies dz = e^{(2i\pi/b)}\,dt$ (t going from R to ϵ). Like in Problem 6(a), let us choose a branch cut of x^b; just take $x^b := r^b e^{ib\theta}$ for $z = re^{i\theta}$ and $0 \leq \theta \leq 2\pi/b$ (recall $b > 1$ so this is possible, since we can define such a branch cut for the slit plane $\mathbb{C}\setminus[0,\infty)$, and such a pie-slice region lies inside the slit plane).

Thus the integral around the pie-slice contour $\oint_\rho f(z)\,dz$ is (very similar to Gamelin pg. 207)

$$\oint_\rho f(z)\,dz = \int_\epsilon^R \frac{1}{1+x^b}\,dx + \int_{\Gamma_R} f(z)\,dz + \int_R^\epsilon \frac{1}{1+t^b(e^{2i\pi/b})^b}\cdot e^{(2i\pi/b)}\,dt + \int_{\gamma_\epsilon} f(z)\,dz.$$

The ML-estimate gives that the Γ_R integral is bounded in absolute value by $\frac{1}{R^b-1}\cdot 2\pi R/b$, which for large enough R is $\leq \frac{1}{R^b/2}\cdot 2\pi R/b = \frac{4\pi}{b}R^{1-b}$ (the M bound uses the reverse triangle inequality to see that $|1+z^b| \geq |1-|z^b|| = R^b - 1$ for any $|z| = R$ for R large enough). Because $b > 1$, the exponent of R is negative, and so this goes to 0 as $R \to \infty$. Similarly, the ML-estimate gives that the γ_ϵ integral is bounded in absolute value by $\frac{1}{1-\epsilon^b}\cdot 2\pi\epsilon/b$. Because b is positive, $\epsilon^b \to$ as $\epsilon \to 0$, so as $\epsilon \to 0$, the numerator goes to 0 and denominator goes to 1, giving us that integral is also 0 as $\epsilon \to 0$.

The residue theorem (pg. 196) on the other hand gives that $\oint_\rho f(z)\,dz = 2\pi i \operatorname{Res}\big[f(z), e^{i\pi/b}\big]$ (since that's the only isolated singularity of f, since $(e^{i\pi/b})^b = -1$, in the pie-slice), where the pole is simple because defining $g(z) = f(z)(z - e^{i\pi/b})$, $\lim_{z\to e^{i\pi/b}} g(z) = \lim_{z\to e^{i\pi/b}} \frac{z-e^{i\pi/b}}{x^b-(-1)}$ which exists and equals

the reciprocal of the derivative of z^b at $e^{i\pi/b}$, i.e. $(\frac{1}{bz^{b-1}})|_{z=e^{i\pi/b}} = -e^{\pi i/b}/b \neq 0$ (i.e. g can be extended to be analytic and non-zero to the singularity), and so the first theorem on pg. 173 tells us that the pole at $z=-1$ is simple. Rule 1 (pg. 196) then gives that $\operatorname{Res}\left[f(z), e^{i\pi/b}\right] = \lim_{z\to e^{i\pi/b}} f(z)(z-e^{\pi i/b}) = \lim_{z\to e^{i\pi/b}} g(z) = -e^{\pi i/b}/b$. Thus, putting the two pieces together we have

$$(1-e^{(2i\pi/b)})\int_0^\infty \frac{1}{1+x^b}\,dx = \lim_{R\to\infty,\epsilon\to 0}\oint_\rho f(z)\,dz = -2\pi i e^{\pi i/b}/b$$

so

$$\int_0^\infty \frac{1}{1+x^b}\,dx = \frac{-2\pi i e^{\pi i/b}}{b\cdot(1-e^{2i\pi/b})} = \frac{-2\pi i}{b\cdot(e^{-\pi i/b}-e^{i\pi/b})} = \frac{\pi}{b\sin(\pi/b)}.$$

Check against Problem 6(a): defining $u = x^b \implies dx = \frac{1}{b}u^{-a}$ for $a := 1-\frac{1}{b}$ (indeed in $(0,1)$, assuming $b>1$), we get

$$\int_0^\infty \frac{1}{1+x^b}\,dx = \tfrac{1}{b}\int_0^\infty \frac{u^{-a}}{1+u}\,du = \frac{\pi}{b\sin(\pi a)} = \frac{\pi}{b\sin(\pi-\pi/b)} = \frac{\pi}{-b\sin(-\pi/b)} = \frac{\pi}{b\sin(\pi/b)}.$$

534 Homework 8

Daniel Rui - 5/27/20

Analytic continuation along a path: **Gamelin definition:** given a path $\gamma(t)$ for $t \in [a,b]$ and a power series $f_a(z)$ centered at $z_0 := \gamma(a)$ with radius of convergence $r(a)$, analytically continuing along γ means that for each $t \in [a,b]$, there is a convergent power series $f_t(z) = \sum_{n=0}^{\infty} a_n(t)(z-\gamma(t))^n$ for $z \in B(\gamma(t), r(t))$ where $r(t)$ is the radius of convergence for the series at t, where furthermore for s near t (explicitly say $s \in T(t)$, where $T(t)$ can be thought of as $B(t, \delta(t))$ for some small $\delta(t) > 0$) $f_s(z) = f_t(z)$ for all z in the intersection of the disks of convergence.

Class definition: given a path $\gamma(t)$ for $t \in [a,b]$ and a power series $f_a(z)$ centered at $z_0 := \gamma(a)$ with radius of convergence $r(a)$, analytically continuing along γ means that we have a collection $\{t_0, t_1, \ldots, t_n\} \subset [a,b]$ s.t. $t_0 = a, t_n = b$ and we have a power series $f_i(z)$ centered at each $\gamma(t_i)$ with radius of convergence $r(t_i)$ s.t. $\gamma(t_{i+1}) \in B(\gamma(t_i), r(t_i))$ and $\gamma(t_i) \in B(\gamma(t_{i+1}, r(t_{i+1})))$, and the union of all $n+1$ disks covers γ, and f_i agrees with f_{i-1} on the overlap of their disks of convergence.

Problem 1 - DO NOT GRADE, JUST FOR MY OWN REFERENCE

Elaboration on proof two theorems on pg. 160 and pg. 161. For first theorem (analytic continuation unique along a path): as Gamelin says in pg. 159, define t_0 to be the infimum of the parameter values t s.t. $f_t(z)$ is not uniquely determined by $f_a(z)$. Then, must be that $f_{t_0}(z)$ is uniquely determined by $f_a(z)$, because if we had $g_{t_0}(z)$ a power series centered at $\gamma(t_0)$ with radius of convergence $r_g(t_0)$, then we know for some sufficiently near $t_{-1} \in [a, t_0)$ (must be in $T(t_0)$ and $T_g(t_0)$ and s.t. $\gamma(t_{-1}) \in B(\gamma(t_0), \min\{r(t_0), r_g(t_0)\})$ which we can guarantee exists by continuity of γ), g_{t_0} and f_{t_0} agree with $f_{t_{-1}}$ on an open set (intersection of two open disks). Thus, the identity theorem gives that g_{t_0} and f_{t_0} must equal wherever they are defined. But by uniqueness of power series (see pg. 140 of Gamelin or Theorem 2.8 in Marshall), must be that $g_{t_0} = f_{t_0}$ with $r(t_0) = r_g(t_0)$.

Then, with this, for some $t_1 \in (t_0, b]$ (must be in $T(t_0)$ and s.t. $\gamma(t) \in B(\gamma(t_0), r(t_0))$ for all $t \in [t_0, t_1]$, which again we can guarantee by continuity of γ) suppose we have another power series $g_{t_1}(z)$. Problem: t_1 may not be "near enough" to t_0 in this new power series (recall that $T(t)$ is dependent on the specific continuum of power series we are considering)...IDK :(very sad. Ok, idea: if prove $\delta(t)$ continuous, get $\delta(t)$ bounded below (uniformity condition). But $\delta(t)$ is probably not continuous :(. Anyways, I asked Logan Gnanapragasam and he did a proof using "sup" direction, not "inf" (very similar to Axler proof of Heine-Borel).

For second theorem, again do "inchworm" argument, but on the infimum s_0 of $s \in [0,1]$ s.t. the analytic continuations along $\gamma_s(z)$ and $\gamma_0(z)$ do NOT coincide. Again contradiction via identity theorem.

OR, just see Marshall (assuming class definition) pg. 217-218. Class definition also explored https://link.springer.com/chapter/10.1007/978-1-4757-3083-8_11.

Infuriating that "analytic continuation" can result in a non-well-defined function. If we define analytic continuation to be $\{f_t, U_t\}_{t\in[a,b]}$ s.t. f_t defined on disk U_t with center $\gamma(t)$, small enough s.t. $f_t(z) = f_s(z)$ for $z \in U_s \cap U_t$ for ALL $s, t \in [a,b]$. Then can stitch this up into f analytic on $\bigcup_{t\in[a,b]} U_t$. But of course this does not allow us to consider arbitrarily long winding around the unit circle, analytically continuing up the branches of say log endlessly.

Problem 2 - V.8.5

Let $f(z) := \sum_{n=0}^{\infty} z^{2^n} = z + z^2 + z^4 + z^8 + \ldots$, and so on. We want to show that f is analytic on $[|z| < 1]$ and cannot be extended analytically to any larger open set.

Method 1: the theorem on pg. 138 says that any power series has a number R s.t. it is convergent absolutely on $[|z| < R]$ and not convergent on $[|z| > R]$. Moreover theorem on pg. 140 says that f is analytic on $[|z| < R]$, and the corollary on pg. 146 says that R is the largest number s.t. f extends to be analytic on $[|z| < R]$. Well, we have that $|f(z)| \leq \sum_{n=0}^{\infty} |z^{2^n}| \leq \sum_{n=0}^{\infty} |z^n| = \sum_{n=0}^{\infty} |z|^n$, so for any $|z| < 1$, the series converges absolutely.

Thus, by the results I cited above, R must be ≥ 1. Can't be any greater, because if f could be extended analytically past the open unit disk, then because we know $f(z) = z + f(z^2)$ (this is hint from Gamelin) for $|z| < 1$, the formula holds as we take $r \to 1$ (we are assuming that f can be extended past the open unit disk, hence we can take limit $z \to 1$), yielding $f(1) = 1 + f(1)$, which is impossible.

Method 2: the Cauchy-Hadamard formula (pg. 142) gives that $R = 1/\limsup_{k\to\infty} |a_k|^{1/k}$, and in our case, all the a_k are 1 for $k = 2^n$ and 0 otherwise (so raising to $1/k$ power useless), meaning $\limsup_{k\to\infty} |a_k|^{1/k} = \limsup_{k\to\infty} |a_k| = 1$, and so $R = 1$ as desired.

edit 6/4/21 — **EPIC FAIL:** I misread the problem as "any larger open disk". Anyways, here is solution https://math.stackexchange.com/questions/1749560/show-that-fz-sum-n-0-inftyz2n-cant-be-analytically-continued-past. **Lemma:** $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is power series with radius of convergence 1, $a_k \geq 0$, then $\lim_{r\nearrow 1} f(r) = \sum_{k=0}^{\infty} a_k$. *Proof:* using that $a_k \geq 0$, we have $f_n(r) = \sum_{k=0}^{n} a_k r^k \leq f(r) = \sum_{k=0}^{\infty} a_k r^k \leq \sum_{k=0}^{\infty} a_k$. By direct comparison test, $f(r)$ is weakly increasing on $(0,1)$, so $\lim_{r\nearrow 1} f(r) = \sup_{r\in(0,1)} f(r)$ exists (could be infinity). Taking limits of both sides of the inequality above we get $\sum_{k=0}^{n} a_k = \lim_{r\nearrow 1} f_n(r) \leq \lim_{r\nearrow 1} f(r) \leq \sum_{k=0}^{\infty} a_k$. Taking $n \to \infty$ yields $\sum_{k=0}^{\infty} a_k \leq \lim_{r\nearrow 1} f(r) \leq \sum_{k=0}^{\infty} a_k$, hence $\lim_{r\nearrow 1} f(r) = \sum_{k=0}^{\infty} a_k$.

Then Gamelin's hint $f(z) = z + f(z^2)$ yields by induction $f(z^{2^n}) = f(z) - \sum_{j=0}^{n-1} z^{2^j}$ for any $n \in \mathbb{N}$. Applying this to $rz_{k,n} := re^{2\pi i \cdot k/2^n}$, $r \in (0,1)$, we have $f(r^{2^n}) = f((rz_{k,n})^{2^n}) = f(rz_{k,n}) - \sum_{j=0}^{n-1} (rz_{k,n})^{2^j}$. But $|f(r^{2^n})| \geq |f(rz_{k,n})| - |\sum_{j=0}^{n-1} (rz_{k,n})^{2^j}| \geq |f(rz_{k,n})| - \sum_{j=0}^{n-1} |r^{2^j}| \geq |f(rz_{k,n})| - n$. Thus, because Lemma above gives $\lim_{r\nearrow 1} f(r^{2^n}) = \lim_{r\nearrow 1} f(r) = \infty$, must be that $\lim_{r\nearrow 1} |f(rz_{k,n})| = \infty$. Any open set strictly larger than open unit disk contains $z_{k,n}$ for some $k, n \in \mathbb{N}$, and we just showed that f, if extended there, would go to ∞, hence NO analytic extension to open set strictly larger than unit disk.

Problem 3(i) - V.8.8

Let $f(z)$ be analytic at z_0 and let $\gamma(t)$, $t \in [a,b]$ be path s.t. $\gamma(a) = z_0$. If f can not be analytically continued along γ, we want to show that there is parameter value t_1 s.t. there is an analytic continuation $f_t(z)$ for all $t \in [a,t_1)$ with radii of convergence $r(t) \to 0$ as $t \nearrow t_1$.

Well, not having an analytically continuation means that there is $t \in [a,b]$ s.t. there is no convergent power series with the properties specified on pg. 159. Let t_1 be the supremum of $t \in [a,b]$ s.t. for all $s \in [a,t]$ there is $f_s(z)$ satisfying the properties of analytic continuation specified on pg. 159 (set of such t non-empty because a must be in it). Recall from the lemma on pg. 159 that $r(t)$ (which exists for each $f_t(z)$ for $t \in [a,t_1)$) is continuous. Then, $r(t)$ not converging to 0 as $t \nearrow t_1$ means that (also considering the fact that $r(t) \geq 0$) there is some $\epsilon > 0$ s.t. $r(t) \geq \epsilon$ for $t < t_1$ sufficiently close to t_1. By continuity of γ, we can choose t_0, t_2 ($t_0 \leq t_1 < t_2$) sufficiently close to t_1 s.t. $\gamma(t) \in B(\gamma(t_0), \epsilon) \subseteq B(\gamma(t_0), r(t_0))$ for every $t \in [t_0, t_2]$. Then, by the theorem on pg. 144, we can express $f_{t_0}(z)$ as a power series (with positive radius of convergence) locally at $\gamma(t)$ (for all $t \in (t_0, t_2]$) by $g_t(z) = \sum_{n=0}^{\infty} \frac{1}{n!} f_{t_0}^{(n)}(\gamma(t)) \cdot (z - \gamma(t))^n$.

By this definition, f_{t_0} and g_t agree locally at $\gamma(t)$, and hence by the identity theorem must be equal to each other at every point their disks of convergence overlap (and again this is for all $t \in (t_0, t_2]$). By uniqueness of analytic continuation (Problem 1, see Logan's pdf), it must be that $f_t = g_t$ for all $t \in (t_0, t_1)$. Then, for all $t \in (t_0, t_2)$, all nearby s (explicitly, all $s \in (t_0, t_2)$ s.t. $\gamma(s) \in B(\gamma(t), r(t))$) agree with f_t on the overlap of their disks of convergence (because f_s agrees with f_{t_0} locally at $\gamma(s) \in B(\gamma(t), r(t)$, and f_t agrees with f_{t_0} everywhere on $B(\gamma(t), r(t))$, i.e. f_t agrees with f_s locally at $\gamma(s)$, meaning by the identity theorem that f_t, f_s agree everywhere their domains overlap). Thus, we have constructed an analytic continuation to $t_2 > t_1$ (assuming that $t_1 < b$), contradicting that t_1 was the sup. We have therefore shown that t_1 must be b, and we are done.

Problem 3(ii) - V.8.9

Let $P(z,w)$ be a polynomial in z and w (degree n in w). Suppose that $f(z)$ is analytic at z_0 and satisfies $P(z, f(z)) = 0$. We want to show that if $f_t(z)$ is any analytic continuation of f along any path starting at z_0, then $P(z, f_t(z)) = 0$ for all t. *Remark:* any analytic function f that satisfies a polynomial equation $P(z, f(z)) = 0$ is called an algebraic function; e.g. the branches of $\sqrt[n]{z}$ are algebraic functions, since they satisfy $z - w^n = 0$.

First let us recall **Gamelin's definition** of analytic continuation on a path: given a path $\gamma(t)$ for $t \in [a,b]$ and a power series $f_a(z)$ centered at $z_0 := \gamma(a)$ with radius of convergence $r(a)$, analytically continuing along γ means that for each $t \in [a,b]$, there is a convergent power series $f_t(z) = \sum_{n=0}^{\infty} a_n(t)(z - \gamma(t))^n$ for $z \in B(\gamma(t), r(t))$ where $r(t)$ is the radius of convergence for the series at t, where furthermore for s near t (explicitly say $s \in T(t)$, where $T(t)$ can be thought of as $B(t, \delta(t))$ for some small $\delta(t) > 0$) $f_s(z) = f_t(z)$ for all z in the intersection of the disks of convergence.

Again as with Problem 3(i) above, let us take t_1 to be the supremum of all $t \in [a, b]$ s.t. for all $s \in [a, s]$, $P(z, f_s(z)) = 0$ for all z in the domain of f_s (set of such t is non-empty since a must be in this set). Then, it must be that $P(z, f_{t_1}(z)) = 0$ everywhere on the domain of f_{t_1}, because by continuity of γ we know that there is $t_0 < t_1$ close enough (i.e. $t_0 \in T(t_1)$ and $\gamma(t_0) \in B(\gamma(t_1), r(t_1))$) so that $f_{t_0} = f_{t_1}$ where their domains overlap, and because $P(z, f_{t_0}(z)) = 0$ on the domain of f_{t_0}, so in particular locally at $\gamma(t_0)$ (which is a point in the intersection of their domains), we see that $P(z, f_{t_1}(z)) = 0$ locally at $\gamma(t_0)$, and so by the identity theorem $P(z, f_{t_1}(z)) = 0$ everywhere on the domain of f_{t_1}.

On the other side, we have $t_2 > t_1$ s.t. $t_2 \in T(t_1)$ and $\gamma(t) \in B(\gamma(t_1), r(t_1))$ for all $t \in [t_1, t_2]$ (again this last assertion is guaranteed by continuity of γ) so that $f_t = f_{t_1}$ where their domains overlap (in particular locally at $\gamma(t)$), and because $P(z, f_{t_1}(z)) = 0$ on the domain of f_{t_1} (in particular locally at $\gamma(t)$), we see that $P(z, f_t(z)) = 0$ locally at $\gamma(t)$, and by the identity theorem (because $P(z, f_t(z))$ is an analytic function on the domain of f_t, as it is just powers of z times powers of an analytic function f_t) we have that $P(z, f_t(z)) = 0$ everywhere on the domain of f_t. Thus, we have found that if $t_1 < b$, it can not possibly be the supremum; contradiction. We have therefore shown that t_1 must be b. ■

Problem 4(a) - VI.1.1(a),1(c)

We find all possible Laurent expansions centered at 0 of the following functions:

(a) $\frac{1}{z^2-z} = \frac{1}{z(z-1)} = \frac{1}{z-1} - \frac{1}{z}$. Because Laurent series converge in annuli, the possible regions are just $[0 < |z| < 1]$ and $[|z| > 1]$. Note that standard power series are convergent in an (open) disk, and negative power series are convergent on the complement of a closed disk. We know from pg. 168 that Laurent expansion on any particular annulus is unique (coefficients determined by analogue of Cauchy integral formula, formula (1.4) on pg. 168).

- On $[0 < |z| < 1]$: we need to choose the negative power series for $-\frac{1}{z}$ (which is literally just itself) and the standard power series for $\frac{1}{z-1}$; the geometric series formula gives $\frac{1}{z-1} = -\sum_{n=0}^{\infty} z^n$, and so their sum is $\sum_{n=-1}^{\infty}(-1)z^n$.
- On $[|z| > 1]$: we need to choose the negative power series for both; the geometric series formula gives $\frac{1}{z-1} = \frac{1}{z} \cdot \frac{1}{1-(1/z)} = \frac{1}{z} \cdot \sum_{n=0}^{\infty} z^{-n} = \sum_{n=1}^{\infty} z^{-n}$, and so their sum $\sum_{n=1}^{\infty} z^{-n} - z^{-1}$ is $\sum_{n=2}^{\infty} z^{-n}$.

(c) $\frac{1}{(z^2-1)(z^2-4)} = \frac{1}{3}(\frac{1}{z^2-4} - \frac{1}{z^2-1})$. The only possible regions are $[|z| < 1]$, $[1 < |z| < 2]$, $[|z| > 2]$. See https://www.desmos.com/calculator/5ovjzpuhlr.

- On $[|z| < 1]$: we need to choose the standard power series for both; the geometric series formula gives $\frac{1}{z^2-4} = -\frac{1}{4}(\frac{1}{1-(z/2)^2}) = -\frac{1}{4}\sum_{n=0}^{\infty}(\frac{z}{2})^{2n}$ and $-\frac{1}{z^2-1} = \frac{1}{1-z^2} = \sum_{n=0}^{\infty} z^{2n}$. Their sum is hence
$$\frac{1}{3}\left(\sum_{n=0}^{\infty}(1 - \tfrac{1}{4^{n+1}})z^{2n}\right).$$
- On $[1 < |z| < 2]$: we need to choose the negative power series for $-\frac{1}{1-z^2}$ and the standard power series for $\frac{1}{z^2-4}$, which recall from above is $-\frac{1}{4}\sum_{n=0}^{\infty}(\frac{z}{2})^{2n}$; the geometric series formula

gives $-\frac{1}{z^2-1} = -\frac{1}{z^2}\frac{1}{1-(1/z)^2} = -\frac{1}{z^2}\cdot\sum_{n=0}^{\infty}(\frac{1}{z})^{2n} = \sum_{n=1}^{\infty}(-1)(\frac{1}{z})^{2n}$, and so their sum is

$$\frac{1}{3}\left(\sum_{n=-1}^{-\infty}(-1)z^{2n} + \sum_{n=0}^{\infty}(-\tfrac{1}{4^{n+1}})z^{2n}\right).$$

- On $[|z| > 2]$: we need to choose the negative power series for both; recall from above that $-\frac{1}{z^2-1} = \sum_{n=1}^{\infty}(-1)(\frac{1}{z})^{2n}$, and similarly $\frac{1}{z^2-4} = \frac{1}{z^2}\cdot\frac{1}{1-(2/z)^2} = \frac{1}{z^2} = \sum_{n=0}^{\infty}(\frac{2}{z})^{2n} = \sum_{n=1}^{\infty}(4^{n-1})z^{-2n}$. Their sum is hence

$$\frac{1}{3}\left(\sum_{n=-1}^{-\infty}(-1+4^{-n-1})z^{2n}\right).$$

Problem 4(b) - VI.1.5

edit 6/4/21 — **EPIC FAIL:** I did this in 336 even and just forgot to bring it over!!!! AHHHHHHHH!!!!

Suppose that f is analytic on the puncture plan $D = \mathbb{C}\setminus\{0\}$. Show that there's a constant c such that $f(z) - c/z$ has a primitive on D (and give a formula for c in terms of an integral of $f(z)$).

Applying the Laurent Series Expansion theorem with $\sigma = \infty$ and $\rho = 0$ and $z_0 = 0$, we get that

$$f(z) = \sum_{k=-\infty}^{\infty} a_k z^k \text{ where } a_n = \frac{1}{2\pi i}\oint_{|z|=r}\frac{f(z)}{z^{n+1}}\,dz$$

for all $z \in D$ and $r \in (0,\infty)$ (where the convergence is uniform on every closed subannulus in D). Recall from 336 Homework 5, §IV.2.5, that we proved that $f(z)$ has a primitive in D if and only if $\int_\gamma f(z)\,dz = 0$ for every closed path γ in D. Now consider $g(z) = f(z) - \frac{a_{-1}}{z} = \sum_{k\neq -1} a_k z^k$: for any γ, it lies in sum subannulus of D (and hence we have the sum is uniformly convergent), so we can exchange the sum and integral via the Theorem on page 135:

$$\int_\gamma g(z)\,dz = \sum_{k\neq -1} a_k \int_\gamma z^k\,dz = 0$$

where the integral around any γ of z^k for $k \neq -1$ is 0 because any z^k for $k \neq 1$ has a primitive (the same as the real antiderivative). Thus, $g(z)$ has a primitive on D, where

$$c = a_{-1} = \frac{1}{2\pi i}\oint_{|z|=r} f(z)\,dz$$

0.0.1 336 Homework 5 - IV.2.5

We want to show that $f(z)$ has a primitive in D if and only if $\int_\gamma f(z)\,dz = 0$ for every closed path γ in D.

($\implies$) From the theorem at the beginning of the section on page 107, we have that $\int_A^B f(z)\,dz = F(B) - F(A)$ where F is the primitive of f (which we are assuming to exist), and where the integral can be taken over any path from A to B in D. Because $A = B$ in a closed path, we see that this theorem gives that the integral over any closed path is 0.

($\impliedby$) on page 82, it says that independent of path $\iff$ exact. But we know that independent of path $\iff$ integral over any closed path is 0, and so because we have that the integral over any closed $\gamma \in D$ is 0, it must be that $f(z)\,dx + if(z)\,dy$ is exact; that is, there is F s.t. $dF = \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy = f(z)\,dz + if(z)\,dy$, so we have

$$\frac{\partial F}{\partial x} = f(z) \quad \text{and} \quad \frac{\partial F}{\partial y} = if(z)$$

and so if $F = u + iv$, we have

$$\partial_x u + i\partial_y v = f(z) = \operatorname{Re} f + i \operatorname{Im} f \quad \text{and} \quad \partial_y u + i\partial_y v = if(z) = i \operatorname{Re} f - \operatorname{Im} f$$

and comparing real and imaginary parts, we see that $\partial_x u = \partial_y v$ and $\partial_x v = -\partial_y u$ and so the C-R equations are satisfied, hence F is analytic on D, and is a primitive of $f(z)$ (it satisfies the formula on the middle of page 107).

Problem 5 - VI.2.1(a)-(g)

We want to find the isolated singularities of the following functions, and determine whether or not they are removable, essential, or poles. If they are poles, we determine their order, and moreover find the principal part at each pole (recall that the principal part for a pole is the finite sum of negative powers of the Laurent series centered at the pole).

(a) $f(z) = z/(z^2-1)^2$. There's a double pole at 1 and -1, since $f(z) = g_1(z)/(z-1)^2 = g_2(z)/(z+1)^2$ where $g_1(z) = z/(z+1)^2$ and $g_2(z) = z/(z-1)^2$ are analytic and non-zero resp. at 1 and -1 (and so the claim about being a double pole follows from the first theorem on pg. 173). To find the principal parts, we just find the first two coefficients in the Taylor series of g_1, g_2 (resp. centered at 1 and -1), which are just the 0th and 1st derivatives of g_1, g_2 at $1, -1$ respectively. Using a tool like Wolfram Alpha to do the busy-work calculations for us, we see that the principal part at $z = 1$ is $\frac{1}{4} \cdot \frac{1}{(z-1)^2} + 0 \cdot \frac{1}{(z-1)}$ and at $z = -1$ it is $-\frac{1}{4} \cdot \frac{1}{(z+1)^2} + 0 \cdot \frac{1}{(z+1)}$.

(b) $f(z) = ze^z/(z^2-1)$. There's a single pole at 1 and -1, since $f(z) = g_1(z)/(z-1) = g_2(z)/(z+1)$ where $g_1(z) = ze^z/(z+1)$ and $g_2(z) = ze^z/(z-1)$ are analytic and non-zero resp. at 1 and -1 (and so the claim about being a single pole follows from the first theorem on pg. 173). To find the principal parts, we just find the first coefficient in the Taylor series of g_1, g_2 (resp. centered at 1 and -1), which are just the 0th derivative of g_1, g_2 at $1, -1$ resp. (i.e. the value of g_1, g_2 at $1, -1$ resp.). We see that the principal part at $z = 1$ is $\frac{1}{2}e \cdot \frac{1}{(z-1)}$ and at $z = -1$ it is $\frac{1}{2e} \cdot \frac{1}{(z+1)}$.

(c) $f(z) = (e^{2z} - 1)/z$. There is a singularity at 0, but it is removable since f is bounded near $z = 0$ (see the Riemann removable singularity theorem on pg. 172), because $\frac{e^{2z}-1}{z} = 2 \cdot \frac{e^{2z}-e^{2\cdot 0}}{2z-0}$ and so $\lim_{z\to 1} \frac{e^{2z}-1}{z} = 2 \cdot \lim_{z\to 1} \frac{e^{2z}-e^{2\cdot 0}}{2z-0} = 2$ times the derivative of e^{2z} at 0, which is $2 \cdot 2 = 4$ (so by

the definition of limit, near $z = 0$, $|f(z)|$ is bounded by say 5).

(d) $f(z) = \tan z = \frac{\sin z}{\cos z}$. Using that $\cos z = \frac{e^{iz}+e^{-iz}}{2}$, $\cos z = 0 \implies e^{2iz} = 1$. So if $z = x + iy$ for $x, y \in \mathbb{R}$, then $|e^{2iz}| = |e^{-2y}| = 1 \implies y = 0$. Thus, the zeroes of $\cos z$ are exactly $\frac{\pi}{2} + \pi k$ for $k \in \mathbb{Z}$, i.e. the isolated singularities of $f(z)$ are exactly at $\frac{\pi}{2} + \pi k$ for $k \in \mathbb{Z}$. Fixing such a $k \in \mathbb{Z}$, the Taylor series at $z_k := \frac{\pi}{2} + \pi k$ of $\cos z$ is exactly $\sum_{n=0}^{\infty} \frac{1}{n!} \cos^{(n)}(z_k)(z - z_k)^n = (-1)^k \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}(-1)^{n+1}(z - z_k)^{2n+1} = (-1)^{k+1}(z - z_k) \cdot (1 - (z - z_k)^2/3! + (z - z_k)^4/5! - \ldots)$. Similarly we have that the Taylor series of $\sin z$ centered at z_k is $(-1)^k \sum_{n=0}^{\infty} \frac{1}{(2n)!}(-1)^n (z - z_k)^{2n}$.

Following the procedure on pg. 153 (about dividing power series using the geometric series formula $\frac{1}{1-(\cdots)} = 1 + (\cdots) + (\cdots)^2 + \ldots$, where the $(\cdots)$ denote an infinite sum), we get that $\frac{\sin z}{\cos z} = (-1)^{k+1} \cdot (-1)^k \frac{1}{(z-z_k)}$ times some (standard, i.e. non-negative exponent) power series starting with the constant term 1. Thus, we get that the Laurent series of $f(z)$ centered at z_k has just one negative exponent term, namely $-\frac{1}{z-z_k}$, hence this is the principal part of the single pole at z_k (z_k is arbitrary, so indeed all singularities are single poles with the principal part of this form).

(e) $f(z) = z^2 \sin(\frac{1}{z})$. The only singularity of $\sin(\frac{1}{z})$ is at $z = 0$, but as the Laurent expansion on the annulus $[|z| > 0]$ (which recall is unique) can (uniquely) be written as $\sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!}(\frac{1}{z})^{2n+1}$, even multiplying by z^2 will leave infinitely many negative exponent terms, so indeed this singularity is an essential singularity.

(f) $f(z) = \cos z/(z^2 - (\pi/2)^2)$. The only singularities are at $\pm\pi/2$. We can use the derivative technique from part (c) again to show that f is bounded near the singularities, and hence by the Riemann removable singularity theorem (pg. 172).

I'll spell out the one at $\pi/2$: $\lim_{z\to\pi/2} \frac{\cos z}{(z^2-(\pi/2)^2)} = \lim_{z\to\pi/2} 1/(z+\pi/2) \cdot \lim_{z\to\pi/2} \frac{\cos z - \cos(\pi/2)}{z-\pi/2} = \frac{1}{\pi} \cdot (-\sin(\pi/2)) = -\frac{1}{\pi}$, so by the definition of limit $|f|$ indeed bounded near $\pi/2$. The one at $-\pi/2$ is much the same.

(g) $f(z) = \mathrm{Log}(1 - \frac{1}{z})$: recall that $\mathrm{Log}(z)$ is analytic on $\mathbb{C} \setminus (-\infty, 0]$ (and not analytic on any set strictly containing this split plane). Well, $1 - \frac{1}{z} \in (-\infty, 0] \iff \frac{1}{z} \in [1, \infty) \iff z \in (0, 1]$ and so we find that $f(z)$ is analytic on $\mathbb{C} \setminus [0, 1]$ (and not analytic on any set strictly containing $\mathbb{C} \setminus [0, 1]$, because if it were, we would have $\mathrm{Log}(z)$ analytic outside the split plane, which it is NOT). Thus, all the singularities are not isolated (see pg. 171; f is not analytic on any punctured disk centered at any $z \in [0, 1]$).

Problem 6 - VI.2.12

Suppose that z_0 is an isolated singularity of $f(z)$ that is not removable. We want to show that it is an isolated singularity of e^f at z_0.

We split into two cases. **Case 1:** the isolated singularity of f at z_0 is a pole (say of order N): then

using the theorem on pg. 173, $f(z) = g(z)/(z - z_0)^N$, where $g(z)$ is analytic at z_0 and $g(z_0) \neq 0$. Denote $g(z_0) = r_0 e^{i\theta_0}$. We prove that e^f has an essential singularity at z_0 by find two sequences that converge to z_0, where the first will show that the singularity is not removable, and the second will show that it is not a pole:

- Define the sequence $z_n = z_0 + \frac{1}{n} e^{i\theta_0/n}$, so that

  $$f(z_n) = g(z_n) n^n e^{-i\theta_0}$$

  and so as $n \to \infty$, $e^{f(z_n)} = e^{g(z_n)e^{-i\theta_0} n^n}$ goes to infinity because $g(z_n) \to r_0 e^{i\theta_0}$ where $r_0 > 0$. This shows that e^f is not bounded near z_0, and hence by Riemann's removable singularity theorem, the singularity of e^f at z_0 can't be removable.

- Define $z_n = z_0 + \frac{1}{n} e^{i(\theta_0+\pi)/n}$ so that

  $$f(z_n) = g(z_n) n^n e^{-i(\theta_0+\pi)}$$

  and so as $n \to \infty$, $e^{f(z_n)} = e^{g(z_n)e^{-i\theta_0} e^{-i\pi} n^n} = e^{-g(z_n)e^{-i\theta_0} n^n} \to 0$ (because $g(z_n) \to r_0 e^{i\theta_0}$ where $r_0 > 0$). With these two sequences, we've shown that $\lim_{z \to z_0} |e^f| = \infty$ is false, and so the singularity of e^f at z_0 cannot be a pole, implying of course that it's an essential singularity.

Case 2: the isolated singularity of f at z_0 is essential: well, by the Casorati-Weierstrass theorem, for every $w_0 \in \mathbb{C}$, there is a sequence $z_n \to z_0$ such that $f(z_n) \to w_0$. Thus we can find two sequences, one for $w_0 = 0$ and the other for $w_0 \in \mathbb{R}$ as big as we desire, meaning (similar to above) that the singularity can't be removable (because there's a sequence $\{z_n\}$ s.t. $f(z_n)$ converges to arbitrarily large w_0, and hence e^f is not bounded near z_0), and it can't be a pole (because there's a sequence s.t. $f(z_n)$ converges to 0), which means that e^f has an essential singularity at z_0.

534 Homework 7 (Midterm)

Daniel Rui - 5/19/20

I use the notation $B(z,r) = D_r(z)$ interchangeably to mean the open disk/ball centered at z with radius r. Also, $\mathbb{N}$ denotes $\{0,1,\ldots\}$ and $\mathbb{N}^+$ denotes $\{1,2,\ldots\}$.

Problem 1a - V.4.2

We want to show that the radius of convergence R of the power series expansion of $f(z) := \frac{z^2-1}{z^3-1}$ about $z = 2$ is $\sqrt{7}$. We follow the lead of the example spanning pg. 146-147. Note that $f(z)$ equals $g(z) := \frac{z+1}{z^2+z+1}$ for all $z \in \mathbb{C} \setminus \{1, \zeta, \overline{\zeta}\}$ where $1, \zeta, \overline{\zeta}$ are the 3 cubic roots of unity, i.e. $\zeta := \cos(120°) + i\sin(120°) = \frac{1}{2}(-1 + i\sqrt{3})$, and moreover $g(z)$ is analytic everywhere in $\mathbb{C} \setminus \{\zeta, \overline{\zeta}\}$. The corollary on pg. 146 says that the radius of convergence of the power series of f analytic at z_0 is the largest $R > 0$ s.t. f extends to be analytic on the disk $\{|z - z_0| < R\}$. Note that $B(2, \sqrt{7}) \subseteq \mathbb{C} \setminus \{\zeta, \overline{\zeta}\}$ (because $|2 - \overline{\zeta}| = |2 - \zeta| = \frac{1}{2}|5 + i\sqrt{3}| = \frac{1}{2}\sqrt{25+3} = \sqrt{7}$), and so $R \geq \sqrt{7}$ because f can be extended to g which is analytic on $\{|z - 2| < \sqrt{7}\}$. Finally, R must be $\leq \sqrt{7}$, because $|f(z)| \to \infty$ as $z \to \zeta$ (or $\overline{\zeta}$), and so f does not extend to be analytic on any $B(2, R)$ for $R > \sqrt{7}$ because $\zeta, \overline{\zeta} \in B(2, R)$ for $R > \sqrt{7}$.

Problem 1b - V.4.12

Suppose that the analytic function $f(z)$ has power series expansion $\sum_{n=0}^{\infty} a_n z^n$. We want to show that if $f(z)$ is an even function, then $a_n = 0$ for all n odd, and that if $f(z)$ is an odd function, then $a_n = 0$ for all n even. In the following proof I will use the fact that if $f_1(z) = \sum_{n=0}^{\infty} a_n z^n$ and $f_2(z) = \sum_{n=0}^{\infty} b_n z^n$ near $z = 0$ (i.e. f_1, f_2 are analytic at 0), then $g(z) := f_1(z) + f_2(z)$ is analytic at 0, i.e. it equals $\sum_{n=0}^{\infty}(a_n + b_n)z^n$ near 0, say on a disk $D_r(0)$ for some $r > 0$ (see pg. 151 of Gamelin).

Even means that $f(z) = f(-z)$, so $g(z) = 0 = f(z) - f(-z) = \sum_{n=0}^{\infty}(a_n - (-1)^n a_n)z^n = \sum_{n \text{ odd}} 2a_n z^n$ near 0, so say on a disk $D_r(0)$ for some $r > 0$. By the theorem on pg. 140 (derivatives of series are obtained by term-by-term differentiation), the coefficients of the power series must be exactly equal to the kth derivative at 0 over $k!$, and because the derivatives of $g \equiv 0$ on $D_r(0)$ at $z = 0$ are all 0, we have that $2a_n$ for n odd must all be 0, hence $a_n = 0$ for all n odd as desired.

Similarly, if f is odd, then $f(z) = -f(-z)$, so $g(z) = 0 = f(z) + f(-z) = \sum_{n=0}^{\infty}(a_n + (-1)^n a_n)z^n = \sum_{n \text{ even}} 2a_n z^n$ near 0, so say on a disk $D_r(0)$ for some $r > 0$. Again by the theorem on pg. 140 (derivatives of series are obtained by term-by-term differentiation), the coefficients of the power series must be exactly equal to the kth derivative at 0 over $k!$, and because the derivatives of $g \equiv 0$ on $D_r(0)$ at $z = 0$ are all 0, we have that $2a_n$ for n even must all be 0, hence $a_n = 0$ for all n even as desired. (Alternatively, we could multiply odd $f(z)$ by z to get even $zf(z)$, and use the claim for even analytic functions to get that the even coefficients of f, i.e. exactly the odd coefficients of $zf(z)$, must be 0)

Problem 2 - V.4.14

Let f be a continuous function on the unit circle $T = \{|z| = 1\}$ Show that f can be approximated uniformly on T by a sequence of polynomials in z if and only if f has an extension F that is continuous on the closed disk $\{|z| \leq 1\}$ and analytic on the interior $\{|z| < 1\}$. Hint. To approximate such an F, consider dilates $F_r(z) = F(rz)$

($\implies$) We have some polynomials $\{p_k\}$ that converge uniformly to f on T. Thus, these polynomials are uniformly Cauchy: for m, n large enough, $|p_m - p_n| < \epsilon$ on T. But polynomials are analytic on $\mathbb{C}$. By the maximum principle on page 88, we have that $|p_m - p_n| \leq \epsilon$ on the boundary of the unit disk, and so $|p_m - p_n| \leq \epsilon$ on the unit disk. Uniformly Cauchy is equivalent to being uniformly convergent to something, so the p_k converge to some function F on the closed unit disk. Uniform convergence of continuous functions on a compact interval results in a continuous function, so F is continuous. Finally, by the theorem in the middle of page 136, analytic functions (such as the $\{p_k\}$) on a domain converge uniformly to an analytic function on the domain; and so F is analytic on the open unit disk. ($\impliedby$) we suppose that there is F continuous on the closed unit disk and analytic on the open unit disk, that is extended to the continuous f on T. Analyticity gives that

$$F(z) = \sum_{n=0}^{\infty} a_n z^n \quad |z| < 1$$

Define (for $r < 1$) $F_r(z) = F(rz) = \sum_{n=0}^{\infty} a_n r^n z^n$ (which is analytic on the open disk of radius $\frac{1}{r}$. All the F_r (as in the infinite sums) converge uniformly on the closed unit disk (via the Weierstrass M-test), and F is uniformly continuous on the closed unit disk (which is compact), and so we get that for r close enough to 1, $|F(z) - F(rz)|$ is uniformly bounded by an arbitrarily small number, and so we get that for large enough N,

$$\left|F(z) - \sum_{n=0}^{N} a_n r^n z_n\right| \leq \left|F_r(z) - \sum_{n=0}^{N} a_n r^n z_n\right| + |F(z) - F_r(z)| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

so we have found a sequence of polynomials uniformly approximating f on T.

Problem 3

Assume that $\{f_k\}_{k=1}^{\infty}$ is a bounded sequence of analytic functions in the unit disk D, i.e. $f_k(z) = \sum_{n=0}^{\infty} a_{n,k} z^n$ and $|f_k(z)| \leq M$ for some $M \in \mathbb{N}$, $\{a_{n,k}\}_{n,k \in \mathbb{N}} \subseteq \mathbb{C}$, all $z \in D$, and all $k \in \mathbb{N}$. We want to show that f_k converges to an analytic function $f(z)$ locally uniformly in D if and only if $a_{n,k}$ converges for every fixed n as $k \to \infty$.

($\implies$) essentially a weaker version of the convergence of the derivatives result from Problem 5 of Homework 6 (which is also Theorem 2 of Lec 5/12 I think) — the theorem on pg. 140 gives that if $f_k(z) = \sum_{n=0}^{\infty} a_{n,k} z^n$ for all $|z| < 1$ then $a_{n,k} = f^{(n)}(0)/n!$ for all $n \in \mathbb{N}$, so indeed the claim about "$a_{n,k}$ converges for every fixed n as $k \to \infty$" is proven by the claim from Problem 5 of Homework

6 (henceforth referred to as CLAIM), which is "if $\{f_k(z)\}$ is a sequence of analytic functions on D that converges uniformly to $f(z)$ on D, then for each m the derivatives $f_k^{(m)}(z)$ converge uniformly to $f^{(m)}(z)$ on each subset of D at a positive distance from ∂D" (explicitly, for any $z \in D$, it is contained in $\overline{D_r(0)} \subseteq D_R(0)$ for some $r, R \in (0,1)$, and by local uniform convergence, $f_k \to f$ uniformly on $\overline{D_R(0)}$ and hence on $D_R(0)$, and because $\overline{D_r(0)}$ is at positive distance from $\partial D_R(0)$, CLAIM gives that in particular that for fixed $n \in \mathbb{N}$, $a_{n,k} = f_k^{(n)}(z)$ converge to some value $a_n := f^{(n)}(z)$ as $k \to \infty$.

Proof to Problem 5 of Homework 6

Let $f(z)$ be analytic on a domain D and suppose $|f(z)| \leq M$ for all $z \in D$. Part 1: we want to show for each $\delta > 0$ and $m \geq 1$, $|f^{(m)}(z)| \leq m!M/\delta^m$ for all $z \in D$ whose distance from ∂D is $\geq \delta$. Part 2: we then want to use this to show that if $\{f_k(z)\}$ is a sequence of analytic functions on D that converges uniformly to $f(z)$ on D, then for each m the derivatives $f_k^{(m)}(z)$ converge uniformly to $f^{(m)}(z)$ on each subset of D at a positive distance from ∂D.

Let us fix $\delta > 0$ and $m \geq 1$ and $z_0 \in D$ s.t. $d(z, \partial D) = d(z_0, \mathbb{C} \setminus D) \geq \delta$. Then, I claim that $B(z_0, \delta) \subseteq D$ because otherwise, $B(z_0, \delta) \cap (\mathbb{C} \setminus D) \neq \varnothing$, and so $d(z_0, \mathbb{C} \setminus D) < \delta$, contradiction. Thus, for all $\lambda \in (0, \delta)$, Cauchy's estimates (pg. 118) gives that because f is analytic for $|z - z_0| \leq \lambda$ and $|f(z)| \leq M$ for $|z - z_0| = \lambda$, we have $|f^{(m)}(z_0)| \leq m!M/\lambda^m$, and taking the limit as $\lambda \nearrow \delta$, we get that $|f^{(m)}(z_0)| \leq m!M/\delta^m$, as desired in Part 1 of this problem.

Now on the claim about uniform convergence of the derivatives on subsets at positive distance from ∂D. Let S be such a subset s.t. $d(S, \partial D) \geq \delta$. Let us also fix arbitrary $\epsilon > 0$. Then, because $\{f_k\}$ converges uniformly to f on D (so f is analytic by first theorem on pg. 136), there is K_ϵ s.t. $k \geq K_\epsilon \implies |f_k(z) - f(z)| = |[f_k - f](z)| < \frac{\epsilon}{2}\delta^m/(m!) =: M$. Because all the f_k and f are analytic, we can use Part 1 of the problem to see that for all $z \in S$ (which are distance $\geq \delta$ from ∂D), $k \geq K_\epsilon \implies |f_k^{(m)}(z) - f^{(m)}(z)| = |[f_k - f]^{(m)}(z)| \leq m! \cdot \frac{\epsilon}{2}(\delta^m/m!)/\delta^m = \frac{\epsilon}{2} < \epsilon$. Thus concludes Part 2.

($\Longleftarrow$) direction of current Problem 3

($\Longleftarrow$) since we suppose for each fixed $n \in \mathbb{N}$, $a_{n,k}$ converges as $k \to \infty$, define a_n to be what it converges to. Let us define $f(z) := \sum_{n=0}^{\infty} a_n z^n$. We will show that this series converges uniformly on every compact set of D, and we will show that the f_k converges locally uniformly to f, which will show that f is analytic (equiv. holomorphic/$\mathcal{C}^1$ by the theorem on pg. 144 of Gamelin, recalling that Gamelin's "analytic" is our "holomorphic") on D, because for any $z \in D$, z is contained in the interior of a compact subset K of D, and by the theorem on pg. 136, $f(z)$ must be analytic on the interior of K, hence must be analytic/holomorphic/$\mathcal{C}^1$ at $z \in D$.

Suppose we have a compact set K. Because the distance between a compact set and a closed set (in our scenario, K and $\mathbb{C} \setminus D$) must be positive, say $\delta > 0$, we have that $K \subseteq \overline{B(0, 1-\delta)}$. Thus, it suffices to prove uniform convergence on every closed disk $\overline{D_r(0)}$.

First note that by (4.4) on pg. 144, for every $\lambda < 1$, because $|f_k(z)| \leq M$ on D for all $k \in \mathbb{N}^+$, we have that for every $k \in \mathbb{N}^+$ and $n \in \mathbb{N}$, $|a_{n,k}| \leq \frac{M}{\lambda^n}$. Taking the limit as $\lambda \nearrow 1$ we get that $|a_{n,k}| \leq M$ (again for every $k \in \mathbb{N}^+$ and $n \in \mathbb{N}$). Also, taking the limit as $k \to \infty$ for fixed $n \in \mathbb{N}$ yields $|a_n| \leq M$. The Weierstrass M-test (pg. 135) tells us that $f(z) := \sum_{n=0}^{\infty} a_n z^n$ converges uniformly on $\overline{D_r(0)}$ because $|a_n z^n| \leq Mr^n$ on $\overline{D_r(0)}$. Let us now fix arbitrary $\epsilon > 0$. Then, fixing arbitrary $r \in (0,1)$, we have on $\overline{D_r(0)}$ that

$$\begin{aligned}
|f_k(z) - f(z)| &= \left| \sum_{n=0}^{\infty} a_{n,k} z^n - \sum_{n=0}^{\infty} a_{n,k} z^n \right| \\
&\leq \left| \sum_{n=0}^{N} (a_{n,k} - a_n) z^n \right| + \left| \sum_{n=N+1}^{\infty} a_{n,k} z^n \right| + \left| \sum_{n=N+1}^{\infty} a_n z^n \right| \\
&\leq \sum_{n=0}^{N} |a_{n,k} - a_n| 1^n + 2 \sum_{n=N+1}^{\infty} Mr^n
\end{aligned}$$

for any $N \in \mathbb{N}$. Choose N large enough so that $\sum_{n=N+1} r^n = r^N/(1-r) < \frac{\epsilon}{4M}$ (which we can do because $r < 1 \implies r^N \to 0$ as $N \to \infty$). So now N is completely fixed. Then, since $a_{n,k} \to a_n$ as $k \to \infty$ for all fixed $n \in \mathbb{N}$, we have that there exist $K_n \in \mathbb{N}$ s.t. for each $n \in [N]$, $k > K_n \implies |a_{n,k} - a_n| < \frac{\epsilon}{2N}$. Then, letting $K = \max_{n \in [N]}\{K_n\}$, we have that $k > K \implies |a_{n,k} - a_n| < \frac{\epsilon}{2N}$ for all $n \in [N]$, and so the above inequality yields $k \geq K \implies |f_k(z) - f(z)| < N \cdot \frac{\epsilon}{2N} + 2M \cdot \frac{\epsilon}{4M} = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ (for all $z \in \overline{D_r(0)}$). Thus, indeed we have uniform convergence on all compact sets, and we are done.

Problem 4

Let $f : D \to D$ be analytic with $f(0) = 0$ and $|f'(0)| < 1$.

(a) We want to show that there is some $r > 0$ s.t. the sequence of iterates $f_n := f^{\circ n} := f \circ f \circ \ldots \circ f$ (n times) converges uniformly on $D_r(0)$.

Recall that in class we defined "holomorphic" to mean $\mathcal{C}^1$, and we also proved that holomorphic $\iff$ analytic. Thus, we have that f' is continuous on D, and because $\alpha := |f'(0)| < \frac{\alpha+1}{2} =: \beta < 1$, we know that there is some $\delta > 0$ s.t. $|f'(z)| < \beta$ for all $z \in D_\delta(0)$ (since by continuity of $|f'|$, the inverse image of $(-\infty, \beta)$ must be open). Then, by the fundamental theorem of calculus (see pg. 76, where because f is analytic we have $df(z) = f_x(z)\,dx + f_y(z)\,dy = f'(z)\,dx + if'(z)\,dy = f'(z)\,dz$; by the way, the "$dz$" notation is introduced on pg. 103, the first page of Chapter IV) gives us that $\int_0^z f'(z)\,dz = \int_0^z df = f(z) - f(0) = f(z)$ (where with "$\int_0^z$" I mean the path integral over the straight line from 0 to z), meaning that for $z \in D_\delta(0)$, the ML-estimate (pg. 105) gives that $|f(z)| \leq \beta \cdot |z|$.

By easy induction, we see that for $z \in D_\delta(0)$ and $n \in \mathbb{N}^+$, $|f_n(z)| \leq \beta^n |z|$ (base case is above, and if $|f_n(z)| \leq \beta^n |z|$, we have that $\beta^n < 1$ and $|z| < \delta$, so $f_n(z) \in D_\delta(0)$, so the "base case" yields $|f_{n+1}(z)| = |f(f_n(z))| \leq \beta |f_n(z)| \leq \beta \cdot \beta^n |z| \leq \beta^{n+1} |z|$). Because $\beta < 1$, we know $\beta^n \to 0$ as $n \to \infty$, so for every $\epsilon > 0$, there is $N \in \mathbb{N}$ s.t. $n \geq N \implies \beta^n \delta < \epsilon$, i.e. for every $z \in D_\delta(0)$,

$n \geq N \implies |f_n(z)| \leq \beta^n \delta < \epsilon$, giving us uniform convergence of $\{f_n\}_{n=1}^{\infty}$ to $g \equiv 0$ on $D_\delta(0)$.

(b) Later in class, we will prove that every bounded sequence of analytic functions has a subsequence that converges locally uniformly. We use this fact to show that the sequence of iterates from (a) converges locally uniformly in all of D.

Since $f : D \to D$, all the f_n are also $D \to D$, so we have for every $z \in D$ and $n \in \mathbb{N}^+$ that $|f_n(z)| \leq 1$, i.e. $\{f_n\}_{n=1}^{\infty}$ is a bounded sequence of analytic functions, and so by the fact from "later in class" written above, there is a subsequence $\{f_n\}_{n\in S}$ where S is some infinite subset of $\mathbb{N}^+$ s.t. $\{f_n\}_{n\in S}$ converges locally uniformly in all of D to say some h (which must be analytic on D, by the same reasoning as the first paragraph of the ($\Longleftarrow$) direction of Problem 3). But we know that $\{f_n\}_{n=1}^{\infty}$ converges uniformly to $g \equiv 0$ on $D_\delta(0)$, so indeed $\{f_n\}_{n\in S}$ must converge uniformly to $g \equiv 0$ on $D_\delta(0)$, i.e. $h(z) = 0$ for all $z \in D_r(0)$ (a non-empty open set). By the identity theorem (pg. 156), h must equal 0 on D. Thus, if $\{f_n\}_{n=1}^{\infty}$ converges locally uniformly on D, then it MUST converge locally uniformly to 0.

Thus, to show that $\{f_n\}_{n=1}^{\infty}$ converges locally uniformly on D, it suffices to assume it does not converge locally uniformly to 0 on D and somehow derive a contradiction. Recall that "converging locally uniformly to 0 on D" means that for all compact sets $K \subseteq D$, all $\epsilon > 0$ there is $N \in \mathbb{N}$ s.t. for all $z \in K$, $n \geq N \implies |f_n(z)| < \epsilon$. Therefore, "not converging locally uniformly to 0 on D" means that there is some compact $K \subset D$ and $\epsilon > 0$ s.t. there does not exist any $N \in \mathbb{N}^+$ s.t. for all $z \in K$, $n \geq N \implies |f_n(z)| < \epsilon$. In other words, there is some compact $K \subset D$ and $\epsilon > 0$ s.t. for any $N \in \mathbb{N}^+$, there is some $z \in K$ and $n \geq N$ s.t. $|f_n(z)| \geq \epsilon$.

Let $T \subseteq \mathbb{N}^+$ be the set of all indices s.t. there is some $z \in K$ s.t. $|f_n(z)| \geq \epsilon$. The above assumption tells us that this set T is infinite. Then, $\{f_n\}_{n\in T}$ is a sequence of bounded analytic functions, so by the fact from "later in class", there must be a subsequence $\{f_n\}_{n\in T'}$ for $T' \subseteq T$ s.t. $\{f_n\}_{n\in T'}$ converges locally uniformly on D. But by construction of $T' \subseteq T$, we know there exists a compact set $K \subseteq D$ and $\epsilon > 0$ such that for every $n \in T' \subseteq T$, there is $z \in K$ s.t. $|f_n(z)| \geq \epsilon$. Thus, $\{f_n\}_{n\in T'}$ can not possibly converge uniformly on K, contradiction.

Problem 5

Let $f(z)$ be an entire function with finitely many zeroes. Define $m(r) := m(f, r) := \min_{|z|=r}\{|f(z)|\}$. We want to show that if f not a polynomial, then $m(r) \to 0$ as $r \to \infty$. Also define $M(g, r) := \max_{|z|=r}\{|g(z)|\}$. Then, it is clear that $m(r) \to 0$ as $r \to \infty$ if and only if $M(\frac{1}{f}, r) \to \infty$ as $r \to \infty$. Let us now suppose for sake of contradiction that $M(\frac{1}{f}, r) \not\to \infty$ as $r \to \infty$. Then, there is some bound $B \in \mathbb{N}$ s.t. there are values r arbitrarily large s.t. $M(\frac{1}{f}, r) \leq B$.

By applying the argument from pg. 154 in Gamelin (or equivalently Lec 5/14) multiple times, we get that $f(z) = (z - z_1) \cdots (z - z_n)h(z)$, where z_i are all the zeroes of f (counting multiplicity) and h is an analytic nowhere zero function (explicitly, the argument from pg. 154/Lec 5/14 gives that we can

write $f(z) = (z - \text{1st zero})^{m_1} h_1(z)$ where h_1 is nonzero at "1st zero", and m_1 is the multiplicity of "1st zero"; then we can do the same and write $h_1(z) = (z - \text{2nd zero})^{m_2} h_2(z)$, where h_2 is nonzero at both "1st zero" and "2nd zero". Continuing like this at all finitely many zeroes of f we get the desired "factorization" with h entire and nowhere zero).

Then, $g := \frac{1}{h}$ is entire and equals $(z - z_1) \cdots (z - z_n)/f(z)$ at every $z \in \mathbb{C} \setminus Z$, where Z the zero set $\{z_1, \ldots, z_n\}$. First, let us define R to be some natural number s.t. $Z \subseteq D_R(0)$ (there's only finitely many zeroes so we can do this). Then, we have for any $r > R$ that

$$\begin{aligned} M(g, r) &\leq \max_{|z|=r} \left\{ \frac{1}{|f(z)|} \right\} \cdot \max_{|z|=r} \{|(z - z_1) \cdots (z - z_n)\} \\ &\leq \max_{|z|=r} \left\{ \frac{1}{|f(z)|} \right\} \cdot \max_{|z|=r} \{|z - z_1\} \cdots \max_{|z|=r} \{|z - z_n|\} \\ &\leq \max_{|z|=r} \left\{ \frac{1}{|f(z)|} \right\} \cdot (R + r)^n \end{aligned}$$

implying that

$$\frac{M(g, r)}{(R + r)^n} \leq M(\tfrac{1}{f}, r),$$

where recall from our "supposition for the sake of contradiction" that the RHS is $\leq B$ for values of r arbitrarily large. What follows next is essentially Problem 1 of Homework 6; i.e. we use the Cauchy estimates to show that g must be a polynomial. Looking at pg. 118 at the Cauchy estimates, we see that for $m \geq n + 1$, $|g^{(m)}(0)| \leq \frac{m!}{r^m} B(R + r)^n$ for values of r arbitrarily large. Because $m \geq n + 1$ and the limit as $r \to \infty$ of $(R + r)^n / r^m$ is 0, and the inequality is true for values of r arbitrarily large, we in fact get that $|g^{(m)}(0)| = 0$ for all $m \geq n + 1$. Because g is entire, it equals its power series centered at 0 everywhere, so in fact g is a polynomial (in fact of degree $\leq n$). But g has no roots ($g = \frac{1}{h}$ where h was an entire nowhere 0 function), so g must be constant, meaning that h must be constant, meaning that $f = (z - z_1) \cdots (z - z_n)$, contradicting that f was not a polynomial. Thus, our initial supposition must have been false; indeed $M(\frac{1}{f}, r) \to \infty \iff m(f, r) \to 0$ as $r \to \infty$.

534 Homework 6

Daniel Rui - 5/12/20

Problem 1 - IV.5.4

Suppose that $f(z)$ is an entire function s.t. $g(z) = f(z)/z^n$ is bounded (by say B) for $|z| \geq R$. We want to show that f must be a polynomial of degree $\leq n$. Well, $|g(z)| \leq B$ for $|z| \geq R$ implies that for any z_0, and for any big enough R_{z_0}, $|z - z_0| < R_{z_0}$ contains the disk $|z| < R$, which means that all $|z - z_0| \geq R_{z_0}$ satisfy $|g(z)| \leq B$. In particular, $|g(z)| \leq B$ for $|z - z_0| = R_{z_0}$, a circle contained in the region $|z| \leq |z_0| + R_{z_0}$, and so $|f(z)| \leq B(|z_0| + R_{z_0})^n$ for $|z - z_0| = R_{z_0}$. $f(z)$ is of course analytic inside this region, as it is entire, and so now we may apply Cauchy's estimates to get that:

$$\left|f^{(n+1)}(z_0)\right| \leq \frac{(n+1)!}{R_{z_0}^{n+1}} B(|z_0| + R_{z_0})^n$$

Letting R_{z_0} go to ∞ (because recall that I said this works for ANY big enough R_{z_0}), we see that $|f^{(n+1)}(z_0)| = 0 \implies f^{(n+1)}(z_0) = 0$. Thus all derivatives of order $(n+1)$ and higher are 0, EVERYWHERE on $\mathbb{C}$ (because z_0 was arbitrary). Now because f is analytic in a neighborhood of z_0, it equals its power series there, which we just showed is a finite degree polynomial. By the corollary on pg. 146, since f is entire, the radius of convergence is in fact ∞, and so- f is that finite degree polynomial $a_0 + a_1 z + \ldots + a_n z^n$ on all of $\mathbb{C}$.

Finally, if $g(z)$ is bounded on all of $\mathbb{C}$, then $f(z)/z^n = a_n + a_{n-1}/z + \ldots + a_0/z^n$ must be bounded at 0, which can only happen if $a_{n-1} = \ldots = a_0 = 0$. Thus, bounded everywhere $\implies f(z) = a_n z^n$.

Problem 2a - IV.6.2

Let $h(t)$ be a continuous function on $[a, b]$. We want to show that the Fourier transform

$$H(z) = \int_a^b h(t) e^{-itz}\, dt$$

is an entire function that satisfies $|H(z)| \leq Ce^{A|y|}, z = z + iy \in \mathbb{C}$, for constants $A, C > 0$. Well, because h is continuous on $[a, b]$ and e^{-itz} is analytic on $z \in \mathbb{C}$, $h(t, z) = h(t)e^{-itz}$ is continuous complex valued function for $t \in [a, b]$ and $z \in \mathbb{C}$, and is an analytic function of z for every fixed $t \in [a, b]$, and so we apply the first theorem on page 121, which yields that $H(z)$ is analytic.

As for the bound, notice that $h(t)$ is a continuous function on a compact interval $[a, b]$, so it is bounded by say B. Then:

$$|H(z)| = \int_a^b |h(t)||e^{-itz}|\, dt \leq \int_a^b B|e^{-itx+ty}|\, dt = \int_a^b Be^{ty}\, dt$$

where e^{ty} is bounded by $e^{A|y|}$ for $A = \max\{|a|, |b|\}$, which means that the whole thing is bounded by

$B(b-a)e^{A|y|}$ (and so taking $C = B(b-a)$ we get the desired bound).

Problem 2b - IV.6.3

For $h(t)$ a continuous function on a subinterval $[a,b]$ of $[0,\infty)$, we want to show that the Fourier transform $H(z)$ defined above is bounded in the lower half plane. Recall from above that we have the inequality

$$|H(z)| = \int_a^b |h(t)||e^{-itz}|\,dt \le \int_a^b B|e^{-itx+ty}|\,dt = \int_a^b Be^{ty}\,dt,$$

where B is the bound of $h(t)$ on $[a,b]$ (we know such a B exists because h is continuous on a compact set). For z in the lower half plane, $y < 0$, because $t \in [a,b] \in [0,\infty)$ is always ≥ 0, e^{ty} on $[a,b]$ is bounded by $e^0 = 1$, and so $|H(z)| \le B(b-a)$ for all z in the lower half plane.

Problem 3a - IV.8.5

Defining $d\overline{z} = dx - i\,dy$, we want to show for a smooth function $f(z)$ that $df = d_z f\,dz + d_{\overline{z}} f\,d\overline{z}$. Recall from Problem 4 of Homework 1 that $f_z(z_0) = \frac{1}{2}(f_x(z_0) - if_y(z_0))$ and $f_{\overline{z}}(z_0) = \frac{1}{2}(f_x(z_0) + if_y(z_0))$, so

$$\begin{aligned} f_z(z_0)\,dz + f_{\overline{z}}(z_0)\,d\overline{z} &= \tfrac{1}{2}(f_x(z_0) - if_y(z_0))(dx + i\,dy) + \tfrac{1}{2}(f_x(z_0) + if_y(z_0))(dx - i\,dy) \\ &= f_x(z_0)\,dx - (i^2)f_y(z_0)\,dy = f_x(z_0)\,dx + f_y(z_0)\,dy =: df. \end{aligned}$$

Problem 3b - IV.8.6

We want to show for a domain D with smooth boundary and f, g analytic on $D \cup \partial D$ that $\int_{\partial D} f(z)\overline{g(z)}\,dz = 2i\iint_D f(z)\overline{g'(z)}\,dx\,dy$. Again looking at Problem 4 of Homework 1, we see that we can write uniquely $f(z) = f(z_0) + f_z(z_0)\cdot(z-z_0) + f_{\overline{z}}(z_0)\cdot(\overline{z-z_0}) + o(|z-z_0|)$ and $\overline{g(z)} = \overline{g(z_0)} + \overline{g_{\overline{z}}(z_0)}\cdot(z-z_0) + \overline{g_z(z_0)}\cdot(\overline{z-z_0}) + o(|z-z_0|)$ and so

$$f(z)\overline{g}(z) := f(z_0)\overline{g(z_0)} + [f\overline{g_{\overline{z}}} + \overline{g}f_z](z_0)\cdot(z-z_0) + [\overline{g}f_{\overline{z}} + f\overline{g_z}](z_0)\cdot(\overline{z-z_0}) + o(|z-z_0|)$$

and so formula (8.4) on pg. 127 gives us that (since $f\overline{g}$ is a smooth function on $D \cup \partial D$)

$$\int_{\partial D} f(z)\overline{g(z)}\,dz = 2i\iint_D [\overline{g}f_{\overline{z}} + f\overline{g_z}](z)\,dx\,dy.$$

We now use the fact that f, g are both analytic on $D \cup \partial D \iff f_{\overline{z}} = g_{\overline{z}} = 0$ on $D \cup \partial D$, and so the RHS equals $2i\iint_D [f\overline{g_z}](z)\,dz$. Since g is analytic, we can also use the notation $g' := g_z$, and so we do in fact recover the desired $2i\iint_D f(z)\overline{g'(z)}\,dz$.

Problem 4a - V.2.8

We want to show that $\sum_{k=1}^{\infty} \frac{z^k}{k^2}$ converges uniformly for $|z| < 1$. Recall that the Weierstrass M-test (pg. 135) says that supposing we have $M_k \ge 0$ and $\sum_{k=1}^{\infty} M_k$ convergent, if $g_k(x)$ are complex-valued functions on a set E s.t. $|g_k(x)| \le M_k$ for all $x \in E$, then $\sum_{k=1}^{\infty} g_k(x)$ converges UNIFORMLY on E. Well, in our case, take $E := [|z| < 1]$; it is clear that $|\frac{z^k}{k^2}| \le \frac{1}{k^2}$ on E and so by the M-test,

since $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges (by the p-test/integral test from elementary calculus), $\sum_{k=1}^{\infty} \frac{z^k}{k^2}$ converges UNIFORMLY on E, as desired.

Problem 4b - V.2.9

We want to show that $\sum_{k=1}^{\infty} \frac{z^k}{k}$ does NOT converge uniformly for $|z| < 1$. Well, suppose it did converge uniformly to a function $S(z)$. Then, there would exist $N \in \mathbb{N}$ s.t. $|S_N(z) - S(z)| \leq 1$ for all $z \in D := [|z| < 1]$, where $S_N(z) := \sum_{k=1}^{N} \frac{z_k}{k}$. Then, because $|S_N(z)| \leq \sum_{k=1}^{N} \frac{1}{k} =: B < \infty$ on D, we see that $|S(z)| \leq B+1$ for all $z \in D$. This of course is not true, because the $\sum_{k=1}^{\infty} \frac{1}{k}$ diverging (which we know from e.g. the p-test/integral test from elementary calculus) means that there is some $M \in \mathbb{N}$ s.t. $\sum_{k=1}^{M} \frac{1}{k} \geq 3(B+1)$, and choosing $z \in [0,1) \subseteq \mathbb{R}$ s.t. $z^k \geq \frac{1}{2}$ for all $k \in \{1, \ldots, M\}$ (for example we could take $z := (1/2)^{1/M}$), we get that $\sum_{k=1}^{\infty} \frac{z^k}{k} \geq \frac{1}{2} \sum_{k=1}^{M} \frac{1}{k} \geq \frac{1}{2} \cdot 3(B+1) > B+1$, which is a contradiction.

Problem 5 - V.2.12

Let $f(z)$ be analytic on a domain D and suppose $|f(z)| \leq M$ for all $z \in D$. Part 1: we want to show for each $\delta > 0$ and $m \geq 1$, $|f^{(m)}(z)| \leq m!M/\delta^m$ for all $z \in D$ whose distance from ∂D is $\geq \delta$. Part 2: we then want to use this to show that if $\{f_k(z)\}$ is a sequence of analytic functions on D that converges uniformly to $f(z)$ on D, then for each m the derivatives $f_k^{(m)}(z)$ converge uniformly to $f^{(m)}(z)$ on each subset of D at a positive distance from ∂D.

Let us fix $\delta > 0$ and $m \geq 1$ and $z_0 \in D$ s.t. $d(z, \partial D) = d(z_0, \mathbb{C} \setminus D) \geq \delta$. Then, I claim that $B(z_0, \delta) \subseteq D$ because otherwise, $B(z_0, \delta) \cap (\mathbb{C} \setminus D) \neq \varnothing$, and so $d(z_0, \mathbb{C} \setminus D) < \delta$, contradiction. Thus, for all $\lambda \in (0, \delta)$, Cauchy's estimates (pg. 118) gives that because f is analytic for $|z - z_0| \leq \lambda$ and $|f(z)| \leq M$ for $|z - z_0| = \lambda$, we have $|f^{(m)}(z_0)| \leq m!M/\lambda^m$, and taking the limit as $\lambda \nearrow \delta$, we get that $|f^{(m)}(z_0)| \leq m!M/\delta^m$, as desired in Part 1 of this problem.

Now on the claim about uniform convergence of the derivatives on subsets at positive distance from ∂D. Let S be such a subset s.t. $d(S, \partial D) \geq \delta$. Let us also fix arbitrary $\epsilon > 0$. Then, because $\{f_k\}$ converges uniformly to f on D (so f is analytic by first theorem on pg. 136), there is K_ϵ s.t. $k \geq K_\epsilon \implies |f_k(z) - f(z)| = |[f_k - f](z)| < \frac{\epsilon}{2}\delta^m/(m!) =: M$. Because all the f_k and f are analytic, we can use Part 1 of the problem to see that for all $z \in S$ (which are distance $\geq \delta$ from ∂D), $k \geq K_\epsilon \implies |f_k^{(m)}(z) - f^{(m)}(z)| = |[f_k - f]^{(m)}(z)| \leq m! \cdot \frac{\epsilon}{2}(\delta^m/m!)/\delta^m = \frac{\epsilon}{2} < \epsilon$. Thus concludes Part 2.

Problem 6

534 Homework 5

Daniel Rui - 5/5/20

Problem 1a - IV.1.8

Suppose the continuous function $g(\theta) = f(e^{i\theta})$ on the unit circle satisfies $|g(\theta)| \leq M$, and $|\int_{|z|=1} f(z)\,dz| = 2\pi M$. We want to show that $f(z) = c\overline{z}$ for some constant c satisfying $|c| = M$. From the middle of page 103, we can parameterize the unit circle as $z(\theta) = e^{i\theta}$ and get that $dz = i(\cos\theta + i\sin\theta)\,d\theta = ie^{i\theta}\,d\theta$, and so the condition above says that

$$2\pi M = \left|\int_{|z|=1} f(z)\,dz\right| = \left|\int_0^{2\pi} f(e^{i\theta})ie^{i\theta}\,d\theta\right| = \left|i\int_0^{2\pi} g(\theta)e^{i\theta}\,d\theta\right| \leq |i|\int_0^{2\pi} |g(\theta)||e^{i\theta}|\,d\theta = \int_0^{2\pi} |g(\theta)|\,d\theta$$

We know that $|g(\theta)| \leq M$, but the equation above tells us that it must be equal to M for all $\theta \in [0, 2\pi)$, because if it were $< M$ at some θ_0, then by continuity, we would have that the integral $\int_0^{2\pi} |g(\theta)| - M\,d\theta < 0 \iff \int_0^{2\pi} |g(\theta)|\,d\theta < 2\pi M$ (we proved a theorem like this on Homework 10 of 334), which contradicts the inequality in the above display. Thus, there is some continuous function h such that $g(\theta) = Me^{ih(\theta)}$, and so

$$2\pi M e^{i\vartheta} = \int_{|z|=1} f(z)\,dz = \int_0^{2\pi} g(\theta)ie^{i\theta}\,d\theta = \int_0^{2\pi} Me^{ih(\theta)}ie^{i\theta}\,d\theta = Mi\int_0^{2\pi} e^{ih(\theta)+i\theta}\,d\theta$$

where $\vartheta \in [0, 2\pi)$ is some constant. Dividing by M (if M is zero, obviously $f(z) = 0$ on the unit circle), the above equation is equivalent to

$$-2\pi i = \int_0^{2\pi} e^{i(h(\theta)+\theta-\vartheta)}\,d\theta$$

Comparing imaginary parts, we see that $-2\pi = \int_0^{2\pi} \sin(h(\theta) + \theta - \vartheta)\,d\theta$. We know that $\sin(\theta)$ takes values in $[-1, 1]$, and if at any point it's not -1, due to continuity the integral will fail to be -2π. Thus, we know that for all $\theta \in [0, 2\pi)$, $\sin(h(\theta) + \theta - \vartheta) = -1$, or $h(\theta) + \theta - \vartheta = \frac{3}{2}\pi + 2k\pi$. Plugging this back in, we have that

$$f(e^{i\theta}) = g(\theta) = Me^{ih(\theta)} = Me^{i(2k\pi+3\pi/2+\vartheta-\theta)} = M \cdot 1 \cdot (-i) \cdot e^{i\vartheta} \cdot e^{-i\theta} = -iMe^{i\vartheta}\overline{e^{i\theta}}$$

and thus on $\{z : |z| = 1\}$, we have that $f(z) = c\overline{z}$ where $c = -iMe^{i\vartheta} \implies |c| = M$.

Problem 1b - IV.1.9

We have $H(w) = \int_\gamma \frac{h(z)}{z-w}\,dz = \int_\gamma \frac{h(z)\,dx}{z-w} + i\int_\gamma \frac{h(z)\,dy}{z-w}$ for $z \in \gamma$ and $w \in \mathbb{C} \setminus \gamma$ where h is a continuous function on the curve γ. We want to show that H is analytic on $\mathbb{C} \setminus \gamma$, and we want to find $H'(w)$.

We have:

$$\begin{aligned}\frac{H(w)-H(w_0)}{w-w_0} &= \frac{1}{w-w_0}\left(\int_\gamma \frac{h(z)}{z-w} - \frac{h(z)}{z-w_0}\,dx + i\int_\gamma \frac{h(z)}{z-w} - \frac{h(z)}{z-w_0}\,dy\right)\\ &= \frac{1}{w-w_0}\left(\int_\gamma \frac{h(z)\cdot(w-w_0)}{(z-w)(z-w_0)}\,dx + i\int_\gamma \frac{h(z)\cdot(w-w_0)}{(z-w)(z-w_0)}\,dy\right)\\ &= \int_\gamma \frac{h(z)}{(z-w)(z-w_0)}\,dx + i\int_\gamma \frac{h(z)}{(z-w)(z-w_0)}\,dy\end{aligned}$$

which goes to $\int_\gamma \frac{h(z)}{(z-w)^2}\,dx + i\int_\gamma \frac{h(z)}{(z-w)^2}\,dy$ as $w \to w_0$ because the integrands converge uniformly (and limits can be pulled inside integrals when we have uniform convergence) as follows: we just look at the real part now (the imaginary part is the same)

$$\begin{aligned}\sup_{z\in\gamma,|w-w_0|\le r} \left|\tfrac{h(z)}{(z-w)(z-w_0)} - \tfrac{h(z)}{(z-w_0)^2}\right| &= \sup_{z\in\gamma,|w-w_0|\le r} \left|\tfrac{h(z)}{(z-w_0)}\right|\left|\tfrac{(w-w_0)}{(z-w)(z-w_0)}\right|\\ &= (w-w_0)\sup_{z\in\gamma,|w-w_0|\le r}\left|\tfrac{h(z)}{(z-w_0)(z-w)(z-w_0)}\right|,\end{aligned}$$

but the supremum of a continuous function over a compact set (we choose r small enough that the denominator is never 0, which we can do because $\mathbb{C}\setminus\gamma$ is open) is finite, so we can bound the above by just $(w-w_0)\cdot C$, which goes to 0 uniformly as $w \to w_0$. (Compare this to problem II.2.6 from Homework 1 Problem 5)Thus, $H(\omega)$ is analytic on $\mathbb{C}\setminus\gamma$ and

$$H'(\omega) = \int_\gamma \frac{h(z)}{(z-w)^2}\,dx + i\int_\gamma \frac{h(z)}{(z-w)^2}\,dy = \int_\gamma \frac{h(z)}{(z-w)^2}\,dz$$

Problem 2 - IV.3.3

Let $f(z) = c_0 + c_1 z + \ldots + c_n z^n$ be a polynomial.

(a) Assuming that the c_k are real, we want to show that

$$\int_{-1}^{1} f(x)^2\,dx \le \pi\int_0^{2\pi} |f(e^{i\theta})|^2\frac{d\theta}{2\pi} = \pi\sum_{k=0}^{n} c_k^2.$$

Gamelin provides the hint that we should tackle the inequality by applying Cauchy's theorem (pg. 110) on the top and bottom half of the unit disk. So let D^+ and D^- denote the domains $\{z\in\mathbb{C} : |z|<1, \operatorname{Im} z>0\}$ and $\{z\in\mathbb{C} : |z|<1, \operatorname{Im} z<0\}$ respectively. Then, applying Cauchy's theorem to f^2 the domain D^+ (which is indeed bounded with piecewise smooth boundary s.t. f^2 is analytic on D^+ and extends smoothly to ∂D^+ because the polynomial f^2 is analytic everywhere on $\mathbb{C}$), we see that $\int_{-1}^{1} f(x)^2\,dx + \int_0^{\pi} f(e^{i\theta})^2\,d\theta = 0$. Similarly for D^-, we see that $\int_1^{-1} f(x)^2\,dx + \int_\pi^{2\pi} f(e^{i\theta})^2\,d\theta = 0$. Thus, we have that $|\int_{-1}^{1} f(x)^2\,dx| = |\int_1^{-1} f(x)^2\,dx| = |\int_0^{\pi} f(e^{i\theta})^2\,d\theta| = |\int_\pi^{2\pi} f(e^{i\theta})^2\,d\theta|$, and so

$$2\int_{-1}^{1} f(x)^2\,dx = 2\left|\int_{-1}^{1} f(x)^2\,dx\right| = \left|\int_0^{\pi} f(e^{i\theta})^2\,d\theta\right| + \left|\int_\pi^{2\pi} f(e^{i\theta})^2\,d\theta\right| \le \int_0^{2\pi} |f(e^{i\theta})|^2\,d\theta.$$

Problem 3 - IV.3.5

Suppose D is bounded domain with piecewise smooth boundary and that $f(z)$ is analytic on $D \cup \partial D = \overline{D}$. We want to show that $\sup_{z\in\partial D} |\overline{z} - f(z)| \geq 2\frac{\text{Area}(D)}{\text{Length}(\partial D)}$, and the moreover this estimate is sharp in the sense that there exists D and $f(z)$ for which the equality holds. That the estimate is harp is easy to see, since taking D to be the unit disk (so area is π and circumference/length of ∂D is 2π) and $f = 0$, $\sup_{z\in\partial D} |\overline{z}| = 1 = 2\frac{\pi}{2\pi}$. For the inequality in general Gamelin suggests considering $\int_{\partial D} \overline{z} - f(z)\, dz$, and using Exercise IV.1.4 (Problem 6b of Homework 4 last week), which reminder states that for a bounded domain D with smooth boundary,

$$\begin{aligned}\int_{\partial D} \overline{z}\, dz &= \int_{\partial D} (x - iy)\, dx + (x - iy)i\, dy = \int_{\partial D} x\, dx + y\, dy + i \int_{\partial D} -y\, dx + x\, dy \\ &= \iint_D (0 - 0)\, dA + i \iint_D (1 - (-1))\, dA = 2i\text{Area}(D).\end{aligned}$$

Well, by this above Exercise and by Cauchy's theorem (pg. 110), which we can apply because f is analytic on $D \cup \partial D$, $\int_{\partial D} \overline{z} - f(z)\, dz = \int_{\partial D} \overline{z}\, dz - \int_{\partial D} f(z)\, dz = 2i\text{Area}(D) - 0$. Then, by ML-estimate (pg. 105), $2\text{Area}(D) = |2i\text{Area}(D)| = |\int_{\partial D} \overline{z} - f(z)\, dz| \leq \sup_{z\in\partial D} |\overline{z} - f(z)| \cdot \text{Length}(\partial D)$. Dividing by the length gives us the desired inequality.

Problem 4 - IV.4.1

We evaluate several integrals using the Cauchy integral formula. We will be using the one for higher derivatives, i.e. the one on pg. 114 that says that for a bounded domain with piecewise smooth boundary and analytic f on $D \cup \partial D$ and $z_0 \in D$, $f^{(m)}(z_0) = \frac{m!}{2\pi i} \int_{\partial D} \frac{f(z)}{(z-z_0)^{m+1}}\, dz$, or equivalently $\frac{1}{m!} 2\pi i f^{(m)}(z_0) = \int_{\partial D} \frac{f(z)}{(z-z_0)^{m+1}}\, dz$. In all the following problems, D will be a (bounded) disk with boundary a circle (indeed smooth), and we adopt the notation $C(x, r)$ for a circle centered at x with radius r, and $D(x, r)$ for the corresponding disk. The variables m, n denote integers.

(a) $\oint_{C(0,2)} \frac{z^n}{z-1}\, dz$ for $n \geq 0$. Here, $f(z) = z^n$, which is indeed analytic over the domain D in question (corresponding disk to the circle we are integrating over), and $z_0 = 1 \in D$. The Cauchy integral formula gives that this integral is hence equal to $2\pi i f(1) = 2\pi i$ (for all $n \geq 0$).

(b) $\oint_{C(0,1)} \frac{z^n}{z-2}\, dz$ for $n \geq 0$. Here, the pole is not in the disk so $\frac{z^n}{z-2}$ is analytic on D (corresponding disk to the circle we are integrating over), and so by Cauchy's theorem of the previous chapter (pg. 110), this integral is 0 for all $n \geq 0$.

(c) $\oint_{C(0,1)} \frac{\sin z}{z}\, dz$. Here, $f(z) = \sin z$, which is indeed analytic over the domain D in question (corresponding disk to the circle we are integrating over), and $z_0 = 0 \in D$. The Cauchy integral formula gives that this integral is hence equal to $2\pi i f(0) = 0$.

(d) $\oint_{C(0,1)} \frac{\cosh z}{z^3}\, dz$. Here, $f(z) = \cosh z$, which is indeed analytic over the domain D in question, and $z_0 = 0 \in D$. The Cauchy integral formula gives that this integral is hence equal to $\frac{1}{2!} 2\pi i f^{(2)}(0) = 2\pi i \cosh(0) = \pi i$.

(e) $\oint_{C(1+i,5/4)} \frac{\operatorname{Log} z}{(z-1)^2}\,dz$. Here, $f(z) = \operatorname{Log} z$ (defined over the slit plane, see pg. 22), which is indeed analytic over the domain D in question (which is $D(1+i, 5/4)$ contained in the slit plane), and $z_0 = 1 \in D$. The Cauchy integral formula gives that this integral is hence equal to $\frac{1}{1!}2\pi i f'(1) = 2\pi i \cdot \frac{1}{1} = 2\pi i$.

(f) $\oint_{C(0,1)} \frac{1}{z^2(z^2-4)e^z}\,dz$. Here, $f(z) = \frac{1}{(z^2-4)e^z}$, which is indeed analytic over the domain D in question (because the poles of $\frac{1}{z^2-4}$ do not lie in the unit disk and hence in the unit disk D, f is indeed just 1 divided by a non-zero analytic function), and $z_0 = 0 \in D$. The Cauchy integral formula gives that this integral is hence equal to $\frac{1}{1!}2\pi i f'(0) = 2\pi i(\frac{(z^2-4)(-e^{-z})-e^{-z}(2z)}{(z^2-4)^2})|_{z=0}) = 2\pi i(\frac{1}{4}) = \frac{\pi i}{2}$.

Problem 5 - IV.4.4

Let D be a bounded domain with smooth boundary ∂D and let $z_0 \in D$. We want to use the Cauchy integral formula to show that there is a constant C such that $|f(z_0)| \leq C \sup_{z\in\partial D} |f(z)|$ for any function $f(z)$ analytic on $D \cup \partial D$. By applying this estimate to $f(z)^n$, taking nth roots, and letting $n \to \infty$, we want to show that this estimate holds with $C = 1$. In essence, this is an alternate proof of the maximum principle for analytic functions on domains with (piecewise) smooth boundary.

First, we use Cauchy's integral formula (pg. 113) to write $f(z_0) = \int_{\partial D} \frac{f(z)}{z-z_0}\,dz$. We then aim to use the ML-estimate. First note that because ∂D is closed and the boundary of a bounded domain, hence bounded, it is compact, and the two disjoint compact sets are are distant, $|z - z_0|$ for all $z \in \partial D$ is bounded below by say $\delta > 0$, and so we can take "M" in the ML-estimate to be $\frac{1}{\delta}\sup_{z\in\partial D}|f(z)|$. Because D has smooth boundary and is bounded, the length of ∂D, say ℓ, is a finite positive real number. Thus by the ML-estimate we have that $|f(z_0)| \leq \frac{\ell}{\delta}\sup_{z\in\partial D}|f(z)|$.

Applying this estimate to f^n for any $n \in \mathbb{N}^+$, we have that $|f(z_0)|^n = |f(z_0)^n| \leq C\sup_{z\in\partial D}|f(z)^n|$. But $\sup_{z\in\partial D}|f(z)^n| = [\sup_{z\in\partial D}|f(z)|]^n$ because x^n and $x^{1/n}$ are increasing and continuous functions attain their supremums on compact sets, like ∂D here. Thus taking nth roots (of real numbers, not complex), we have that $|f(z_0)| \leq C^{1/n}\sup_{z\in\partial D}|f(z)|$. This works for all $n \in \mathbb{N}^+$. We know $C \in (0, \infty)$, and for all numbers in that interval, raising to the $\frac{1}{n}$th power as $n \to \infty$ approaches 1. Thus indeed $|f(z_0)| \leq \sup_{z\in\partial D}|f(z)|$, and we are done.

Problem 6a - IV.5.1

We want to show that if u is harmonic and bounded (say by B), then u is constant. Looking at pg. 83, we know that u is harmonic on $\mathbb{C}$ means that there is some v s.t. $f = u + iv$ is analytic on $\mathbb{C}$. Consider $e^f = e^u e^{iv} = e^u(\cos v + i\sin v)$ which is analytic on $\mathbb{C}$ (i.e. entire) because f is and e^x is as well (and composition preserves analyticity). The function e^f is clearly bounded because $|e^f| = |e^u(\cos v + i\sin v)| = |e^u| = e^u \leq e^B$. Thus by Liouville's theorem on page 118, e^f is constant, which means that f must be constant (because e^x doesn't take on the same value for two different x), which of course implies that u is constant.

Problem 6b - IV.5.2

We want to show that if $f(z)$ is an entire function s.t. there is a non-empty disk s.t. f does not attain any values in the disk, then f is constant. Let's say that the (open) disk is centered at z_0 with radius $\epsilon > 0$. The assumption is that f (an entire function) does not take values inside this disk; that is, $|f(z) - z_0| \geq \epsilon$ for all $z \in \mathbb{C}$. Then consider $g(z) = \frac{1}{f(z)-z_0}$, which is entire because the denominator is entire and never 0. Furthermore, $|g(z)| = \frac{1}{|f(z)-z_0|} \leq \frac{1}{\epsilon}$. So $g(z)$ is bounded, so Liouville's theorem gives that $g(z)$ is constant, which of course implies that $f(z)$ is constant ($C = \frac{1}{f(z)-z_0} \implies f(z) = \frac{1}{C+z_0}$).

534 Homework 4

Daniel Rui - 4/28/20

Problem 1 - III.3.2

We want to show that a complex-valued function $h(z)$ on a star-shaped domain D is harmonic iff $h(z) = f(z) + \overline{g(z)}$ for analytic f and g on D.

($\implies$) By definition, $h = u + iv$ is harmonic if u and v are harmonic on D. The theorem on pg. 83 tells us that any harmonic function $u(x, y)$ on D has a harmonic conjugate on D; and so let a and b be the harmonic conjugates of u and v resp., so that $\alpha = u + ia$ and $\beta = v + ib$ are analytic on D. Obviously, we can recover h by $h = \frac{1}{2}(\alpha + \overline{\alpha} + i(\beta + \overline{\beta}))$. Thus we can set $f = \frac{1}{2}(\alpha + i\beta)$ and $\overline{g} = \frac{1}{2}(\overline{\alpha} + i\overline{\beta}) \iff g = \frac{1}{2}(\alpha - i\beta)$ where f and g are just sums of scalar multiples of analytic functions, and so clearly analytic.

($\impliedby$) Let $f = a + bi$ and $g = c + di$ (which are both analytic on D so a, b, c, d are all harmonic), and so $h = f + \overline{g} = (a + c) + i(b - d)$. By linearity of differentiation, a and c being harmonic imply that $(a + c)$ is harmonic, and similarly $(b - d)$ is too. Thus, h is harmonic because its real and imaginary parts are both harmonic.

Problem 2 - III.3.4

Let $u(z)$ be harmonic on $\{a < |z| < b\}$. We want to show that there is a constant C s.t. $u(z) - C \log |z|$ has a harmonic conjugate on the annulus, and furthermore that

$$C = \frac{r}{2\pi} \int_0^{2\pi} \frac{\partial u}{\partial r}(re^{i\theta})\, d\theta$$

where r is some fixed number in (a, b). The lemma on pg. 83 says that if u is harmonic, then the differential $-\frac{\partial u}{\partial y}\, dx + \frac{\partial u}{\partial x}\, dy$ is closed (i.e. $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$), which implies by Green's theorem that $\int_{\partial U} -\frac{\partial u}{\partial y}\, dx + \frac{\partial u}{\partial x}\, dy = 0$ for any domain U s.t. $\overline{U} \subseteq D$ (this is explained in more detail on pg. 78). This means that for any r_1, r_2 s.t. $a < r_1 < r_2 < b$, that

$$\int_{|z|=r_2} -\frac{\partial u}{\partial y}\, dx + \frac{\partial u}{\partial x}\, dy - \int_{|z|=r_1} -\frac{\partial u}{\partial y}\, dx + \frac{\partial u}{\partial x}\, dy = 0$$

and so along any circle with radius in (a, b), the line integral of $-\frac{\partial u}{\partial y}\, dx + \frac{\partial u}{\partial x}\, dy$ along that circle will be some constant c (assuming all circles are traversed in a counterclockwise fashion). Let $w = u - \frac{c}{2\pi} \log |z| = u - \frac{c}{2\pi} \log(\sqrt{x^2 + y^2})$, which is harmonic because u is harmonic, and $\log(\sqrt{x^2 + y^2})$ has $\partial_x = \frac{x}{x^2+y^2} \implies \partial_{xx} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2}$ and $\partial_y = \frac{y}{x^2+y^2} \implies \partial_{yy} = \frac{x^2+y^2-2y^2}{(x^2+y^2)^2}$ and so $\partial_{xx} + \partial_{yy} = 0$.

We do this so that

$$\begin{aligned}\int_{|z|=r} -\frac{\partial w}{\partial y}\,dx + \frac{\partial w}{\partial x}\,dy &= \int_{|z|=r} -\frac{\partial u}{\partial y} + \frac{c}{2\pi}\frac{y}{x^2+y^2}\,dx + \frac{\partial u}{\partial x} - \frac{c}{2\pi}\frac{x}{x^2+y^2}\,dy \\ &= \int_{|z|=r} -\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy + \frac{c}{2\pi}\int_{|z|=r} \frac{y}{x^2+y^2}\,dx - \frac{x}{x^2+y^2}\,dy \\ &= c + \frac{c}{2\pi}\int_0^{2\pi} \frac{r\sin\theta}{r^2}(-r\sin\theta)\,d\theta - \frac{r\cos\theta}{r^2}(r\cos\theta)\,d\theta = c - \frac{c}{2\pi}2\pi = 0\end{aligned}$$

where we used that $x = r\cos\theta \implies dx = (-r\sin\theta)\,d\theta$, and a similar identity for y. Now, we can define $v(z) = \int_C -\frac{\partial w}{\partial y}\,dx + \frac{\partial w}{\partial x}\,dy$ where C is the curve going from a point, say $r+0i$ counterclockwise along the circle of radius r, then on radially inward and outward to z. This is well defined, because any counterclockwise path can be transformed continuously to C, and furthermore because the path integral around the circle is 0, any clockwise paths ALSO give the same value. Therefore, this differential form is independent of path. This is the key part of the proof. Finally, pg. 82 reminds us that independent of path $\iff$ exact, i.e. there is v with $dv := \frac{\partial v}{\partial x}\,dx + \frac{\partial v}{\partial y}\,dy = -\frac{\partial w}{\partial y}\,dx + \frac{\partial w}{\partial x}\,dy$, and so the C-R equations are satisfied for $w + iv$, which means that v is the harmonic conjugate of w.

To get the alternative form of $C := \frac{c}{2\pi}$, we use Exercise II.3.8 (done below), which is about the polar form C-R equations, the first of which says that $\frac{\partial w}{\partial r} = \frac{\partial u}{\partial r} - \frac{c}{2\pi}\frac{1}{r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \implies \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{c}{2\pi}\frac{1}{r}$, and so

$$\begin{aligned}\frac{r}{2\pi}\int_0^{2\pi} \frac{\partial u}{\partial r}(re^{i\theta})\,d\theta &= \frac{r}{2\pi}\int_0^{2\pi} \frac{1}{r}\frac{\partial v}{\partial \theta}(re^{i\theta}) + \frac{c}{2\pi}\frac{1}{r}\,d\theta \\ &= \frac{1}{2\pi}\int_{|z|=r} dv + \frac{1}{2\pi}\int_0^{2\pi} \frac{c}{2\pi}\,d\theta \\ &= 0 + \frac{c}{2\pi} = C\end{aligned}$$

Exercise II.3.8 - POLAR FORM OF C-R EQUATIONS

We have $f = u(x,y) + iv(x,y)$ where $x = r\cos\theta, y = r\sin\theta$. The chain rule says that

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \partial_x u\cos\theta + \partial_y u\sin\theta \quad \text{and} \quad \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = \partial_x u(-r\sin\theta) + \partial_y u(r\cos\theta)$$

and similarly

$$\partial_r v = \partial_x v\cos\theta + \partial_y v\sin\theta \quad \text{and} \quad \partial_\theta v = \partial_x v(-r\sin\theta) + \partial_y v(r\cos\theta)$$

The C-R equations ($\partial_x v = -\partial_y u$ and $\partial_y v = \partial_x u$) can be easily substituted into the above to get that

$$\partial_r u = \frac{1}{r}\partial_\theta v \quad \text{and} \quad \partial_\theta u = -r\partial_r v.$$

Problem 4a - III.5.5

Let $f(z)$ be a bounded analytic function on the right half-plane. Suppose f extends continuously to the imaginary axis and satisfies $|f(iy)| \leq M$ for all $y \in \mathbb{R}$. As the hint suggests, we consider the function $g(z) := \frac{f(z)}{(z+1)^\epsilon}$. We know $f(z)$ is bounded, so say $|f(z)| \leq B$ on the right half plane. Fix some small $\epsilon > 0$, and take R to be some constant large enough s.t. $\frac{B}{R^\epsilon} \leq M$. In particular, we ask that $R > 1$. Now as is obvious from the Pythagorean theorem, for any z in the right half-plane, $|z+1| > |z|$, and so on the boundary of any half-circle with radius $r > R$, we have that

$$\left|\frac{f(z)}{(z+1)^\epsilon}\right| = \frac{|f(z)|}{|z+1|^\epsilon} \leq \frac{B}{|z|^\epsilon} = \frac{B}{r^\epsilon} \leq \frac{B}{R^\epsilon} \leq M.$$

But $|g(z)| \leq \frac{|f(z)|}{R^\epsilon} \leq |f(z)| \leq M$ on the imaginary axis, so the axis together with the large half-circle bound a region D. The maximum principle tells us that $|g(z)| \leq M$ on $\partial D \implies |g(z)| \leq M$ for all $z \in D$. Because D can be any half-disk with $r > R$, every point in the right-half plane can be in one of these D, and so $|\frac{f(z)}{(z+1)^\epsilon}| \leq M$ for all z in the right half-plane.

Finally, because $(z+1)^\epsilon$ approaches 1 as $\epsilon \to 0$, it is impossible that $|f(z)| > M$ for some z because then it would mean that for some small enough ϵ, $|\frac{f(z)}{(z+1)^\epsilon}| > M$ as well, which contradicts what we just proved. Thus, we have successfully shown that $|f(z)| \leq M$ for all z in the right half-plane.

Problem 4b - III.5.6

Let $f(z)$ be a bounded analytic function on the right half-plane. Suppose that $\limsup_{z \to iy} |f(z)| \leq M$ for all $y \in \mathbb{R}$. That is to say, for every $y \in \mathbb{R}$ and $\eta > 0$, there is $\delta_{y,\eta} > 0$ s.t. $z \in B(iy, \delta_{y,\eta}) \implies |f(z)| \leq M + \eta$. We want to show that $|f(z)| \leq M$ for all z in the right half-plane. It should be remarked that this is a technical improvement to the previous exercise, Problem 4a above.

As before, we consider the function $g(z) := \frac{f(z)}{(z+1)^\epsilon}$. We know $f(z)$ is bounded, so say $|f(z)| \leq B$ on the right half plane. Fix some small $\epsilon > 0$, and take R to be some constant large enough s.t. $\frac{B}{R^\epsilon} \leq M$. In particular, we ask that $R > 1$. Exactly like in Problem 4a, on the boundary of any half-circle with radius $r > R$, we have that

$$\left|\frac{f(z)}{(z+1)^\epsilon}\right| = \frac{|f(z)|}{|z+1|^\epsilon} \leq \frac{B}{|z|^\epsilon} = \frac{B}{r^\epsilon} \leq \frac{B}{R^\epsilon} \leq M. \qquad (1)$$

Unfortunately, we no longer have any information regarding the function ON the imaginary axis; however the limsup condition does give us a bound on the function on the line segment $S_\Delta := \{z \in \mathbb{C} : |z| \leq r, \mathrm{Re}(z) = \Delta\}$ for some small enough Δ, as follows. Let us first fix arbitrary $\eta > 0$ and radius $r > R$. Since S_0 is a compact set in the metric space $\mathbb{C}$, and $\{B(iy, \delta_{y,\eta})\}_{y \in \mathbb{R}, |y| \leq r}$ is an open cover of S_0, the Lebesgue number lemma tells us that there is some $\delta > 0$ s.t. any set of diameter $< \delta$ with non-empty intersection with S_0 (such as $B(iy, \frac{\delta}{3})$ for any $y \in \mathbb{R}, |y| \leq r$) lies within one of the sets from the open cover.

Thus, in particular we have that $\bigcup_{y\in\mathbb{R},|y|\leq r} B(iy, \frac{\delta}{3}) \subseteq \bigcup_{y\in\mathbb{R},|y|\leq r} B(iy, \delta_{y,\eta})$, and so from the definition of $\delta_{y,\eta}$, we have that $z \in \bigcup_{y\in\mathbb{R},|y|\leq r} B(iy, \frac{\delta}{3}) \implies |f(z)| \leq M + \eta$. We can now take any $\Delta \leq \frac{\delta}{4}$; then,

$$S_\Delta \subseteq \bigcup_{y\in\mathbb{R},|y|\leq r} B(iy, \tfrac{\delta}{3}), \qquad (2)$$

so $|g(z)| \leq \frac{|f(z)|}{R^\epsilon} \leq |f(z)| \leq M + \eta$ on S_Δ.

Finally, consider the domain $D_{\Delta,r} := \{z \in \mathbb{C} : |z| < r, \mathrm{Re}(z) > \Delta\}$. We just proved that $|g(z)| \leq M+\eta$ on S_Δ, and equation (1) above established that $|g(z)| \leq M \leq M + \eta$ on the boundary of any half-circle with radius r. Thus, we have that $|g(z)| \leq M + \eta$ on $\partial D_{\Delta,r}$, and because g is analytic on the right half-plane, we can use the maximum principle which tells us that $|g(z)| \leq M + \eta$ on $\partial D_{\Delta,r} \implies |g(z)| \leq M + \eta$ for all $z \in D_{\Delta,r}$. Looking at the sentence containing expression (2) from the previous paragraph, recall that we could have taken ANY $\Delta \leq \frac{\delta}{4}$; since any $z \in D_{0,r}$ is in some $D_{\Delta,r}$ for Δ small enough, this in fact proves that $z \in D_{0,r} \implies |g(z)| \leq M + \eta$. Because $\eta > 0$ was chosen arbitrarily, we get that $z \in D_{0,r} \implies |g(z)| \leq M$. Lastly, because $r > R > 1$ was chosen arbitrarily, and because any fixed z in the right half-plane lie in $D_{0,r}$ for all r sufficiently large, we have that $|g(z)| \leq M$ for all z in the right half-plane.

Exactly as we said in Problem 4a, because $(z + 1)^\epsilon$ approaches 1 as $\epsilon \to 0$, it is impossible that $|f(z)| > M$ for some z in the right half-plane because then it would mean that for some small enough ϵ, $|\frac{f(z)}{(z+1)^\epsilon}| > M$ as well, which contradicts what we just proved. Thus, we have successfully shown that $|f(z)| \leq M$ for all z in the right half-plane.

Problem 5 - III.5.9

Let D be an unbounded domain, $D \neq \mathbb{C}$, and let $u(z)$ be a real-valued harmonic function on D that extends continuously to the boundary ∂D. Suppose that $u(z)$ is bounded below on D, and that $u(z) \geq 0$ on ∂D. We want to show that $u(z) \geq 0$ on D. The hint suggests to suppose that $0 \in \partial D$ and to consider functions of the form $u(z) + \rho \log|z|$ on $D \cap \{|z| > \epsilon\}$. Note that we can not just apply the maximum principle (or in this case the minimum principle, i.e. the maximum principle for $-u(z)$) because in the book it is only proven for BOUNDED domains D.

Because $D \neq \varnothing, \mathbb{C}$, $\partial D \neq \varnothing$, so we can always translate the complex plane so that $0 \in \partial D$. By continuity of extension to the boundary, we know that for all $\epsilon > 0$, there is $\delta > 0$ s.t. $z \in D \cap B(0, \delta) \implies u(z) > u(0) - \epsilon \geq -\epsilon$. Now let us fix $\epsilon > 0$ and its corresponding $\delta > 0$. We also fix arbitrary $\rho > 0$. Then, because $u(z)$ is bounded below on D, by say B, and because $\rho \log x \to \infty$ as $x \to \infty$, any $R > 0$ large enough satisfies $u(z) + \rho \log|z| > 0$ for $|z| > R$. Then,

$$u(z) + \rho \log|z| \geq -\epsilon + \rho \log \delta$$

for $z \in \partial[D \cap \{\frac{\delta}{2} < |z| < R\}]$ because $\partial[D \cap \{\frac{\delta}{2} < |z| < R\}] \subseteq \partial D \cup \partial[\{\frac{\delta}{2} < |z| < R\}]$, and $u(z) \geq -\epsilon$ on $\{|z| = \frac{\delta}{2}\}$, $u(z) + \rho \log|z| > 0$ on $\{|z| = R\}$, $u(z) \geq 0$ on ∂D, and $\rho \log|z| \geq \rho \log \delta$ on $\{|z| \geq \delta\}$.

Then, because $u(z) + \rho \log|z|$ is a real-valued harmonic function (because u is harmonic and $\log|z|$ is harmonic, as I explained in Problem 2 above), the maximum principle tells us that $u(z) + \rho \log|z| \geq -\epsilon + \rho \log \delta$ on $D \cap \{\frac{\delta}{2} < |z| < R\}$. Because any large enough R works, we have that the lower bound holds on $D \cap \{\frac{\delta}{2} < |z|\}$. For any fixed z in this set, we have $u(z) \geq -\epsilon + \rho(\log \delta - \log|z|)$, and because $\rho > 0$ was arbitrary, we can let $\rho \to 0$, and get that $u(z) \geq -\epsilon$ for all $z \in D \cap \{\frac{\delta}{2} < |z|\}$. Because this lower bound does not depend on δ, and because we could have chosen δ arbitrarily small, we have that $u(z) \geq -\epsilon$ on $D \cap \{0 < |z|\} = D$. Finally, because $\epsilon > 0$ was arbitrary, we get that $u(z) \geq 0$ on D as desired.

Problem 6a - IV.1.2

Let γ be the unit circle $\{|z| = 1\}$, with the usual counterclockwise orientation. We want to evaluate the following integrals for $m = 0, \pm 1, \pm 2, \ldots$:

(a) $\int_\gamma z^m \, dz$: as explained in Gamelin on pg. 103, "$dz = ie^{i\theta} \, d\theta$", so we have

$$\int_\gamma z^m \, dz = \int_0^{2\pi} (e^{i\theta})^m i e^{i\theta} \, d\theta = \int_0^{2\pi} i e^{i\theta(m+1)} \, d\theta,$$

where the integral on the right is 0 unless $m = -1$, in which case it is 2π. This is actually Gamelin already did on the example spanning pg. 103-104.

(b) $\int_\gamma \overline{z}^m \, dz$: similar to part (a), we have

$$\int_\gamma \overline{z}^m \, dz = \int_0^{2\pi} (e^{-i\theta})^m i e^{i\theta} \, d\theta = \int_0^{2\pi} i e^{i\theta(1-m)} \, d\theta,$$

where the integral on the right is 0 unless $m = 1$, in which case it is 2π.

(c) $\int_\gamma z^m |\, dz|$: as Gamelin explains in the example spanning pg. 104-105, along a circle of radius R we have $|dz| = R \, d\theta$, so in our case $\int_\gamma z^m |\, dz| = \int_0^{2\pi} (e^{i\theta})^m \, d\theta$ which is 0 unless $m = 0$, in which case it is 2π.

Problem 6b - IV.1.4

For a bounded domain D with smooth boundary, we know from the definition of line integral presented on page 102:

$$\begin{aligned}\int_{\partial D} \overline{z} \, dz &= \int_{\partial D} (x - iy) \, dx + (x - iy) i \, dy = \int_{\partial D} x \, dx + y \, dy + i \int_{\partial D} -y \, dx + x \, dy \\ &= \iint_D (0 - 0) \, dA + i \iint_D (1 - (-1)) \, dA = 2i\text{Area}(D)\end{aligned}$$

where we used Green's theorem twice where obviously the P and Q in each application had continuous 1st-partials, and the domain is given to have a smooth boundary.

Problem 6b - IV.1.5

We show that

$$\left| \oint_{[|z-1|=1]} \frac{e^z}{z+1}\, dz \right| \leq 2\pi e^2.$$

We use the ML-estimate. The length L is the circumference of a circle with radius 1, i.e. 2π. Now we want to bound $|\frac{e^z}{z+1}|$ for $z \in [|z-1|=1] = \{1+e^{i\theta} : \theta \in \mathbb{R}\}$. Note that for any $z \in [|z-1|=1]$, $|e^z| = |e^{\operatorname{Re} z + i \operatorname{Im} z}| = |e^{\operatorname{Re} z}| = |e^{1+\cos\theta}|$ for some $\theta \in \mathbb{R}$, and because $\cos\theta \leq 1$ for any $\theta \in \mathbb{R}$, we see that $|e^z| \leq e^2$ for the z we are concerned with. For the denominator, $|z+1| \geq \operatorname{Re}(z+1) = 1+\operatorname{Re}(z) = 1+1+\cos\theta$, again for some $\theta \in \mathbb{R}$. Because $\cos\theta \geq -1$ for all $\theta \in \mathbb{R}$, we see that $|z+1| \geq 1$ for any $z \in [|z-1|=1]$. Thus, $|\frac{e^z}{z+1}| \leq e^2 =: M$ for $z \in [|z-1|=1]$, so the ML-estimate gives that indeed the above integral is $\leq 2\pi e^2$.

534 Homework 3

Daniel Rui - 4/21/20

Problem 1 - I.3.6

We define the chordal distance $d(z, w)$ between two points $z, w \in \hat{\mathbb{C}}$ to be the length of the straight line segment joining the points P, Q on the unit sphere whose stereographic projections are z, w respectively.

(a) We want to to show that d is a metric on $\hat{\mathbb{C}}$. In essence, this comes from the fact that the 2-norm in $\mathbb{R}^3$ restricted to the sphere is a metric:

- Zero: $d(z, w) = 0 \iff |P - Q| = 0 \iff P = Q \iff z = w$ where the last $\iff$ comes from the fact that stereographic projection is a bijection between the sphere and $\hat{\mathbb{C}}$.
- Symmetry: $d(z, w) = |P - Q| = |Q - P| = d(z, w)$.
- Triangle inequality: for any $m \in \hat{\mathbb{C}}$ with corresponding point R on the sphere, we have $d(z, w) = |P - Q| \leq |P - R| + |R - Q| = d(z, m) + d(m, w)$.

(b) Recall from pg. 12 of Gamelin that the point $z = x + iy \in \mathbb{C}$ corresponds to $(X, Y, Z) = (\frac{2x}{|z|^2+1}, \frac{2y}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1})$ on the sphere, so for $w = u + iv \in \mathbb{C}$ with corresponding point (U, V, W),

$$\begin{aligned} d(z, w) &= \sqrt{(X - U)^2 + (Y - V)^2 + (Z - W)^2} \\ &= \sqrt{(X^2 + Y^2 + Z^2) - 2(XU + YV + ZW) + (U^2 + V^2 + W^2)} \\ &= \sqrt{2 - 2\left(\frac{4xu + 4yv + (|z|^2 - 1)(|w|^2 - 1)}{(|z|^2 + 1)(|w|^2 + 1)}\right)} \\ &= \sqrt{2\left(\frac{-4xu - 4yv + 2|z|^2 + 2|w|^2}{(|z|^2 + 1)(|w|^2 + 1)}\right)} = 2\sqrt{\frac{-2xu - 2yv + |z|^2 + |w|^2}{(|z|^2 + 1)(|w|^2 + 1)}}. \end{aligned}$$

It is easy to check that the numerator of the fraction inside the square root equals $|z - w|^2$: $|z-w|^2 = (z-w)(\overline{z}-\overline{w}) = |z|^2 - w\overline{z} - \overline{w}z + |w|^2 = |z|^2 + |w|^2 - 2(w\overline{z} + \overline{w\overline{z}}) = |z|^2 + |w|^2 - 2\,\mathrm{Re}(w\overline{z}) = |z|^2 + |w|^2 - 2(xu + yv)$. Thus, for any $z, w \in \mathbb{C}$,

$$d(z, w) = \frac{2|z - w|}{\sqrt{(|z|^2 + 1)(|w|^2 + 1)}}.$$

(c) We want to find $d(z, \infty)$ for any $z \in \hat{\mathbb{C}}$. Recall that ∞ corresponds to the north pole $N = (0, 0, 1)$ on the sphere. Using the formula referenced in part (b), we have

$$d(z, \infty) = \sqrt{X^2 + Y^2 + (Z - 1)^2} = \sqrt{X^2 + Y^2 + Z^2 - 2Z + 1} = \sqrt{2 - 2Z} = \frac{2}{\sqrt{|z|^2 + 1}}$$

Problem 2a - II.7.3

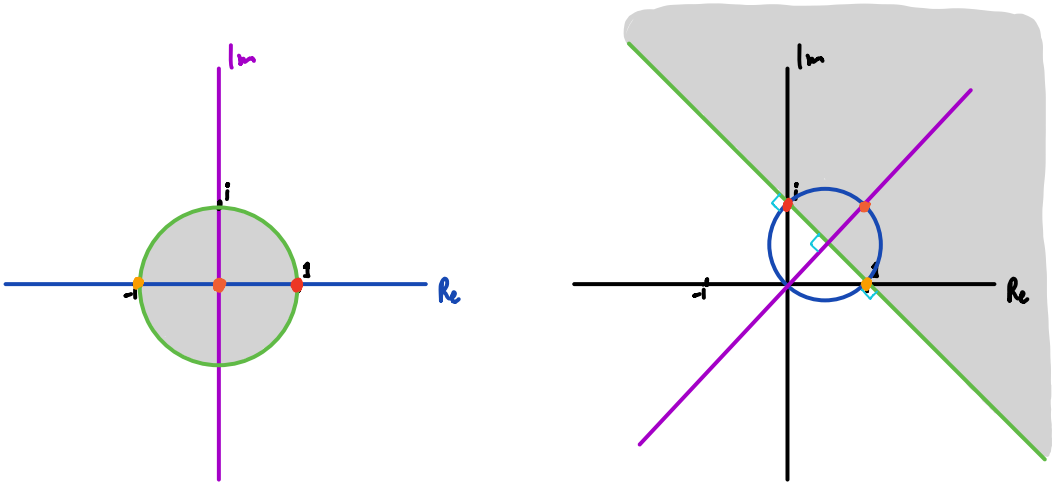
Consider the LFT f that maps $1 \mapsto i$, $0 \mapsto 1+i$, and $-1 \mapsto 1$. We want to determine the image of the unit circle $C := \{|z| = 1\}$, the image of the open unit disk $D := \{|z| < 1\}$, and the image of the imaginary axis. We include a sketch.

We use the technique described on pg. 66-67 of Gamelin. As Gamelin says on pg. 66, "to determine the image of a circle under a LFT, it suffices to determine the images of three points on that circle (since 3 points determine a circle and LFTs map circles to circles, the image of the circle is the circle passing through the three image points), and moreover LFTs map orthogonal circles to orthogonal circles (by conformality)".

First, we see that the real line (the "circle" determined by the points $-1, 0, 1$) is mapped to the circle determined by the points $1, 1+i, i$ (see the dark blue line in the following pictures). The circle centered at $\frac{1}{2}+\frac{1}{2}i$ with radius $\frac{\sqrt{2}}{2}$ does in fact contain the points $1, 1+i, i$, and also 0; we'll denote this $f(\hat{\mathbb{R}})$.

The unit circle is orthogonal to the real line at -1 and 1 (the yellow and red dots in the picture), and so the image of the unit circle C must be a circle, orthogonal to the circle $f(\hat{\mathbb{R}})$ at the points 1 and i. The only such circle is in fact the straight line between 1 and i, i.e. the line $\mathrm{Re}(z) + \mathrm{Im}(z) = 1$. Because the image of C is this line, the image of the open unit disk D does not cross this line, and so it must be either below or above this line. Because $0 \in D$ is mapped to $1+i$ which is above the line, D must be mapped to the "half" of the plane above the line $\mathrm{Re}(z) + \mathrm{Im}(z) = 1$.

Finally, the imaginary axis is orthogonal to the unit circle, and hence the image of the imaginary axis must be orthogonal to the line $\mathrm{Re}(z) + \mathrm{Im}(z) = 1$. It must also pass through the point $f(0) = 1+i$, and the only circle that passes through $1+i$ that is orthogonal to $\mathrm{Re}(z) + \mathrm{Im}(z) = 1$ is in fact the straight line that passes through 0 and $1+i$, i.e. the line $\mathrm{Re}(z) = \mathrm{Im}(z)$.



Problem 2b - II.7.9

We want to show that the LFTs that are real on the real axis are precisely those that can be expressed in the form $\frac{az+b}{cz+d}$ for $a, b, c, d \in \mathbb{R}$. Suppose that $f(z) = \frac{az+b}{cz+d}$ sends the extended real line to the extended real line (we don't know yet if it's surjective). Well, first we can make a a real number (because we can always multiply all coefficients by a constant and keep the LFT the same). We now have three equations:

$$f(0) = \frac{b}{d} \quad \text{and} \quad f(\infty) = \frac{a}{c} \quad \text{and} \quad f(1) = \frac{a+b}{c+d}$$

Case 1: all non-zero. If $c \neq 0$ and $d \neq 0$ and $c + d \neq 0$, we have that $\frac{b}{d} = r_1 \in \mathbb{R} \iff b = r_1 d$, and similarly $a = r_2 c$ and $a + b = r_3(c + d)$. First, we fixed a as a real number, so c is also a real number. Thus this last equation can be written:

$$a - r_3 c = r_3 d - r_1 d \implies d = \frac{a - r_3 c}{r_3 - r_1}$$

where $r_1 \neq r_3$ because if it did, then $\frac{a+b}{c+d} = \frac{b}{d} \implies da + bd = bc + bd \implies ad - bc = 0$ which is forbidden. Thus d is real and then b is real.

Case 2: some zero. Only one of the three terms can be infinite (because LFTs are one-to-one), so three further cases:

- $d = 0$, and $c \neq 0$ and $c + d \neq 0$. Again, because we fixed a to be real, c is real. d is obviously real, so now the third equation becomes $a + b = r_3 c \implies b = r_3 c - a$ and so b is real too.
- $c = 0$, $d \neq 0$ and $c + d \neq 0$. We still have $b = r_1 d$, and $a + b = r_3 d$. a is fixed to be real, and $c = 0$ is real. Substituting the first equation into the third we get $a + r_1 d = r_3 d \implies d = \frac{a}{r_3 - r_1}$ where again $r_3 - r_1 \neq 0$. Thus d is real and so b is real too.
- $c + d = 0$, $c \neq 0, d \neq 0$. a is still real implying that c is real, and $b = r_1 d$. But $d = -c$ so d is real, and therefore b is real too.

Conversely, if a, b, c, d are all real, then it's very easy to see that the LFT takes $\mathbb{R} \cup \{\infty\}$ to $\mathbb{R} \cup \{\infty\}$ (we can split up by cases; if $c = 0$ then we have just f is a linear function, so $f(\mathbb{R}) = \mathbb{R}$ and $f(\infty) = \infty$; if $c \neq 0$, then we can set $f(-\frac{d}{c}) = \infty$ and $f(\infty) = \frac{a}{c}$ and everything else accordingly).

Problem 4 - III.1.8

We prove Green's theorem for the rectangle defined by $x_0 < x < x_1$ and $y_0 < y < y_1$ first directly, then using the result for triangles. The statement of Green's theorem for such a rectangle is as follows: for $R = \{(x, y) \subseteq \mathbb{R}^2 : x \in (x_0, x_1), y \in (y_0, y_1)\}$, we have that for $P, Q \in \mathcal{C}^1$ on $\overline{R} = R \cup \partial R$ (i.e. there is some open set containing $\overline{R}$ on which P, Q are defined and $\mathcal{C}^1$), then the two integrals

$$\begin{aligned}\int_{\partial R} P\,dx + Q\,dy &= \int_{x_0}^{x_1} P(x, y_0)\,dx + \int_{y_0}^{y_1} Q(x_1, y)\,dy + \int_{x_1}^{x_0} P(x, y_1)\,dx + \int_{y_1}^{y_0} Q(x_0, y)\,dy \\ &= \int_{x_0}^{x_1} P(x, y_0) - P(x, y_1)\,dx + \int_{y_0}^{y_1} Q(x_1, y) - Q(x_0, y)\,dy\end{aligned}$$

(where the path we took for ∂R started at (x_0, y_0) and walked counterclockwise around) and

$$\iint_R (\partial_x Q - \partial_y P)\, dx\, dy = \int_{y_0}^{y_1} \int_{x_0}^{x_1} (\partial_x Q - \partial_y P)\, dx\, dy$$

are equal. Because Q, P are $\mathcal{C}^1$ on $\overline{R}$, the first partials are continuous on $\overline{R}$, hence L^1-integrable (continuous on a compact set implies bounded), Fubini and the "undergraduate" fundamental theorem of calculus gives that

$$\begin{aligned}\int_{y_0}^{y_1} \int_{x_0}^{x_1} (\partial_x Q - \partial_y P)\, dx\, dy &= \int_{y_0}^{y_1} \int_{x_0}^{x_1} \partial_x Q\, dx\, dy - \int_{x_0}^{x_1} \int_{y_0}^{y_1} \partial_y P\, dy\, dx \\ &= \int_{y_0}^{y_1} Q(x_1, y) - Q(x_0, y)\, dy - \int_{x_0}^{x_1} P(x, y_1) - P(x, y_0)\, dx,\end{aligned}$$

which is exactly the expression we found above for $\int_{\partial R} P\, dx + Q\, dy$. Now assuming Green's theorem for triangles, let T_1 be the triangle defined by the vertices $\{(x_0, y_0), (x_0, y_1), (x_1, y_1)\}$ and let T_2 be the be the triangle defined by the vertices $\{(x_0, y_0), (x_1, y_1), (x_1, y_0)\}$. Let D be the diagonal shared edge of $\partial T_1, \partial T_2$ oriented right-upwards, i.e. the line segment going from (x_0, y_0) to (x_1, y_1). As explicitly stated by Gamelin on pg. 72, traversing D in the opposite orientation makes the integral negative, so we have the following (traversing each triangle in a counterclockwise fashion) from Green's theorem for triangles:

$$\iint_{T_1} (\partial_x Q - \partial_y P)\, dx\, dy = \int_{\partial T_1} P\, dx + Q\, dy = \int_D P\, dx + \, dy + \int_{x_1}^{x_0} P(x, y_1)\, dx + \int_{y_1}^{y_0} Q(x_0, y)\, dy$$

and

$$\iint_{T_2} (\partial_x Q - \partial_y P)\, dx\, dy = \int_{\partial T_2} P\, dx + Q\, dy = -\int_D P\, dx + \, dy + \int_{x_0}^{x_1} P(x, y_0)\, dx + \int_{y_0}^{y_1} Q(x_1, y)\, dy.$$

But because T_1, T_2 are open and disjoint and their union equals R (up to a null set w.r.t. standard $\mathbb{R}^2$-Lebesgue measure, namely the straight line segment between (x_0, y_0) and (x_1, y_1)), summing the two integrals above yields that

$$\iint_R (\partial_x Q - \partial_y P)\, dx\, dy = \int_{x_1}^{x_0} P(x, y_1)\, dx + \int_{y_1}^{y_0} Q(x_0, y)\, dy + \int_{x_0}^{x_1} P(x, y_0)\, dx + \int_{y_0}^{y_1} Q(x_1, y)\, dy,$$

where the RHS is exactly the expression for $\int_{\partial R} P\, dx + Q\, dy$ we wrote at the very beginning of this question.

Problem 5a - III.2.1

Recall that for $h(x, y)$ is $\mathcal{C}^1$, the differential $dh = \partial_x h\, dx + \partial_y h\, dy$. A differential $P\, dx + Q\, dy$ being exact means that $P\, dx + Q\, dy = \, dh$ for some $\mathcal{C}^1$ function $h(x, y)$. Moreover Gamelin showed on pg. 77 that for continuous P, Q and a domain D, $\int P\, dx + Q\, dy$ is independent of path in D if and only if $P\, dx + Q\, dy$ is exact. Lastly, differential being closed means that $\partial_y P = \partial_x Q$ (i.e. the integrand in Green's theorem is 0). We want to determine whether the following line integrals are independent of

path:

(a) $x\,dx + y\,dy$. First we do a sanity check to see if $\partial_y P = \partial_x Q$; if not, than the differential is not even closed. Well, in this case, $\partial_y P = 0 = \partial_x Q$, so at least the differential is closed. Anti-differentiating, we see that if h is s.t. dh equals this differential, then $h(x,y) = \frac{1}{2}x^2 + f(y)$ and $h(x,y) = \frac{1}{2}y^2 + g(x)$; thus taking $h(x,y) = \frac{1}{2}(x^2+y^2)$ we do in fact see that $dh = x\,dx + y\,dy$.

(b) $x^2\,dx + y^5\,dy$. The sanity check again reveals $\partial_y P = 0\partial_x Q$, and in fact similar logic with anti-differentiating like in part (a) yields quickly that $h(x,y) = \frac{1}{3}x^3 + \frac{1}{6}y^6$ satisfies $dh = x^2\,dx + y^5\,dy$.

(c) $y\,dx + x\,dy$. The sanity check reveals $\partial_y P = 1 = \partial_x Q$; again with the anti-differentiation, $h(x,y) = xy$ satisfies $dh = y\,dx + x\,dy$.

(d) $y\,dx - x\,dy$. The sanity check reveals $\partial_y P = 1 \neq -1 = \partial_x Q$, and so this differential is not closed and hence there must be some path γ around which the integral is not zero. I claim that in fact the unit circle is one such path; to verify this, note that since P, Q are $\mathcal{C}^1$ on $\mathbb{C}$, and the unit disk D is a bounded domain with ∂D a smooth closed curve, we can use Green's theorem (pg. 73) which tells us that $\oint_{\partial D} y\,dx - x\,dy = \iint_D -2\,dx\,dy = -2\pi \neq 0$, as desired.

Problem 5b - III.2.2

We want to show that the differential $\frac{-y\,dx+x\,dy}{x^2+y^2}$ for $(x,y) \neq (0,0)$ is closed, but not independent of path on any annulus centered at 0. To verify closedness, we just perform the following calculation:

$$\partial_y P = \frac{(x^2+y^2)(-1) - (-y)(2y)}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2} = \frac{(x^2+y^2)(1) - (x)(2x)}{(x^2+y^2)^2} = \partial_x Q.$$

Now for any annulus $\{\mathbf{x} \in \mathbb{R}^2 : r_1 < |\mathbf{x}| < r_2\}$, we show that the integral over the circle $C_r := |\mathbf{x}| = r$ for $r \in (r_1, r_2)$ is non-zero. Note that $\oint_{C_r} \frac{-y\,dx+x\,dy}{x^2+y^2} = \oint_{C_r} \frac{-y\,dx+x\,dy}{r^2} = \frac{1}{r^2}\oint_{C_r} -y\,dx + x\,dy$. Since P, Q are $\mathcal{C}^1$ on $\mathbb{C}$, and the disk $D_r = \{\mathbf{x} \in \mathbb{R} : |\mathbf{x}| < r\}$ is a bounded domain with $\partial D_r = C_r$ a smooth closed curve, we can use Green's theorem (pg. 73) which tells us that $\frac{1}{r}\oint_{C_r} -y\,dx + x\,dy = \frac{1}{r^2}\iint_{D_r}(1-(-1))\,dx\,dy = \frac{1}{r^2}2\pi r^2 = 2\pi \neq 0$, as desired.

Problem 6a - III.2.3

Let P, Q be smooth (so $\mathcal{C}^1$) functions on the annulus $A_{a,b} := \{\mathbf{x} \in \mathbb{R}^2 : a < |\mathbf{x}| < b\}$ that satisfy $\partial_y P = \partial_x Q$. We show directly using Green's theorem that $\oint_{C_r} P\,dx + Q\,dy$ (where we are borrowing the notation D_r and C_r from Problem 5b above) is independent of the radius $r \in (a,b)$. For any $r_1, r_2 \in (a,b)$, $r_1 < r_2$, we have that A_{r_1,r_2} is a bounded domain with boundary consisting of two disjoint smooth closed curves, C_{r_2} (oriented counter-clockwise, i.e. positively) and C_{r_1} (oriented clockwise, i.e. negatively), and moreover P, Q are $\mathcal{C}^1$ on A_{r_1,r_2} so Green's theorem tells us that $\oint_{C_{r_2}} P\,dx + Q\,dy - \oint_{C_{r_1}} P\,dx + Q\,dy = \iint_{A_{r_1,r_2}} \partial_x Q - \partial_y P\,dx\,dy = 0$, implying that $\oint_{C_{r_2}} P\,dx + Q\,dy = \oint_{C_{r_1}} P\,dx + Q\,dy$ for any $r_1, r_2 \in (a,b)$ as desired.

Problem 6b - III.2.4

Let P, Q be smooth (so $\mathcal{C}^1$) functions on some domain D that satisfy $\partial_y P = \partial_x Q$. Let γ_0, γ_1 be two closed paths in D s.t. the straight line segment from $\gamma_0(t)$ to $\gamma_1(t)$ lies in D for every parameter value t in say $[0,1]$. Then in fact γ_0 can be continuously deformed to γ_1, by taking $\gamma_s(t) = (1-s)\gamma_0(t) + s\gamma_1(t)$ (it is clear that $\gamma_s(t)$ depends continuously on s and t — it may help to recall pg. 70 that Gamelin uses path to mean continuous function); then from the theorem on the bottom of pg. 81, we know that $\oint_{\gamma_0} P\,dx + Q\,dy = \oint_{\gamma_1} P\,dx + Q\,dy$. We are asked to use this fact in an alternate proof of the problem from Problem 6a above.

Well again on $A_{a,b}$ for any $r_1, r_2 \in (a,b)$, the closed paths γ_0, γ_1 corresponding to the circles C_{r_1}, C_{r_2} (so $\gamma_0(t) = r_1(\cos 2\pi t, \sin 2\pi t)$ and $\gamma_1(t) = r_2(\cos 2\pi t, \sin 2\pi t)$) do in fact satisfy that for every fixed $t \in [0,1]$, the straight line between $\gamma_0(t)$ and $\gamma_1(t)$ is contained in $A_{a,b}$ (since the straight line $\{r(\cos 2\pi t, \sin 2\pi t) : r \in (r_1, r_2)\}$ is in fact contained in $A_{a,b}$ since $|r(\cos 2\pi t, \sin 2\pi t)| = r \in (r_1, r_2) \subseteq (a,b)$), meaning that by the above paragraph, $\oint_{C_{r_1}} P\,dx + Q\,dy = \oint_{C_{r_2}} P\,dx + Q\,dy$ as desired.

534 Homework 2

Daniel Rui - 4/14/21

Problem 1a - II.3.3

We want to show that if $f, \overline{f}$ are both analytic on a domain (recall from pg. 38 a domain is an open set s.t. any two points can be connected by a broekn line segment) D (recall that by "analytic" we mean $\mathcal{C}^1$ on D), then f is constant. The last theorem in this section in the book (pg. 50) tells us that if a function is analytic and real-valued on a domain D, then that function is constant. Well, here we are given that f and $\overline{f}$ are analytic on D, which means that $\frac{f+\overline{f}}{2} = \operatorname{Re} f$ and $\frac{f-\overline{f}}{2i} = \operatorname{Im} f$ are both analytic on D as well (just sums and scalar multiplication of analytic functions). $\operatorname{Re} f$ and $\operatorname{Im} f$ are therefore real-valued and analytic on D, hence constant on D, which of course means that $f = \operatorname{Re} f + i \operatorname{Im} f$ is also constant on D.

Problem 1b - II.3.4

We want to show that if f is analytic on a domain D and if $|f|$ is constant, then f is constant. As the hint says, we can write $\overline{f} = |f|^2/f$ but only if $f \neq 0$ everywhere on D. If this is the case, then $|f|^2/f$ is analytic ($|f|$ is a constant so analytic, and f is never 0). As per Problem 1a above, we have f and $\overline{f}$ both analytic, so f is constant.

If however f does equal 0 somewhere, then $|f|$ equals 0 there as well, but because $|f|$ is constant on D, $|f|$ is 0 everywhere on D, which forces $f = 0$ everywhere on D as well (thereby being constant).

Problem 3 - II.5.3

Define $u(z) = \operatorname{Im}(\frac{1}{z^2}) = \operatorname{Im}(\frac{(\overline{z})^2}{|z|^4}) = \frac{-2xy}{(x^2+y^2)^2}$ for $z \neq 0$ and set $u(0) = 0$.

(a) We want to show that all partial derivatives of u w.r.t. x exist at all points of $\mathbb{C}$ (and also those w.r.t. y). We will prove the claim for x, and the claim for y will follow by symmetry (swapping x and y in the definition of $u(z)$ leaves it unchanged). We first prove by induction that the nth x-partial $\partial_x^n u$ evaluated at $(x, y) \neq (0, 0)$ is equal to $\frac{y \cdot p(x,y)}{(x^2+y^2)^{n+2}}$ for some polynomial p (such a quotient of polynomials is always differentiable for $(x, y) \neq (0, 0)$). For the base case $n = 1$, we have that

$$[\partial_x u](x, y) = \frac{(x^2 + y^2)^2(-2y) - (-2xy)2(2x)(x^2 + y^2)}{(x^2 + y^2)^4} = \frac{y \cdot (-2(x^2 + y^2) + 8x^2)}{(x^2 + y^2)^3}$$

which indeed fits the pattern, and assuming the claim for $n = N$, we have

$$\begin{aligned}[\partial_x^{N+1} u](x, y) &= \frac{(x^2 + y^2)^{N+2} y[\partial_x p](x, y) - yp(x, y)(N + 2)(2x)(x^2 + y^2)^{N+1}}{(x^2 + y^2)^{2N+4}} \\ &= \frac{y \cdot \Big((x^2 + y^2)[\partial_x p](x, y) - p(x, y)(N + 2)(2x)\Big)}{(x^2 + y^2)^{N+3}}.\end{aligned}$$

This expression indeed again fits the pattern, so we are done with the induction. All that's left is to observe that for u, the presence of y in the numerator means that for all $(0, y)$, $u(0, y) = 0$ and so $\partial_x u$ at $(0, 0)$ is 0. We can again use induction to see that for all $n \in \mathbb{N}$, $[\partial_x^n u](0, 0) = 0$ (we already did the base case, and assuming we prove the claim for $n = N$, we know that $[\partial_x^N u](0, 0) = 0$ and also the presence of y in the numerator gives that $[\partial_x^N u](0, y) = 0$ for all $y \neq 0$, meaning that indeed $[\partial_x^{N+1} u](0, 0) = 0$).

(b) We want to show that $\partial_{xx} u + \partial_{yy} u = 0$. We have from above that $[\partial_x u](x, y) = \frac{y \cdot (6x^2 - 2y^2)}{(x^2+y^2)^3}$, so using the above induction formula we have

$$\begin{aligned}[\partial_{xx} u](x, y) &= \frac{y \cdot \Big((x^2 + y^2)(12x) - (6x^2 - 2y^2)(3)(2x)\Big)}{(x^2 + y^2)^4} \\ &= 12yx \frac{x^2 + y^2 - 3x^2 + y^2}{(x^2 + y^2)^4} = 24yx \frac{y^2 - x^2}{(x^2 + y^2)^4}.\end{aligned}$$

By symmetry, we have that $[\partial_{yy} u](x, y) = 24xy \frac{x^2 - y^2}{(x^2+y^2)^4}$, and so indeed $\partial_{xx} u + \partial_{yy} u = 0$ (the above computations were for $(x, y) \neq (0, 0)$; for $(x, y) = (0, 0)$, we know from part (a) that both derivatives are 0).

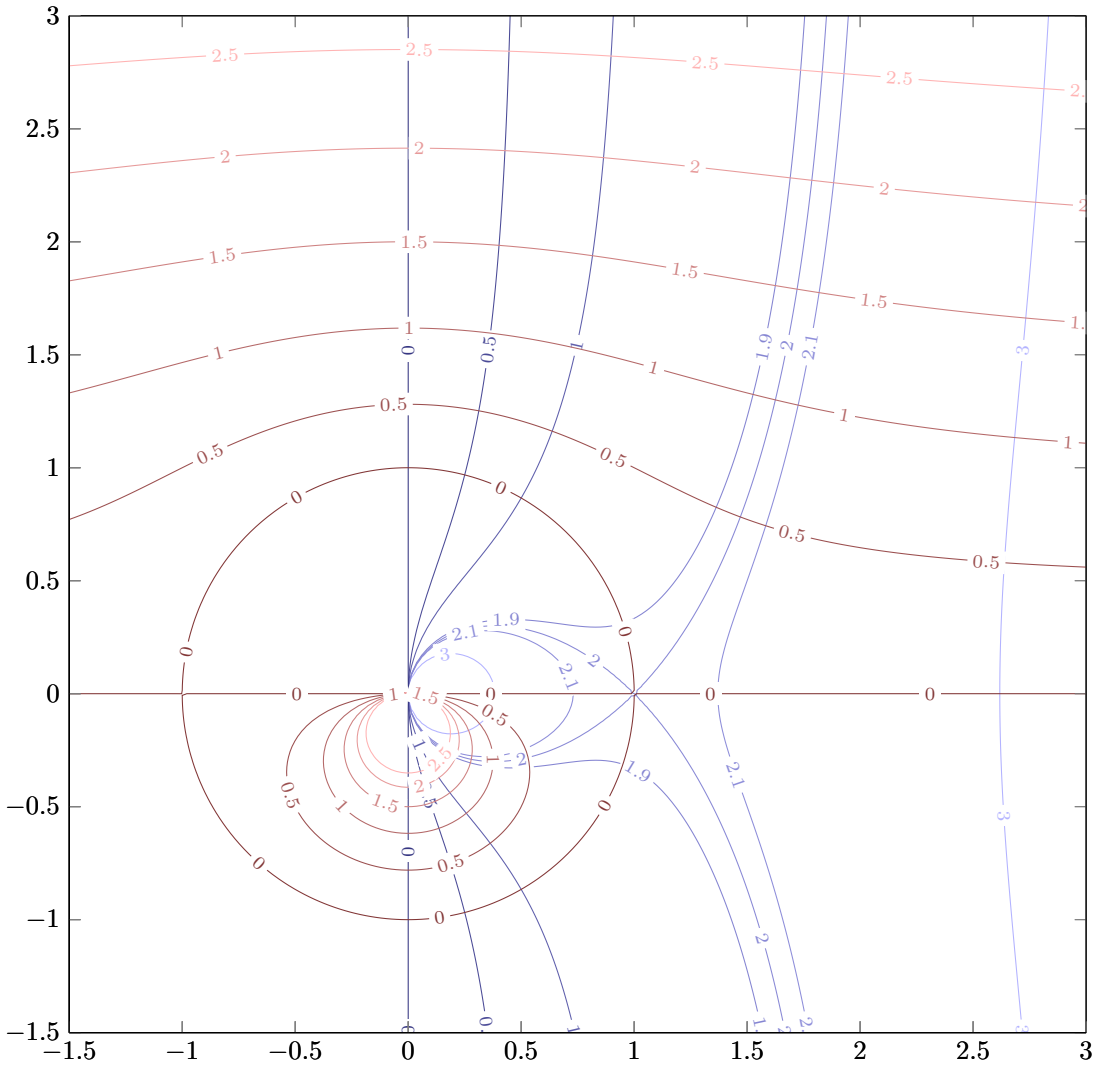
(c) We want to show that u is not harmonic on $\mathbb{C}$. It suffices to show that u is not continuous at 0; approaching $(0, 0)$ by (λ, λ) as $\lambda \to 0^+$, we see that $u(\lambda, \lambda) = \frac{-2\lambda^2}{(2\lambda^2)^2} = \frac{-1}{2\lambda^2}$ approaches $-\infty$ as $\lambda \to 0^+$. This disagrees of course with $u(0, 0) = 0$, and so u is not even continuous at 0, much less differentiable.

(d) We show that $\partial_{yx} u$ does not exist at $(0, 0)$. Well by symmetry and the above computation of $\partial_x u$ we know that $[\partial_y u](x, y) = \frac{x \cdot (6y^2 - 2x^2)}{(x^2+y^2)^3}$ (and of course at $(0, 0)$ it's 0). That means that for $y = 0$, we have that as $x \to 0^+$, $[\partial_y u](x, 0) = \frac{-2x^3}{x^6} = -\frac{2}{x^3}$ approaches $-\infty$, which means that the x-partial derivative of $\partial_y u$ doesn't exist at $(0, 0)$, which is exactly the desired claim.

Problem 4 - II.6.7

For the function $f(z) = z + \frac{1}{z} = u + iv$, we want to visualize the families of level curves of u and v, and use those to determine the images of $f(z)$ of the top D^+ & bottom D^- half of unit disk, and the upper U & lower L half-plane outside of the unit disk. Observe that $f(x, y) = x + iy + \frac{x - iy}{x^2+y^2}$, so $u(x, y) = x + \frac{x}{x^2+y^2}$ and $v(x, y) = y - \frac{y}{x^2+y^2}$.

I have plotted the level curves of u and v on the next page (blue is u and red is v). I only plotted level curves $= c$ for $c > 0$, but for $c < 0$ we have by symmetry that it's just the $= |c|$ curve flipped over the imaginary or real axis respectively. From the picture, we see that the image of D^- via f is the upper half plane (the level curves $v = c$ for $c > 0$ "fill" D^- — and also U — and the level curves of u "scan through" those of v). Thus D^- and U are both mapped to the upper half plane H^+, and by symmetry D^+ and L are mapped to the lower half plane H^-. We provide a more rigorous treatment of these facts below the picture.



We prove rigorously that f is a bijective map from D^- or U to H^+. Let $z_0 \in H^+$. Then we want to see for what $z \in \mathbb{C}$ it is true that $f(z) = z + \frac{1}{z} = x_0 + iy_0$. Well, $f(z) = z_0 \iff z^2 - z_0 z + 1 = 0$, and we know from the fundamental theorem of algebra (and previous experience in algebra from grade school) that quadratics either have 1 root of multiplicity two (iff the discriminant $b^2 - 4ac$ or in our case $z_0^2 - 4$ equals 0), or 2 distinct roots. Because the roots of $z^2 - 4 = 0$ are exactly ± 2, and we chose $z_0 \in H^+$, all our cases the quadratic has 2 distinct roots. Note that the level curve $v = 0$ is exactly the real axis and the unit circle, so both roots of the quadratic must be in $D^+ \cup D^- \cup U \cup L$.

Note that by Vieta's formulas, the product of the roots is 1, i.e. they are each other's inverses. Well we know that $z \mapsto \frac{1}{z}$ maps $re^{i\theta} \mapsto \frac{1}{r}e^{i(-\theta)}$ so we know if one root is in D^-, then the other must be in U (and if one is in D^+, the other must be in L). The following argument proves that it must be the D^-, U case and not the D^+, L case: we know that the sum of the roots must be $z_0 \in H^+$, but for any $r = x + iy \in D^+$, we have $\mathrm{Im}(r + \frac{1}{r}) = y - \frac{y}{|r|}$, but because $y > 0$ and $|r| < 1$, this value is negative and hence $r + \frac{1}{r} \in H^-$.

Thus, this shows that for any $z_0 \in H^+$, there is a unique point in D^- and another unique point in U s.t. f evaluated at that point is z_0. This shows that f is indeed a bijection from $D^- \to H^+$ and also from $U \to H^+$, and by symmetry as I said earlier it is a bijection from $D^+ \to H^-$ and $L \to H^-$.

Problem 5a - I.3.2

We want to show that if the point P on the sphere corresponds to z under stereographic projection, then the antipodal point $-P$ on the sphere corresponds to $-\frac{1}{\overline{z}} = -\frac{z}{|z|^2}$. Let us say that the point $P = (X, Y, Z)$. The formula for stereographic projection on pg. 12 of Gamelin tell us the corresponding point of $\mathbb{C}$ is $z := \frac{X}{1-Z} + i\frac{Y}{1-Z}$, which has norm-squared equal to $\frac{X^2+Y^2}{(1-Z)^2} = \frac{1-Z^2}{(1-Z)^2} = \frac{1+Z}{1-Z}$, implying that $-\frac{z}{|z|^2} = \frac{-X}{1+Z} + i\frac{-Y}{1+Z}$. The formula on pg. 12 tells us that the corresponding point of $-P = (-X, -Y, -Z)$ is indeed $\frac{-X}{1+Z} + i\frac{-Y}{1+Z}$.

Problem 5b - I.3.4

We want to show that a rotation of the sphere of 180° about the X-axis corresponds under stereographic projection to the inversion $z \mapsto \frac{1}{z}$ of $\mathbb{C}$. The formula for the rotation of 180° about the X-axis is $(X, Y, Z) \mapsto (X, -Y, -Z)$. So we just need to show that if $z \in \mathbb{C}$ corresponds to (X, Y, Z), then $\frac{1}{z} = \frac{\overline{z}}{|z|^2}$ corresponds with $(X, -Y, -Z)$.

The formula from pg. 12 gives that (X, Y, Z) corresponds with $z := \frac{X}{1-Z} + i\frac{Y}{1-Z}$, and that $(X, -Y, -Z)$ corresponds with $\frac{X}{1+Z} - i\frac{Y}{1+Z}$; indeed we have that

$$\frac{\overline{z}}{|z|^2} = \frac{\frac{X}{1-Z} + i\frac{Y}{1-Z}}{\frac{X^2+Y^2}{(1-Z)^2}} = \frac{X}{1-Z} + i\frac{Y}{1-Z} \cdot \frac{1-Z}{1+Z} = \frac{X}{1+Z} - i\frac{Y}{1+Z},$$

as desired.

Problem 6 - II.7.12

We classify the conjugacy classes of LFTs (linear fractional transformations) by establishing the following:

(a) an non-identity LFT $f(z)$ has either one or two fixed points. We want to see what $z \in \hat{\mathbb{C}}$ are s.t. $f(z) = \frac{az+b}{cz+d} = z \iff az + b = cz^2 + dz \iff cz^2 + (d-a)z - b = 0$. We know from the factor theorem (over any integral domain!) from algebra class that non-zero degree 2 polynomials have ≤ 2 roots (if the quadratic was identically zero then $c = b = 0, d = a$ implying that $f(z) = z$, but we specified that $f(z)$ was a non-identity LFT). But we have to be more careful here, since ∞ could be a fixed point. Well, observe that ∞ is a fixed point if and only if $c = 0$, since $c \neq 0 \implies f(\infty) = \frac{a}{c}$ and $c = 0 \implies f(\infty) = \infty$, so our above argument about roots of a quadratic in $\mathbb{C}$ gives that $c \neq 0 \implies \leq 2$ roots, and $c = 0 \implies \leq 1$ fixed point in $\mathbb{C}$ (degree 1 polynomial has ≤ 1 root) and ∞ as a fixed point.

Hence, in all cases we have ≤ 2 fixed points; and in both cases we have ≥ 1 fixed points: in the first case $c \neq 0$, the fundamental theorem of algebra guarantees that a non-constant polynomial has a root, and in the second case $c = 0$ we always have ∞ is a fixed point.

(b) an LFT $f(z)$ with exactly two fixed points is conjugate to a dilation $z \mapsto \lambda z$ for $\lambda \in \mathbb{C}$ s.t. $\lambda \neq 0, 1$. That is to say, there is an LFT $h(z)$ s.t. $g := h \circ f \circ h^{-1} = [z \mapsto \lambda z]$. Let the two distinct fixed points be p_1, p_2. Then define $h(z)$ to be the LFT that maps $p_1 \mapsto 0, p_2 \mapsto \infty$; explicitly this is given by $h(x) = \frac{z-p_1}{z-p_2}$. Then, $g := h \circ f \circ h^{-1}$ (which is also an LFT because compositions of LFTs are LFTs) satisfies $0 \mapsto p_1 \mapsto p_1 \mapsto 0$ and $\infty \mapsto p_2 \mapsto p_2 \mapsto \infty$, and so writing $g(z) = \frac{\alpha z+\beta}{\gamma z+\delta}$, we see that $g(0) = 0 \implies \beta = 0$ and $g(\infty) = \infty \implies \gamma = 0$ (because if $\gamma \neq 0$, $g(\infty)$ would be $\frac{\alpha}{\gamma} \in \mathbb{C}$), and so $g(z) = \lambda z$ for some $\lambda \in \mathbb{C}$. Note that $\lambda \neq 0$ because the zero-map is not an LFT, and $\lambda \neq 1$ because then $f = h^{-1} \circ g \circ h = h^{-1} \circ h = \mathrm{id}$, contradicting that f has EXACTLY two fixed points. This proves conjugacy to a non-zero and non-identity dilation as desired.

Lastly, we want to see if λ is unique. Suppose $[z \mapsto \lambda z]$ and $[z \mapsto \lambda' z]$ are conjugate and $\lambda \neq \lambda'$. Then, we have $h \circ [z \mapsto \lambda z] = [z \mapsto \lambda' z] \iff h(\lambda z) = \lambda' h(z)$. So then $h(0) = \lambda' h(0) \implies h(0) = 0$ or ∞, and $h(\infty) = \lambda' h(\infty) \implies h(\infty) = 0$ or ∞. If we have that $h(0) = \infty, h(\infty) = 0$, writing $h(z) = \frac{\alpha z+\beta}{\gamma z+\delta}$, we see that $h(0) = \infty \implies \delta = 0$ and $h(\infty) = 0 \implies \alpha = 0$, and so $h(z) = \frac{\mu}{z}$ for some $\mu \in \mathbb{C}$. Then, we have $h(az) = \frac{\mu}{\lambda z} = \lambda' \frac{\mu}{z}$ for all $z \in \mathbb{C}$, which happens if and only if $\lambda' = \frac{1}{\lambda}$. And in fact $[z \mapsto \lambda z]$ is conjugate via $h(z) = \frac{1}{z}$ to $[z \mapsto \frac{1}{\lambda} z]$; to check this, we compute that $h \circ [z \mapsto \lambda z] \circ h^{-1}$ is indeed the map that maps $z \mapsto \frac{1}{z} \mapsto \frac{\lambda}{z} \mapsto \frac{z}{\lambda}$. Hence, no, λ is not unique since if $z \mapsto \lambda z$ is conjugate to f, then $z \mapsto \frac{1}{\lambda} z$ is also conjugate to f (we know that "is conjugate to" is an equivalence relation from algebra class).

(c) an LFT $f(z)$ with one fixed point is conjugate to the translation $z \mapsto z+1$. That is to say, there is an LFT $h(z)$ s.t. $g := h \circ f \circ h^{-1} = [z \mapsto z+1]$. Let the fixed point be p_1, and define $h(z) = \frac{1}{z-p_1}$ so that it maps $p_1 \mapsto \infty$. Then, g maps $\infty \mapsto p_1 \mapsto p_1 \mapsto \infty$, so ∞ is fixed point of g. Like I said in the first paragraph of part (b), writing $g(z) = \frac{\alpha z+\beta}{\gamma z+\delta}$, we see that $g(\infty) = \infty \implies \gamma = 0$, so $g(z) = \alpha' z + \beta'$. Note that g must also have exactly one fixed point, because if not, say it had some $p \in \mathbb{C}$ as a fixed point, then $f(h^{-1}(p)) = h^{-1}(g(h(h^{-1}(p)))) = h^{-1}(\infty)$, implying that $h^{-1}(p)$ is a fixed point of f. But f has only one fixed point, implying $h^{-1}(p) = p_1 \iff h(p_1) = p$, but contradiction because $h(p_1) = \infty \notin \mathbb{C}$!

Thus, $g(z) = \alpha' z + \beta'$ can not equal z for any $z \in \mathbb{C}$, implying that $\alpha' = 1$ and $\beta' \neq 0$ (if $\alpha' \neq 1$, we would have a fixed point $z = \frac{\beta'}{1-\alpha'}$ of g, and if $\beta' = 0$, $z = 0$ would be a fixed point of g). Thus, $g(z) = z + \beta'$ for $\beta' \neq 0$. Then, I claim that defining $h'(z) = \frac{1}{\beta'} z$, we have that $h' \circ g \circ h'^{-1}$ is the map that maps $z \mapsto \beta' z \mapsto \beta' z + \beta' \mapsto z + 1$, i.e. g is conjugate to $[z \mapsto z+1]$ via h'. Thus, again using that "is conjugate to" is an equivalence relation, we see that $f(z)$ is conjugate to $[z \mapsto z+1]$, and we are done.

Extra Credit

If $p(z)$ is a quadratic polynomial, say $az^2 + bz + c$ for $a, b, c \in \mathbb{C}$, we would like to investigate for which initial values Newton's method converges. Recall from elementary calculus on $\mathbb{R}$ that Newton's method is a root-finding technique for a function f, where we start with an initial point x_0, and iterate $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ (i.e. x_{n+1} is the intersection of the x-axis and the tangent line of f at the point x_n).

The first thing I did was write some Python that generated some nice pictures (can copy-paste to your own machine if you have Python, or run it on the web at deepnote.com/Google Colab):

```
import numpy as np
import matplotlib.pyplot as plt
import cmath

a, b, c = 1, 2, -7-6*1j

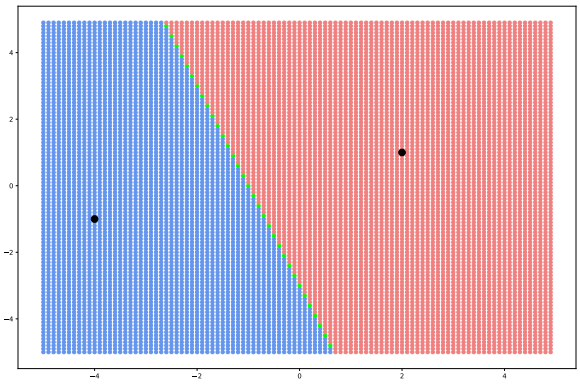
def newtonMethod(z):
    return z - (a*z*z + b*z + c) / (2.0*a*z + b)

def nIterations(n,init):
    zn = init
    for i in range(n):
        zn = newtonMethod(zn)
    return zn

r1, r2 = (-b + cmath.sqrt(b*b - 4.0*a*c)) / (2*a), (-b - cmath.sqrt(b*b -
     4.0*a*c)) / (2*a)
d = abs(r1-r2)

plt.figure(figsize=(15,10))
for x in np.arange(-5,5,0.1):
    for y in np.arange(-5,5,0.1):
        z = x + y*1j
        output = nIterations(10,z)
        if (abs(r1-output) < 0.2 * d):
            plt.plot([x], [y], marker='o', markersize=5, color="pink")
        elif (abs(r2-output) < 0.2 * d):
            plt.plot([x], [y], marker='o', markersize=5, color="olive")
        else:
            plt.plot([x], [y], marker='o', markersize=5, color="red")

plt.plot([r1.real], [r1.imag], marker='o', markersize=10, color="black")
plt.plot([r2.real], [r2.imag], marker='o', markersize=10, color="black")
plt.savefig('ec1plot.pdf')
```



Having done this computation and generated this image, we can make the very reasonable conjecture that given a quadratic polynomial $p(z) = az^2 + bz + c$ with roots r_1, r_2 (say $r_1 \neq r_2$), then for an initial point z_0, iterating yields z_n that converge to r_1 iff $|z_0 - r_1| < |z_0 - r_2|$, converge to r_2 iff $|z_0 - r_2| < |z_0 - r_2|$, and no convergence iff $|z_0 - r_1| = |z_0 - r_2|$. If $r_1 = r_2 =: r$, thinking about the limiting case of the picture we conjecture that every z_0 yields iterations z_n that converge to r.

Since the roots show up so often, it will probably be easier to work with factored form instead of $az^2 + bz + c$ form; thus, let $p(z) = a(z - r_1)(z - r_2)$ for some $a, r_1, r_2 \in \mathbb{C}$. Notice that the Newton-method update $-\frac{p(z)}{p'(z)} = -\frac{a(z-r_1)(z-r_2)}{2az-a(r_1+r_2)} = -\frac{(z-r_1)(z-r_2)}{2z-(r_1+r_2)}$ does not involve a at all, so in fact we do not need to care about it. Let us now tackle the case when $p(z) = (z - r)^2$. Well, we have that $z_{n+1} = z_n - \frac{(z_n-r)^2}{2(z_n-r)} = z_n - \frac{1}{2}(z_n - r) = \frac{1}{2}(z_n + r)$. Now we could expand this out and get that $z_n = \frac{1}{2^n}z_0 + (1 - \frac{1}{2^n})r$ and see that $z_n \to r$ that way, but it's much easier (and much more generalizable) to show that $|z_n - r| \to 0$; you see, for the recursive formula above, if we replace all the "z" terms with "$z - r$" terms, we get that $z_{n+1} - r = \frac{1}{2}(z_n - r)$, and we see immediately that $|z_{n+1} - r| = \frac{1}{2}|z_n - r| \implies |z_n - r| = \frac{1}{2^n}|z_0 - r|$, which obviously goes to 0 as $n \to \infty$.

In essense, our "$z \to z - r$" trick above shifted the problem from $p(z) = (z - r)^2$ to the easier (easiest?) case of $p(z) = z^2$; i.e. we "moved the camera" so that the root at r was now at 0. We use this trick to do the $r_1 \neq r_2$ case. The easiest case in this scenario would arguably be $p(z) = (z - 1)(z + 1)$. The Newton-method update is $z_{n+1} = z_n - \frac{z_n^2-1}{2z_n} = \frac{z_n^2+1}{2z_n} = \frac{1}{2}(z_n + \frac{1}{z_n})$. We would now like see what this formula says about the quantities $|z_n - 1|$ and $|z_n + 1|$. Letting $u_n = z_n - 1$ and $v_n = z_n + 1$, the formula becomes respectively $u_{n+1} + 1 = \frac{1}{2}(u_n + 1 + \frac{1}{u_n+1}) \iff u_{n+1} = \frac{1}{2}(\frac{u_n^2-1+1}{u_n+1}) = \frac{1}{2}(\frac{u_n^2}{u_n+1})$ and similarly $v_{n+1} = \frac{1}{2}(\frac{v_n^2}{v_n-1})$. These don't look promising immediately, but notice that the denominators are actually both z_n! So dividing, we get that $\frac{u_{n+1}}{v_{n+1}} = (\frac{u_n}{v_n})^2 \implies w_n := \frac{u_n}{v_n} = (\frac{z_0-1}{z_0+1})^{2^n}$. Therefore, $|z_0 - 1| < |z_0 + 1| \iff |w_n| \to 0 \iff |z_n - 1| \to 0 \iff z_n \to 1$; and $|z_0 + 1| < |z_0 - 1| \iff |w_n| = |1 - \frac{2}{z_n+1}| \to \infty \iff |1| + |\frac{2}{z_n+1}| \to \infty \iff |z_n + 1| \to 0 \iff z_n \to -1$; and of course $|z_0 - 1| = |z_0 + 1| \iff z_n$ do not converge.

Finally, given arbitrary $p(z) = (z - r_1)(z - r_2)$, to "move the camera" to our previous case $p(z) = (z - 1)(z + 1)$, we translate everything by $-\frac{r_1+r_2}{2}$, and do a rotation/dilation, i.e. multiplication by $1/(r_1 - (\frac{r_1+r_2}{2})) = \frac{2}{r_1-r_2}$ to bring $r_1 \mapsto 1$ and $r_2 \mapsto -1$. That is, letting $w_n = \frac{2(z_n-(r_1+r_2)/2)}{r_1-r_2} = \frac{2z_n-(r_1+r_2)}{(r_1-r_2)}$, we have that

$$z_{n+1} = \frac{z_n^2 - r_1 r_2}{2z_n - (r_1 + r_2)} = \frac{1}{4}\left(2z_n + (r_1 + r_2) + \frac{(r_1 - r_2)^2}{2z_n - (r_1 + r_2)}\right) \iff w_{n+1} = \frac{1}{2}\left(w_n + \frac{1}{w_n}\right),$$

and so we inherit the above results. The last part comes from noticing that z_n no converge $\iff w_n$ no converge, and that $|z_n - r_1| = \frac{1}{2}|w_n - 1|$ and $|z_n - r_2| = \frac{1}{2}|w_n + 1|$.

Summary: Newton's method for an initial condition z_0 and quadratic polynomial $p(z)$ converges to the closest root of p to z_0; if there's no closest root, i.e. if both roots are equally far away, then Newton's method does not converge.

534 Homework 1

Daniel Rui - 4/7/21

Problem 1

(a) Let $|a| \leq 1$ and $|b| < 1$. We want to show that $|\frac{a-b}{1-a\overline{b}}| \leq 1$. First note that the denominator can not be 0, since $|a| \leq 1, |b| < 1 \implies |a\overline{b}| < 1 \implies a\overline{b} \neq 1$. Recall that for any $z, w \in \mathbb{C}$, the conjugates of $z + w$, $z \cdot w$, and $\frac{1}{z}$ are respectively $\overline{z} + \overline{w}$, $\overline{z} \cdot \overline{w}$, and $\frac{1}{\overline{z}}$, and that $z\overline{z} = |z|^2$. Now, let us define $z = \frac{a-b}{1-a\overline{b}}$. The previous identities tells us that $\overline{z} = \frac{\overline{a}-\overline{b}}{1-\overline{a}b}$, and so

$$|z|^2 = z\overline{z} = \frac{a\overline{a} - b\overline{a} - \overline{b}a + b\overline{b}}{1 - a\overline{b} - \overline{a}b + a\overline{a}b\overline{b}} = \frac{|a|^2 - b\overline{a} - \overline{b}a + |b|^2}{1 - a\overline{b} - \overline{a}b + |a|^2|b|^2}.$$

Because a complex number times its conjugate is a square of a real number, both numerator and denominator are real and ≥ 0, and since $|a|, |b|$, and 1 are all real, this means that $-b\overline{a} - \overline{b}a$ is a real number. Thus, to show that the numerator is $\leq$ the denominator, it suffices to show that $|a|^2 + |b|^2 \leq 1 + |a|^2|b|^2 \iff 1 - |a|^2 - |b|^2 + |a|^2|b|^2 \geq 0$. Well, we have that

$$|a|^2|b|^2 - |a|^2 - |b|^2 + 1 = (1 - |a|^2)(1 - |b|^2),$$

which is obviously ≥ 0 since we assumed that $|a| \leq 1$ and $|b| < 1$. Furthermore, it is clear that it equals 0 if and only if $|a| = 1$.

To conclude, we have shown that the desired inequality is true, with equality iff $|a| = 1$.

(b) We want to calculate $(\pm\frac{1}{2} \pm i\frac{\sqrt{3}}{2})^6$ for all combinations of signs. Note that $\frac{1}{2} \pm i\frac{\sqrt{3}}{2} = \cos(\frac{\pi}{3}) + i\sin(\pm\frac{\pi}{3}) = e^{\pm i\frac{\pi}{3}}$, implying that $(\frac{1}{2} \pm i\frac{\sqrt{3}}{2})^6 = e^{\pm 2i\pi} = 1$ (references to $e^{i\theta}$ can be omitted if instead we use say de Moivre's theorem from high school, which can be derived from the angle addition formulas for sine and cosine — this is assuming we are using the Euclidean geometry definitions of sine and cosine from middle/high school). Obviously, to get the other two sixth powers, we just multiply by -1, which gives us that $(-\frac{1}{2} \pm i\frac{\sqrt{3}}{2})^6 = (-1)^6(\frac{1}{2} \mp i\frac{\sqrt{3}}{2})^6 = e^{\mp 2i\pi} = 1$. In other words, all four of the numbers $\pm\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$ raised to the sixth power yield 1 (i.e. they are 4 out of the six roots of unity, the other two being ± 1).

Problem 2

We want to prove that

$$\left|\sum_{k=1}^{n} a_k b_k\right|^2 = \sum_{k=1}^{n} |a_k|^2 \sum_{k=1}^{n} |b_k|^2 - \sum_{1 \leq j < k \leq n} |a_j\overline{b_k} - a_k\overline{b_j}|^2.$$

We proceed by induction on n. For the base case $n = 1$, it is clear that $|a_k b_k|^2 = |a_k|^2|b_k|^2$. Now suppose that we have that claim for $n = N - 1$ for some $N \in \mathbb{N}$. We would like to prove it for $n = N$.

Observe that

$$\begin{aligned}\mathrm{LHS}_N := \left|\sum_{k=1}^{N} a_k b_k\right|^2 &= \left(a_N b_N + \sum_{k=1}^{N-1} a_k b_k\right)\left(\overline{a_N b_N} + \overline{\sum_{k=1}^{N-1} a_k b_k}\right) \\ &= |a_N b_N|^2 + a_N b_N \sum_{k=1}^{N-1} \overline{a_k b_k} + \overline{a_N b_N} \sum_{k=1}^{N-1} a_k b_k + \left|\sum_{k=1}^{N-1} a_k b_k\right|^2,\end{aligned}$$

and

$$\begin{aligned}\mathrm{RHS}_N &:= \sum_{k=1}^{N} |a_k|^2 \sum_{k=1}^{N} |b_k|^2 - \sum_{1\le j<k\le N} |a_j\overline{b_k} - a_k\overline{b_j}|^2 \\ &= \sum_{k=1}^{N-1} |a_k|^2 \sum_{k=1}^{N-1} |b_k|^2 + |a_N|^2 \sum_{k=1}^{N-1} |b_k|^2 + |b_N|^2 \sum_{k=1}^{N} |a_k|^2 \\ &\quad - \sum_{1\le j<k\le N-1} |a_j\overline{b_k} - a_k\overline{b_j}|^2 - \sum_{j=1}^{N-1} |a_j\overline{b_N} - a_N\overline{b_j}|^2 \\ &= \left|\sum_{k=1}^{N-1} a_k b_k\right|^2 + |a_N|^2 \sum_{k=1}^{N-1} |b_k|^2 + |b_N|^2 \sum_{k=1}^{N} |a_k|^2 - \sum_{j=1}^{N-1} |a_j\overline{b_N} - a_N\overline{b_j}|^2,\end{aligned}$$

where

$$\begin{aligned}\sum_{j=1}^{N-1} |a_j\overline{b_N} - a_N\overline{b_j}|^2 &= \sum_{j=1}^{N-1} (a_j\overline{b_N} - a_N\overline{b_j})(\overline{a_j}b_N - \overline{a_N}b_j) \\ &= \sum_{j=1}^{N-1} (a_j\overline{b_N}\overline{a_j}b_N - a_N\overline{b_j}\overline{a_j}b_N - \overline{a_N}b_j a_j\overline{b_N} + \overline{a_N}b_j\overline{a_N}b_j) \\ &= \sum_{j=1}^{N-1} (|a_j|^2|b_N|^2 - a_N b_N\overline{a_j b_j} - \overline{a_N b_N}a_j b_j + |a_N|^2|b_j|^2) \\ &= |b_N|^2 \sum_{j=1}^{N-1} |a_j|^2 + |a_N|^2 \sum_{j=1}^{N-1} |b_j|^2 - a_N b_N \sum_{j=1}^{N-1} \overline{a_j b_j} - \overline{a_N b_N} \sum_{j=1}^{N-1} a_j b_j.\end{aligned}$$

Substituting this in our most recent expression for RHS_N, we see that

$$\mathrm{RHS}_N = \left|\sum_{k=1}^{N-1} a_k b_k\right|^2 + |b_N|^2|a_N|^2 + a_N b_N \sum_{j=1}^{N-1} \overline{a_j b_j} + \overline{a_N b_N} \sum_{j=1}^{N-1} a_j b_j,$$

which is in fact our most recent expression for LHS_N. Thus, in fact $\mathrm{LHS}_N = \mathrm{RHS}_N$, and our proof is complete.

Problem 3

Let $a = e^{2\pi i\alpha}$ with $\alpha \notin \mathbb{Q}$. We want to show that the the sequence $\{a_n\}_{n=1}^{\infty}$ is dense in the unit circle $\mathbb{T} := \{|z| = 1\}$ (obviously the sequence is contained within $\mathbb{T}$).

Well, first recall that $e^{i\theta}$ is a 2π-periodic function ($\theta \in \mathbb{R}$) that moreover doesn't repeat before the period is over (i.e. for any $\theta_0 \in \mathbb{R}$, $e^{i\theta} : [\theta_0, \theta_0 + 2\pi) \to \mathbb{C}$ is an injective function, because $e^{i\theta}$ just walks around the unit circle which does not repeat points until the next period starts, i.e. when the point $e^{i\theta_0}$ is reached again at $\theta = \theta_0 + 2\pi$). Let us call this "doesn't repeat before the period over" fact the "non-repetition property".

Now we consider the sequence $\{a_n\}_{n=1}^{\infty}$. I claim that the sequence can never repeat: if it did, say $a^n = a^m$ for some $n, m \in \mathbb{N}$ where $n \neq m$, then by 2π-periodicity and the non-repetition property, it must be that $2\pi\alpha n \equiv 2\pi\alpha m \pmod{2\pi}$, i.e. there is $k \in \mathbb{Z}$ s.t. $2\pi\alpha(n-m) = 2\pi k \iff \alpha(n-m) = k$. But $n, m, k \in \mathbb{Z}$ and $n \neq m$, implying that $\alpha = \frac{k}{n-m} \in \mathbb{Q}$; contradiction.

To prove that the sequence is dense in $\mathbb{T}$, it suffices to show that for any fixed $z_0 \in \mathbb{T}$ and $\epsilon > 0$, there is always some $a^n \in \{a^n\}_{n=1}^{\infty}$ s.t. $|a^n - z_0| < \epsilon$. Choose some $N \in \mathbb{N}$ s.t. $N < 2\pi\frac{1}{\epsilon}$ (e.g. $N = \lceil 2\pi\frac{1}{\epsilon} \rceil + 1$), and let us divide $\mathbb{T} = \{e^{2\pi i\theta} : \theta \in [0, 1)\}$ (equality holds because of 2π-periodicity of $e^{i\theta}$) into N arcs $A_1, \ldots, A_N$ say by letting $A_i := \{e^{2\pi i\theta} : \theta \in [\frac{i-1}{N}, \frac{i}{N})\}$. These arcs are all disjoint from each other by the non-repetition property.

Because the sequence never repeats, the pigeonhole principle guarantees that among $\{a_n\}_{n=1}^{N+1}$, there must be some A_i containing at least two $a^n \in \{a_n\}_{n=1}^{N+1}$, say a^m and a^n where $m < n$ and of course $m, n \in [N+1]$. Let $\eta \in [-\pi, \pi)$ be the angle (in radians) between a^m and a^n; that is to say, $\eta \equiv 2\pi\alpha(n-m) \pmod{2\pi}$ s.t. $\eta \in [-\pi, \pi)$ (in fact, $|\eta| < \frac{2\pi}{N}$ because a^n and a^m are in the same A_i) and $a^m e^{i\eta} = a^n$. Since $a^n \neq a^m$, $|\eta| > 0$. Let us now suppose w.l.o.g. that $\eta > 0$ (if it were negative, we would just go clockwise around the unit circle instead, i.e. going from A_i to A_{i-1}, etc). **We prove that** given $a^k \in A_i$, there is some amount of steps we can take s s.t. $a^k e^{s\eta} = a^{k+s(n-m)}$ s.t. $a^{k+s(n-m)} \in A_{i+1}$ (where indices of the A_i are taken modulo N).

Let us define $\theta_0 \in [0, 2\pi)$ s.t. $\theta_0 \equiv 2\pi\alpha k \pmod{2\pi}$ (i.e. the argument of a^k expressed as a number in $[0, 2\pi)$). Then, $a^k \in A_i$ means that $\theta_0 \in [2\pi\frac{i-1}{N}, 2\pi\frac{i}{N})$. Because $\eta > 0$, the sequence $\{\theta_0 + t\eta\}_{t=0}^{\infty}$ is increasing, and so it will eventually exceed $2\pi\frac{i}{N}$. Letting s be the first $t \in \mathbb{N}$ s.t. $\theta_0 + t\eta \geq 2\pi\frac{i}{N}$, I claim that $e^{\theta_0 + s\eta} = a^{k+s(n-m)} \in A_{i+1}$. This is because $\eta < \frac{2\pi}{N}$, so $\theta_0 + (s-1)\eta < 2\pi\frac{i}{N} \implies \theta_0 + s\eta < 2\pi\frac{i+1}{N}$. By definition of s, $\theta_0 + s\eta \geq 2\pi\frac{i}{N}$, and so indeed $\theta_0 + s\eta \in [2\pi\frac{i}{N}, 2\pi\frac{i+1}{N}) \iff e^{\theta_0 + s\eta} = a^{k+s(n-m)} \in A_{i+1}$, as desired.

Clearly, this argument shows that for any arc A_i, there is some $a^k \in \{a^n\}_{n=1}^{\infty}$ s.t. $a_k \in A_i$ (given $a^1 \in A_{i_0}$, we know that $a^{1+s(n-m)} \in A_{i_0+1}, a^{1+2s(n-m)} \in A_{i_0+2}$, and so on. Eventually we'll get to A_i). Letting A be the arc containing z_0 (fixed above), we know that there is some $k \in \mathbb{N}$ s.t. $a^k \in A$. The arclength along the circle between a^k and z_0 is less than the arclength of A, which is $\frac{2\pi}{N}$, which recall is $< \epsilon$. But the shortest path between a^k and z_0 is a line, where the length of this line is $|a^k - z_0|$, implying that $|a^k - z_0| < \epsilon$, as desired.

Problem 4

Let $U, V \subseteq \mathbb{C}$ be open, $f : U \to \mathbb{C}$ and $g : V \to C$ functions, and suppose that $f(U) \subseteq V$.

(a) We want to show that f is totally/real differentiable (when viewed as a function $\mathbb{R}^2 \to \mathbb{R}^2$) at $z_0 \in U$ if and only if there are complex numbers a, b s.t. $f(z) = f(z_0)+a(z-z_0)+b(\overline{z-z_0})+o(|z-z_0|)$ (where the o is w.r.t. the limit as $z \to z_0$; i.e. $o(|z - z_0|)$ represents a function ψ s.t. $\frac{\psi(z)}{|z-z_0|} \to 0 \iff |\frac{\psi(z)}{|z-z_0|}| \to 0$ as $z \to z_0$). We will furthermore show that a, b are uniquely determined by the above equation, and satisfy $a = \frac{1}{2}(f_x - if_y)$ and $b = \frac{1}{2}(f_x + if_y)$.

Let us write $z_0 = x_0 + iy_0$. Recall (from Lec 4/2/21) that the definition of total/real differentiability at $(x_0, y_0) \in U \subseteq \mathbb{R}^2$ is that

$$f((x, y)) = f((x_0, y_0)) + L((x - x_0, y - y_0)) + r((x - x_0, y - y_0))$$

where the remainder $r : \mathbb{R}^2 \to \mathbb{R}^2$ satisfies $\lim_{(x,y)\to(x_0,y_0)} \frac{|r((x-x_0,y-y_0))|}{|(x-x_0,y-y_0)|} = 0$, and where L is a $\mathbb{R}$-linear function, i.e. $L((x - x_0, y - y_0)) = (\alpha(x - x_0) + \beta(y - y_0), \alpha'(x - x_0) + \beta'(y - y_0))$ for some $\alpha, \alpha', \beta, \beta' \in \mathbb{R}$. The condition on r makes it clear that $r(z - z_0)$ (now thinking of r as a function $\mathbb{C} \to \mathbb{C}$) is in fact $o(|z - z_0|)$. Thus, to prove the claim, all we need to do is show that L can be written as $a(z - z_0) + b(\overline{z - z_0})$ for some $a, b \in \mathbb{C}$ and that any $a(z - z_0) + b(\overline{z - z_0})$ (after converting from $\mathbb{C}$ to $\mathbb{R}^2$) is an $\mathbb{R}$-linear function.

Suppose we are given $a, b \in \mathbb{C}$. Let us write $a = a_x + ia_y$ and $b = b_x + ib_y$ for $a_x, a_y, b_x, b_y \in \mathbb{R}$. Then, $a(z - z_0) + b(\overline{z - z_0}) = a_x(x - x_0) - a_y(y - y_0) + ia_x(y - y_0) + ia_y(x - x_0) + b_x(x - x_0) + b_y(y - y_0) - ib_x(y - y_0) + ib_y(x - x_0)$. Simplifying and converting from $\mathbb{C}$ to $\mathbb{R}^2$, it is easy to see that this will be of the form $(\alpha(x - x_0) + \beta(y - y_0), \alpha'(x - x_0) + \beta'(y - y_0))$.

For the other direction, $L((x - x_0, y - y_0)) = (\alpha(x - x_0) + \beta(y - y_0), \alpha'(x - x_0) + \beta'(y - y_0))$ thought of as a complex number is $\alpha(x - x_0) + \beta(y - y_0) + i\alpha'(x - x_0) + i\beta'(y - y_0)$. But $x - x_0 = \frac{1}{2}(z - z_0 + \overline{z - z_0})$, and $y - y_0 = -\frac{1}{2}i(z - z_0 - \overline{z - z_0})$. Substituting these identities in and simplifying, we will get that $L((x - x_0, y - y_0)) = a(z - z_0) + b(\overline{z - z_0})$ for some complex numbers $a, b \in \mathbb{C}$.

As for uniqueness, if $f(z) = f(z_0) + a(z - z_0) + b(\overline{z - z_0}) + o(|z - z_0|) = f(z_0) + a'(z - z_0) + b'(\overline{z - z_0}) + o(|z - z_0|)$, then we get that $(a - a')(z - z_0) + (b - b')(\overline{z - z_0}) + o(|z - z_0|)$ (difference of two $o(|z - z_0|)$ functions is still $o(|z - z_0|)$ by the triangle inequality), or in other words $(a-a')(z-z_0)+(b-b')(\overline{z - z_0})$ is $o(|z-z_0|)$. Hence, we have $\lim_{z\to z_0} \frac{|(a-a')(z-z_0)+(b-b')(\overline{z-z_0})|}{|z-z_0|} = 0$. Now fixing $z \in \mathbb{C}$, the above limit being 0 means that the following limit ($\lambda \in \mathbb{R}$) is also 0: $\lim_{\lambda\to 0} \frac{|(a-a')(\lambda(z-z_0))+(b-b')(\overline{\lambda(z-z_0)})|}{|\lambda(z-z_0)|} = 0$. But this fraction is simply $\frac{|(a-a')(z-z_0)+(b-b')(\overline{z-z_0})|}{|z-z_0|}$, since we can move all the λ's in front by linearity! In other words, we have that for all $z \in \mathbb{C}$, $|(a - a')(z - z_0) + (b - b')(\overline{z - z_0})| = 0$, which forces $a - a' = 0 = b - b'$, as desired.

Finally, taking $z = z_0 + \lambda$ where $\lambda \in \mathbb{R}$, we have that $f(z) = f(z_0) + a\lambda + b\lambda + o(|\lambda|) \implies \lim_{\lambda\to 0} \frac{f(z)-f(z_0)}{\lambda} = a + b$. But this limit is by definition $f_x(z_0)$, so we have $f_x(z_0) = a + b$. Similarly, we can take $z = z_0 + \lambda i$ where again $\lambda \in \mathbb{R}$, to get that $f(z) = f(z_0) + a\lambda i - b\lambda i + o(|\lambda|) \implies f_y(z_0) = \lim_{\lambda\to 0} \frac{f(z)-f(z_0)}{\lambda} = i(a-b)$. Thus, indeed a, b satisfy $a = \frac{1}{2}(f_x(z_0) - if_y(z_0))$ and $b = \frac{1}{2}(f_x(z_0) + if_y(z_0))$.

(b) In the above problem, the notation $f_z(z_0), \frac{\partial f}{\partial z}(z_0), \partial f(z_0)$ for a and $f_{\overline{z}}(z_0), \frac{\partial f}{\partial \overline{z}}(z_0), \overline{\partial} f(z_0)$ are used. Now if f is totally/real differentiable at $z_0 \in U$ and g is totally/real differentiable at $w_0 = f(z_0) \in V$, we want to show that $(g \circ f)_z(z_0) = g_w(w_0)f_z(z_0) + g_{\overline{w}}(w_0)[\overline{f}]_z(z_0)$ and $(g \circ f)_{\overline{z}}(z_0) = g_w(w_0)f_{\overline{z}}(z_0) + g_{\overline{w}}(w_0)[\overline{f}]_{\overline{z}}(z_0)$.

From part (a), we have that $f(z) = f(z_0) + a(z - z_0) + b(\overline{z - z_0}) + o(|z - z_0|)$ and $g(w) = g(w_0) + c(w - w_0) + d(\overline{w - w_0}) + o(|w - w_0|)$ for some $a, b, c, d \in \mathbb{C}$. Then,

$$\begin{aligned}(g \circ f)(z) = g(f(z)) &= g(w_0) + c(f(z) - f(z_0)) + d(\overline{f(z) - f(z_0)}) + o(|f(z) - f(z_0)|) \\ &= g(w_0) + c(a(z - z_0) + b(\overline{z - z_0}) + o(|z - z_0|)) \\ &\quad + d(\overline{a}(\overline{z - z_0}) + \overline{b}(z - z_0) + o(|z - z_0|)) + o(|f(z) - f(z_0)|) \\ &= g(w_0) + (ca + d\overline{b})(z - z_0) + (cb + d\overline{a})(\overline{z - z_0}) + o(|z - z_0|) + o(|f(z) - f(z_0)|).\end{aligned}$$

Well as $z \to z_0$, we have by continuity that $f(z) \to f(z_0)$, and so

$$\begin{aligned}\frac{o(|f(z) - f(z_0)|)}{|z - z_0|} &= \frac{o(|f(z) - f(z_0)|)}{|f(z) - f(z_0)|} \cdot \frac{|f(z) - f(z_0)|}{|z - z_0|} \\ &\leq \frac{o(|f(z) - f(z_0)|)}{|f(z) - f(z_0)|} \cdot \frac{|a||z - z_0| + |b||z - z_0| + o(|z - z_0|)}{|z - z_0|} \to 0\end{aligned}$$

as $z \to z_0$ because $\lim_{f(z)\to f(z_0)} \frac{o(|f(z)-f(z_0)|)}{|f(z)-f(z_0)|} = 0$ and $\lim_{z\to z_0} \frac{|a||z-z_0|+|b||z-z_0|+o(|z-z_0|)}{|z-z_0|} \to |a| + |b|$. Thus, we have that

$$(g \circ f)(z) = g(w_0) + (ca + d\overline{a})(z - z_0) + (cb + d\overline{b})(\overline{z - z_0}) + o(|z - z_0|),$$

and by the uniqueness from part (a) and the notation we introduced at the start of this problem, we have that indeed

$$(g\circ f)_z(z_0) = g_w(w_0)f_z(z_0)+g_{\overline{w}}(w_0)[\overline{f}]_z(z_0) \quad \text{and} \quad (g\circ f)_{\overline{z}}(z_0) = g_w(w_0)f_{\overline{z}}(z_0)+g_{\overline{w}}(w_0)[\overline{f}]_{\overline{z}}(z_0)$$

where we also used the fact that $f(z) = f(z_0) + a(z - z_0) + b(\overline{z - z_0}) + o(|z - z_0|) \implies \overline{f}(z) = \overline{f}(z_0) + \overline{b}(z - z_0) + \overline{a}(\overline{z - z_0}) + o(|z - z_0|)$.

Problem 5

We have that

$$\begin{aligned}\frac{H(z)-H(z_0)}{z-z_0} &= \frac{1}{z-z_0}\left(\int_0^1 \frac{h(t)}{t-z}\,dt - \int_0^1 \frac{h(t)}{t-z_0}\,dt\right)\\ &= \frac{1}{z-z_0}\left(\int_0^1 \frac{h(t)(z-z_0)}{(t-z)(t-z_0)}\,dt\right) = \int_0^1 \frac{h(t)}{(t-z)(t-z_0)}\,dt\end{aligned}$$

which goes to $\int_0^1 \frac{h(t)}{(t-z_0)^2}\,dt$ as $z \to z_0$ because the integrands converge uniformly:

$$\sup_{0\le t\le 1} \left|\frac{h(t)}{(t-z)(t-z_0)} - \frac{h(t)}{(t-z_0)^2}\right| = \sup_{0\le t\le 1} \left|\frac{h(t)}{(t-z_0)}\right| \cdot \left|\frac{(z-z_0)}{(t-z)(t-z_0)}\right| = |z-z_0| \sup_{0\le t\le 1} \left|\frac{h(t)}{(t-z_0)(t-z)(t-z_0)}\right|$$

but $|z-z_0|$ goes to 0 as $z \to z_0$ and the sup of a continuous function over a compact set is finite (it's a continuous function because it's a quotient of continuous functions where the denominator is non-zero). Thus we see that $H'(z)$ exists everywhere in $\mathbb{C} \setminus [0,1]$.

We now show that $H'(z)$ is continuous. We have $H'(z) = \int_0^1 \frac{h(t)}{(t-z)^2}\,dt$, which is defined on $\mathbb{C} \setminus [0,1]$ because then $t-z$ will not be 0. Observe that

$$|H'(z) - H'(z_0)| \le \int_0^1 |h(t)| \left|\frac{(t-z)^2 - (t-z_0)^2}{(t-z)^2(t-z_0)^2}\right| dt = \int_0^1 |h(t)| \left|\frac{z^2 - z_0^2 - 2t(z-z_0)}{(t-z)^2(t-z_0)^2}\right| dt$$

where on $[0,1]$, $|h(t)||\frac{z^2-z_0^2-2t(z-z_0)}{(t-z)^2(t-z_0)^2}| \le |z-z_0| \sup_{0\le t\le 1} |h(t)||\frac{z+z_0-2t}{(t-z)^2(t-z_0)^2}|$, where the sup is again finite because it's the sup of a continuous function over a compact set (continuous function because again it's a quotient of continuous functions where the denominator is non-zero). Thus, for any $\epsilon > 0$, there is $\delta > 0$ s.t. $|z-z_0| < \delta \implies |H'(z) - H'(z_0)| < \epsilon$, and hence $H'(z)$ is continuous at every point of $\mathbb{C} \setminus [0,1]$. Thus, $H(z)$ is $\mathcal{C}^1$, i.e. holomorphic on $\mathbb{C} \setminus [0,1]$ as desired.