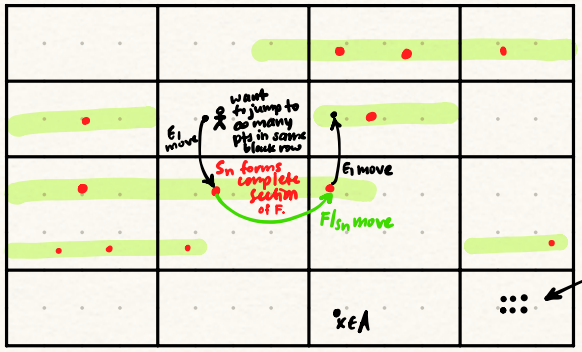
3/13/26 24.2 ∄ subadd fn commuting E_1, E_2 (not just for $E_1 \cap E_2$ ⊴CBER as proven in 23.4 using $G \geq G_0$ graphing of hf ⊴CBER $E_0 \leq E_1 \cap E_2$)

Tamas' picture



$F \subseteq_{CBER} E_2$ subrow(s) of each row, each infinite.

in fact, S_n ∞ many pts in each F-class

Claim 24.3: $(E_1 \vee F|_{S_n}) \cap E_2$ ⊴CBER.

i.e. just using E_1-move (= arb ∫ col motion) and



reach ∞ many pts in each row = E_2-class.

black cells = $E_1 \cap E_2$-classes are finite

columns are E_1 classes, each infinite.

$A = \{x \in X : [x]_{E_1 \cap E_2}$ is finite$\}$.

"Whole philosophy of book contact arguments to use variably seq of markers (complete sections)"

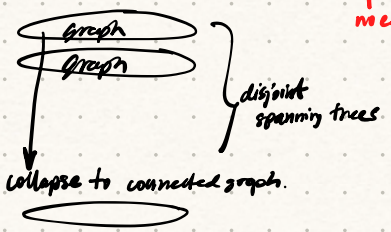
Imagine black cells only have 1 ∀x. ⇔ Antoni joke $E_1 \cap E_2 = id$

Imagine graphing of E_2

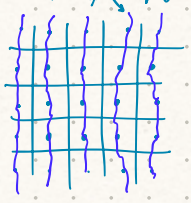
Imagine G_2 has ≤1 edge b/t any 2 columns = G_1-classes. like



Want to kill some edges

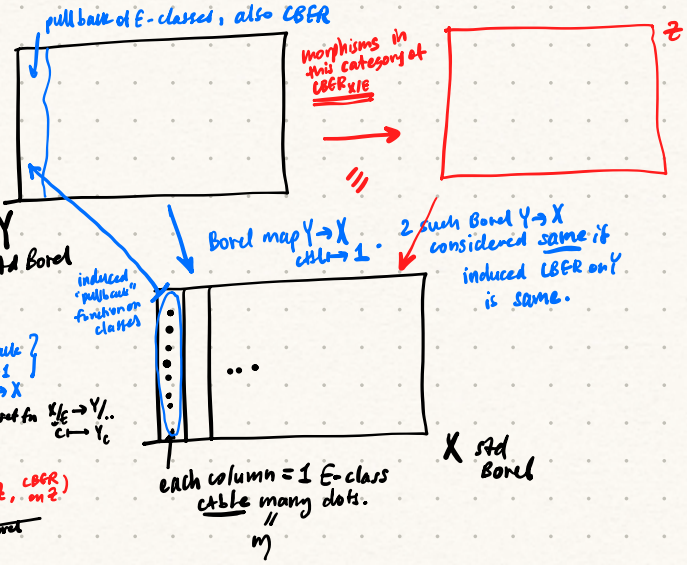


Suppose $\phi_n(G_1) \leq \phi_n(G_2)$



randomly delete edges percolation?

3/22/26 R. Chen's L^2 pf of Gaboriau Thm



Category CBER X/E

Obj: $\begin{bmatrix} Y \\ \downarrow \\ X \end{bmatrix}$ i.e. $\{(Y, \text{CBER on } Y)$: arises from pullback of ctbl→1 Borel $Y \to X\}$

Y std Borel — so has associated set fn $X/E \to Y/\sim$, $C \mapsto Y_C$

Mor: (Y, CBER on Y) —Borel→ (Z, CBER on Z) via (X, E ⊴CBER on X), Borel, Borel

has induced set function $X/E \to Y/\text{its CBER}$, $C \mapsto Y_C$

mapping equivalent things in class Y_C ↦ equivalent things in class Z_C.

Cleany: category CBER X/E

Data: std Borel Y arising from pullback of E via Borel map $Y \to X$

Obj: $(Y, \text{CBER on } Y)$, has Data associated set fn $C \mapsto Y_C$ (a X/E) equiv class $\subseteq Y$.

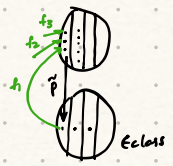
Mor: Data $(Y, \text{CBER on } Y) \xrightarrow{\text{Borel}}$ Data $(Z, \text{CBER on } Z)$

w/ property maps $Y_C \to Z_C$ $\forall C \in X/E$.

equiv things above specific class C ↦ equiv things above specific class C

CBER

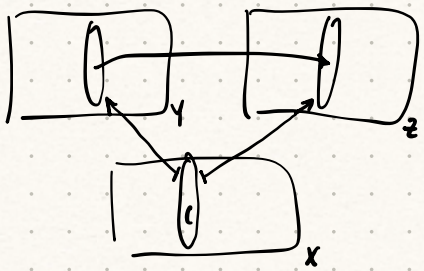
$\forall$ ctbl→1 Borel $\tilde{p}: Y \to X$, Borel ctbl maps $f_\cdots : X \to Y$, injective partial, Union Nonsense



$\int_X |A \cap \tilde{p}^{-1}(x)| \, d\mu(x) \longrightarrow \#n : f_n(x) \in A$

$\int_X \sum_{y \in Y_{[x]}} 1_A(y) \, 1_{(\tilde{p}(y)=x)} \, dx$.

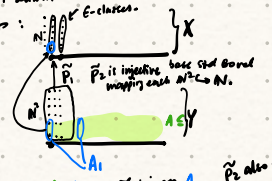
ctbl sum ctble



(this Claude fr spelling this out for us)

in general inv not $\Rightarrow \infty$ Borel transversal on pos. meas. set.

this does not admit inv msr but I draw it nonetheless:



$A = \bigsqcup_n A_n$ $\tilde{p}_1$ inj on A_n

Lusin Novikov!

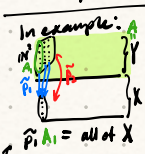
$(\tilde{p}_1)_* \mu_Y(A_n) = \int_X |A_n \cap \tilde{p}_1^{-1}(x)| \, d\mu(x)$.

$\equiv 1_{\tilde{p}_1(A_n)}(x)$

$= \mu(\tilde{p}_1(A_n))$

$\tilde{p}_2$ also inj on each A_n

In example:



$\tilde{p}_1 A_1$ = all of X

Borel bijection preserves E-class ⇒ same μ!

$\tilde{p}_2 A_1$ =



So in my $\mathbb{N} \times$ base example, $\mathbb{N}^2 \times$ base

clearly impossible (does not admit inv μ)

b/c $\mu(\tilde{p}_i A_n) = \mu(X)$

but $\sum_n \mu(\tilde{p}_i A_n) = \mu(X)$ → $\mu(X) = \infty$ or $\mu(X) = 0$

$X = \bigsqcup_n \tilde{p}_i A_n$ from picture

Key Example:

X Std Borel $E \subseteq X^2$ CBER $G \subseteq E \subseteq X^2$ (directed) graphing of E

$G \to X/E$ (map onto)

G-edge $(x,y) \mapsto$ E-class containing the 2 endpts x, y of edge (x,y)

$C_{X/E}(G) \to C_{X/E}(T)$

Ψ

fin supp $m : G_c \to C$ for some $C \in X/E$ (edges $\in G$ of compt G)

finite formal sum $\sum_{\text{fin edges } e \text{ in } 1 \text{ compt } G} (e_{\text{coeff}}) \cdot e^G$

Borel fiberwise linear map defined on basis elts (extend by linearity)

$e^G \mapsto \sum$ T-edges (a,b) in unique path from $s(e^G)$ to $t(e^G)$ $1 \cdot (a,b)$

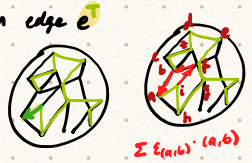


E-class = G-comp = T-comp

See picture

This Red map is surjective (as Chen puts it, $\exists$ section):

given edge e^T



take any path

$\sum \varepsilon_{(a,b)} \cdot (a,b)$

Hmm, maybe Borel have ordering on X, T edges (a,b) $a<b \Rightarrow \varepsilon_{(a,b)} = +1$, $\varepsilon_{(b,a)} = -1$.

$T \subseteq E^2$ as a set only has pairs (α, β) $\alpha < \beta$.

$(b,a) \mapsto \cancel{(b,c)} + (c,z) + \dots + (h,i) + (-1)(a,i)$

$(f,b) \mapsto (-1)(e,f) + (-1)(d,e) + (a,z) + (-1)(c,z) + (-1)(b,c)$

$(i,f) \mapsto (-1)(h,i) + (-1)(g,h) + (-1)(f,g)$

$(i,f) + (f,b) + (b,a) \mapsto (-1)(a,i)$

(I messed up, but same argument should

$(a,b) + (b,f) + (f,i) \mapsto (a,i)$ ✓

3/23/26

Borel fiberwise linear section

recall $=: C_{X/E}(T) \xrightarrow{\text{Y}}$ $f =$ "G-path of" (BFS) $\xrightarrow{\cong} C_{X/E}(G)$

$\bigsqcup_{C \in X/E} C(T_C)$

define for basis elts $1 \cdot e^T_{(x,y)} \mapsto$ do BFS on G (the compt $\ni x$). (Lusin Novikov: Borel way of going thru all nbrs, use Borel ordering of X to determine which branches to pursue first)

then extend by linearity to general elt $\sum_{\text{fin edges } e^T \text{ in } 1 \text{ compt } T} (C_{\text{coeff}}) \cdot e^T$

also why C? would 1 work? 2/2?

injective b/c going $C_{X/E}(T) \hookrightarrow C_{X/E}(G) \to C_{X/E}(T)$ is identity as proven

Borel coloring of intersection graph. See 223D Feldman Moore Lec20

V = finite sets $\subseteq$ some E-comp of X.

E: 2 such ↑ adjacent iff $\cap \neq \emptyset$. ("they intersect")

Pf: Lusin Novikov $\Rightarrow \exists \{f_1, \dots : X \xrightarrow{\text{Borel}} X\}$ s.t. $xEy \Leftrightarrow \exists n : x \xrightarrow{f_n} y$.

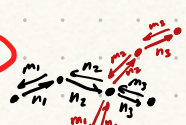
Def $c : V \to \mathbb{N}^{<\infty}$

finset $\{x_1 \dots x_k\} \leftrightarrow$ tuple order by precursor Borel order on X $(x_{(1)} \dots x_{(k)}) \mapsto (n_1, \dots n_{k-1}, m_{k-1}, \dots m_1) \in \mathbb{N}^*$.

order statistic! $x_{(1)} \xrightarrow{f_{n_1}} x_{(2)} \xrightarrow{f_{n_2}} \dots \xrightarrow{f_{n_{k-1}}} x_{(k)}$

($n_i \in \mathbb{N}$ s.t. smallest)

Issue:



black 4-set $\supseteq$ red 4set, but same coloring c!

Fix subgraph (V, F) w/ $F \subseteq E$

$\{$finset$_1 \xrightarrow[\text{adj}]{}$ finset$_2 \Leftrightarrow \supseteq$ and same c$\}$

This subgraph is locally finite!

$\Rightarrow \exists$ Borel $c' : V \to \mathbb{N}$ proper coloring of (V, F). Then (c, c') is Borel pt. of (V, E).

(c(y) < n: dtle Borel coloring $Y \to \mathbb{N}$ above)

$Y_n := \{ y \in Y : \|f(y)\|_{L^\infty(\mathbb{Z}_{p(y)}; \mathbb{C})},\ |\mathrm{supp}(f(y))|,\ c(y) < n \}$

$\{e^T \in T$ — automatic b/c $f(e^T)$ = formal Σ representation of a path in G (coeffs are ±1), i.e. L^∞ norm = 1.

$|\mathrm{supp}(f(y))| \in Y_{p(y)}$: meaning that the G-path has len < n. $\}$

$Z_n := \{ z \in Z$ ($\in Z := G$, $e^G \in G$) : automatic again, unique T-path of e^G has len < n, $c(z) < n \}$

$f^{-1}(C_{X/E}(Z_n)) := \{ y \in Y$ ($e^T \in T$) : $f(e^T) \in C_{X/E}(Z_n) \}$ ("G-path of e^T")

N-coloring

$\tilde{Y}_N^{\text{color}=1} := \{ e^T_{N,1} \in T :$ G-path(e^T) ($=f$) len < n, $c(e^T) = 1 \}$ + !T-path (each edge of G-path(e^T)) len < n.

c is proper χ of intersection graph

all the $e^T_{N,1} \rightsquigarrow$ many G-paths $f(e^T_{N,1})$ that are all edge disjoint

$f_1 : \mathbb{C}(\tilde{Y}^N_C) \longrightarrow \mathbb{C}(Z^N_C)$

$T_C \ni e^T$ if $\in \tilde{Y}^N_1 \longmapsto$ G-path $f(e^T)$ (by def $\tilde{Y}^N$, guaranteed len < N) ; G_C

else $\longmapsto 0$

$\{e^T \in \tilde{Y}^N_C \subseteq Y := T\}$ form ON basis of $\ell^2(\tilde{Y}^N_C)$. f_1 maps ON basis to orthogonal elts of $\ell^2(Z^N_C)$

General fact: ONB $\{v_n\}$ $\rightsquigarrow$ f_1 extend to $\tilde{f}_1 : \ell^2(\tilde{Y}^N_C) \to \ell^2(Z^N_C)$ $\forall C \in X/E$

φ maps ONB $v_n \longmapsto$ OG w_n

$\Rightarrow \varphi(\text{unit vec}) = \sum \text{coeff}\ \varphi(v_n) = \sum \text{coeff} \cdot w_n$

$\Downarrow$

$\|\varphi(\text{unit vec})\|^2 = \sum \text{coeff}^2 \|w_n\|^2 \le N^2 \underbrace{\sum \text{coeff}^2}_{=1}$ (Pythag)

$\Rightarrow \|\varphi\|_{\ell^2 \to \ell^2} \le N$. bdd lin operator!

Clean Def:

$Y \stackrel{\text{T}}{=} \bigcup \tilde{Y}_n$ finite many pieces $\mu(\tilde{Y}_n) \le n$. Same w/ $Z = \bigcup \tilde{Z}_n$

$\tilde{Y}^N_C := \{ e^T \in T_C :$ G-path(e^T) ($=f$) len < N, $c^T(e^T) < N$, !T-path (each edge of G-path(e^T)) len < N, $c^G(\cdot) < N$ ($\to \in \tilde{Z}_N$) $\}$ $\leftrightarrow \forall e^G$ in G-path(e^T), $e^G \in Z^N_C$

T_C: T-comp assoc to E-class $C \subseteq X$. in our case, T treeing of E, G graphing. T_C = spanning tree of C, G_C = spanning graph of C

$Z^N_C := \{ e^G \in G_C :$!T-path(e^G) len < N, $c^G(e^G) < N \}$ ($\tilde{Z}_N$)

Property: $Y = \bigcup \tilde{Y}^N$, $Z = \bigcup Z^N$.

$\mathrm{id} \circlearrowright \mathbb{C}(\tilde{Y}^N_C) \underset{g = \text{"!T-path" and then proj}}{\overset{f = \text{"G-path"}}{\rightleftarrows}} \mathbb{C}(Z^N_C)$ (i.e. $\mathrm{id} = g \circ f$) (if $g(e^G_c) = \sum \circ \cdot e^T_c$ not in $\tilde{Y}^N_C$, just set its coeff $\circ \in \mathbb{C}$ to 0)

Same argument for g.

$\Rightarrow$ Both f, g extend to bdd lin (fiberwise) on ℓ^2

$\mathrm{id} \circlearrowright \mathbb{C}(\tilde{Y}^N_C) \underset{g = \text{"!T-path" and then proj}}{\overset{f = \text{"G-path"}}{\rightleftarrows}} \mathbb{C}(Z^N_C)$

first f = "G-path", then $\mathrm{proj}_{\ker(g)^\perp}$

General fact Hilbert spaces

$H_1 \underset{\psi}{\overset{\varphi}{\rightleftarrows}} H_2$

$\psi \circlearrowright \varphi$ = first φ, then $\mathrm{proj}_{\ker(\psi)^\perp}$ then ψ! b/c $\mathrm{proj}_{\ker(\psi)^\perp} + \mathrm{proj}_{\ker(\psi)} = 1$

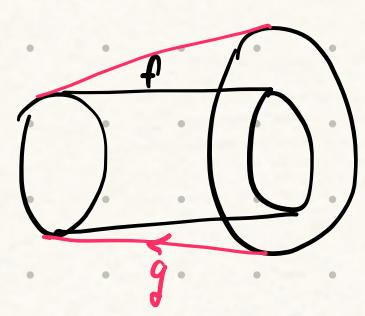
This is key magic of pf! ($\ker(g)$ is not combinatorial, its lin algebra fractional coeffs)

$\langle g \circ f z, z \rangle$ not nec $\le \langle z, z \rangle$, $= f(g(z))$

but $\langle \mathrm{proj}_{\ker(g)^\perp} \circ f \circ g\, z, z \rangle \le \langle z, z \rangle$

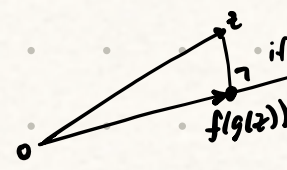
$\mathrm{id} \circlearrowright H_1 \underset{g}{\overset{f}{\rightleftarrows}} H_2$

$f \circ g$ may not $= \mathrm{id}$, but difference b/c $f \circ g$ and id is in $\ker(g) \le H_2$ (i.e. $f \circ g = \mathrm{id} +$ error in $\ker(g)$).



$g(z - f(g(z))) = g(z) - \underbrace{g \circ f}_{=\mathrm{id}}(g(z)) = 0$.

$\Downarrow$



if f lands in $\ker(g)^\perp$

Linear Algebra in Combinatorics

I'm reminded of Igor 206A Erdős-Moser conjecture linear algebra pf

Focusing on where the LA comes in: poset $M(n)$ e.g. $M(4)$

rank = Σ.

Thm: sizes of rows (indexed by rank) unimodal

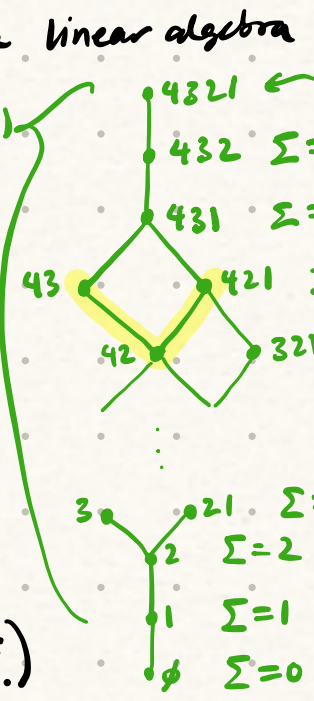
$r_0 \le \dots \le r_{\lfloor \frac{1}{2}\binom{n}{2} \rfloor} \ge \dots \ge 1$, $r_0 = 1$, $r_{\frac{n(n+1)}{2}} = \binom{n}{2}$

no injection $R_n(i) \hookrightarrow R_n(i+1)$ known.

but can define $U_i : \mathbb{Q}\langle R(i) \rangle \to \mathbb{Q}\langle R(i+1) \rangle$

e.g. $\{4,2\} \longmapsto 1 \cdot \{4,3\} + 1 \cdot \{4,2,1\}$.

can show U is inj (by defining lowering op U, show U, D satisfy sl_2 identity, show pos evals.)



size of row/level 10. $r_{10} = 1$

$r_3 = 2$, $r_2 = 1$, $r_1 = 1$, $r_0 = 1$

$t \to s$ $\Leftrightarrow$ $t > s$ lexicographical order e.g. $431000\ldots$ / $43000\ldots$

3/22/16 Application 2 from Igor's notes 2020 (see also pg 2 of 2002 Gerard Cornuejols)

Lovasz perfect graph thm: G perfect $\Leftrightarrow$ $\forall$ induced $H \le G$, $|H| \le \alpha(H)\,\omega(H)$ — "max" notation b/c $K \le \bar{K}$ looks natural

($\chi(G) = \omega(G)$ b/c $\exists K$ largest clique) — $\bar{\chi}(H)$ largest indep set, $\bar{K}(H)$ $\geq$ 2-rep set

"It want to discover the right RMS in course of pf, phrase as 'Look for property P s.t. G perfect $\Leftrightarrow$ $\forall$ induced $H \le G$, H has prop P' (yes, by def, P = "$\chi = \omega$" ✓, but we look for others.

($\Leftarrow$) by induction: $G - I \subseteq G$, by induction has prop P (b/c all its induced subgraphs are) hence $G - I$ known to be perfect by induction.

Then don't use prop P for entirety of pf until end, when we reach the concluding thing:

G not perfect, all its induced subgraphs are perfect $\Rightarrow |G| \ge \alpha\omega + 1$ (this is enough for our pf to go thru)

Or really: G s.t. all induced subs perfect, then [G not perf $\Rightarrow |G| \ge \alpha\omega + 1$] to contrapos [$|G| \le \alpha\omega \Rightarrow$ G perf] ($\Leftarrow$ is obvious.) — this is our prop P!

Study minimal imperfect b/c want to study decomposition; namely "all induced subs perf $\Rightarrow$ global perf!" recurse to conclude "local $\Rightarrow$ global"

[$\forall H \subsetneq G$, $|H| \le \alpha\omega \cdot \omega(H)$ $\Rightarrow$ G perf.] (induction)

($\Rightarrow$) $\forall$ graph G, $|G| \le \chi(G) \cdot \bar{\alpha}(G)$

perfect $\Rightarrow$ $= \bar{K}(G)$

($\Leftarrow$) ~~tot G be minimal imperfect (again, can phrase this pf by induction on $|G|$)~~

Obs 1: $\forall$ indep set $I \subseteq V$, $\chi(G - I) = \bar{K}(G - I) = \bar{K}(G) =: \omega$

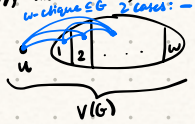
(to this = is $\Leftarrow$ G not perfect.)

↑ by perfectness from induction; ↑ obviously; ≤ obviously; if <, then color I extra color $\Rightarrow \chi(G) \le \bar{K}(G) \Rightarrow$ G perfect (so ☹, ordone in this case) ($\bar{K} \le \chi$ always)

In English, this means (max) have this G which has size ω clique, but $\chi = \omega + 1$ kill any indep $I \subseteq V$, still $G - I$ has K_ω! and $G - I$ can be colored by ω colors

Obs 2: kill any singleton $\{u\} \subseteq V(G)$, have ω-coloring of $G - \{u\}$

ω-clique $\subseteq G$ 2 cases: $\not\ni u \Rightarrow \cap$ all but 1 classes; $\ni u \Rightarrow \cap$ all classes



Pf: fix max indep $I_0 \subseteq V(G)$. (" $\{u_1 \dots u_{\alpha\omega}\}$ ")

$G - u_1$ has ω-$\bar{\chi}$ (enough to kill ω) $\Rightarrow \bigsqcup I_1 \dots I_\omega$ (avoiding u_1). by Obs 1

$G - u_2$ —— $\bigsqcup I_{\omega+1} \dots I_{2\omega}$ (avoiding u_2)

Each I_i indep $\Rightarrow$ (Obs 1) $G - I_i$ still has ω-clique K_i

Theme: ortho ↔ $\not\perp$!

$\langle I_i, K_i \rangle = 0$ ($I_i \not\perp K_i$) thought of as indicator fns on $\{u_1 \dots u_n\} = V(G)$.

Claim: Let K be ω-clique in G. Then K misses exactly 1 ("$\not\perp$") I_i ($i = 0 \dots \alpha\omega$)

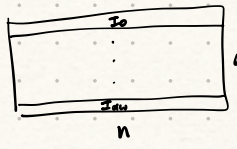
pf: $K \not\perp I_0 \overset{\text{Obs 2 case 2}}{\Rightarrow} K \cap$ all $I_{\ge 1}$ — K avoid u_1, K avoid u_2

$K \cap I_0$ at $u_i \Rightarrow K \cap$ all I_j on rows $G - u_{\neq i}$ and on row $G - u_i$, Obs 2 says K misses exactly one I_j on that row.

K misses exactly 1 cell! b/c its two occupied smoothing u_i

Finaly, LA:

$I_0 \leftrightarrow 1_{I_0} : V(G) \to \{0,1\}$. s.t. $\dots v_i \dots$

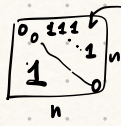
$M_I =$ 

$= M_K$ 

By Claim: K_i miss $I_i \Rightarrow K_i$ hit all $I_{\neq i}$. (necessarily b/c I is indep. in exactly 1 spot.)

this type of matrix = assembly of inner products

also appear in other LA in Combi problem e.g. Omar Mohamed YTvideo, Addtown → LA: $\det \begin{bmatrix} >1 & 1 \\ 1 & >1 \end{bmatrix} \neq 0$.

matrix mult $M_I M_K =$ 

(i.e. $1_{n \times n} - \text{diag}(1_n)$ matrix)

which is indeed full rank $\Rightarrow n \ge \alpha\omega + 1$.

$\det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \neq 0$

3/26/26 let's start from conceptual basics. Lots of formalism can come later.

Mathlogger principle of Examples first

(Hilbert quote "The art of doing mathematics consists in finding that special case which contains all the germs of generality")

$\mathcal{X}$ std Borel

E CBER on $\mathcal{X}$

G graphing of E

T treeing of E (I have Borel ordering on $\mathcal{X}$, $T \subseteq \{(x,x') \in \mathcal{X}^2 : x < x'\}$)

Define !T-path (taking in: edge $e^G \in G$ (directed)) $\longmapsto$ unique T-path in the sense of formal sum w/ ± 1 coeffs of green edges $e^T = (x,x')$, $x<x'$ in picture



comps of G = E-classes



$e^G = (a,b)$ $\longmapsto$!T-path function $\longmapsto$ $(a,f) + (-1)(d,f) + (-1)(c,d) + (c,e) + (-1)(b,c)$

edges e^T whose T-comp $C_T = C_G :=$ G-comp $\ni e^G$

$= C_E$ E-class $\in X/E$

Extend by linearity to functions:

!T-path: formal sums $\mathbb{C}$-coeff (or $\mathbb{R}$) of e^G in same G-comp (or in fact e in same E-class) $\longmapsto$ formal sums $\mathbb{C}$-coeff of e^T in same T-comp

(This fn is "Borel" in all the way we need it to be later)

(formalized as fn $\mathbb{C}_{X/E}(G) \to \mathbb{C}_{X/E}(T)$ (or $\mathbb{C}_{X/E}(E)$))

① Claim 1: this fn !T-path is surjective

Pf: $\mathbb{C}_{X/E}(G) \underset{\text{G-path}}{\overset{\text{!T-path}}{\rightleftarrows}} \mathbb{C}_{X/E}(T)$ we define

f so that $\circlearrowleft$!T-path $\circ$ G-path $= \circlearrowleft$ id (diagram commutes)

automatically shows !T-path surj and G-path inj. ☺

② restrict to suitably bdd setting



$e^T \longmapsto (c,a) + (a,d) + (d,z) + (z,b) + (b,e)$ (or if $(z,b) \notin G$ but (b,z) is, then $(-1)\cdot(b,z)$) — Check that indeed, applying !T-path, get e^T.

Choose say lex-least G-path

(cf. Feldman-Moore $\mathbb{N}$-coloring of edges, so each path gets score $\in \mathbb{N}^*$, well ordered)

② try integral calculation. Realize want $\langle \tilde{f}(g(z)), z \rangle \le \langle z, z \rangle = 1$ (work for any $\tilde{f}$ which satisfies $\tilde{f} \circlearrowright$ = $\circlearrowleft$ id)

Fix: do ℓ^2 stuff ③ instd of $\tilde{f} = f$, use $\tilde{f} = \text{proj}_{\ker(g)^\perp}$ of f.

$\forall e^T \in T$, $1 = \langle \underbrace{\text{!T-path}(\text{G-path}(e^T))}_{f}, e^T \rangle$ (formal sum's e^T coefficient)

$= \sum_{e^G \in \text{G-comp} \subseteq \mathcal{X}} \langle \text{G-path}(e^T), e^G \rangle \cdot \langle \text{!T-path}(e^G), e^T \rangle$ ($e^G \in [e^T]_T$ T-comp; $d\mu_{\mathcal{X}/E}(e^T)$)

$= \int_{(e^T, e^G) \in T \times_{X/E} G} d\mu_{T \times_{X/E} G}$.

symmetrically $\forall e^G \in G$, $1 \ge \langle \text{G-path}(\text{!T-path}(e^G)), e^G \rangle$ (if)

all the $e^T_{N,i}$ $\rightsquigarrow$ many G-paths $f(e^T_{N,i})$ that are all edge disjoint

$f_1 : \mathbb{C}(\tilde{Y}_c^N) \to \mathbb{C}(\tilde{Z}_c^N)$ (= $\mathbb{C}^{T_c}$ to $\mathbb{C}^{G_c}$)

$e^T \mapsto$ G-path $f(e^T)$ if $e^T \in \tilde{Y}_c^N$ (by def $\tilde{Y}_c^N$, guaranteed len $< N$), else $\mapsto 0$

$\{e^T \in \tilde{Y}_c^N \subseteq Y := T\}$ form ONB of $\ell^2(\tilde{Y}_c^N)$. f_1 maps ONB to orthogonal ONB elts of $\ell^2(\tilde{Z}_c^N)$

General fact: ONB fun $\rightsquigarrow$ f_1 extend to $\tilde{f}_1 : \ell^2(\tilde{Y}_c^N) \to \ell^2(\tilde{Z}_c^N)$ $\forall c \in X/E$

φ maps ONB $u_n \mapsto$ OG w_n

$\Rightarrow \varphi(\text{unit vec}) = \sum \text{coeff} \cdot \varphi(v_n) = \sum \text{coeff} \cdot w_n$

$\Downarrow$

$\|\varphi(\text{unit vec})\|^2 \underset{\text{Pythag}}{=} \sum \text{coeff}^2 \|w_n\|^2 \le N^2 \underbrace{\sum \text{coeff}^2}_{=1}$

$\Rightarrow \|\tilde{f}_1\|_{\ell^2 \to \ell^2} \le N$. bdd lin operator!

Same argument for $\tilde{g}$.

$\Rightarrow$ Both $\tilde{f}, \tilde{g}$ extend to bdd lin (fiberwise) on ℓ^2

$Y := \bigcup_n \mathcal{Y}_n$ finite measure pieces $\mu(\mathcal{Y}_n) \le n$. Same w/ $Z := \bigcup_n \mathcal{Z}_n$

N-coloring (F-M type argument) s.t. same color $\Rightarrow$ $f(e^T), f(\tilde{e}^T)$ disjoint supps $\subseteq G$.

$\tilde{Y}_c^N := \{ e^T \in T_c : \text{G-path}(e^T) = f$ len $< N$, $c^T(e^T) < N \}$

T_c = T-comp assoc to E-class $c \in X$. in our case, T treeing of E, G graphing, G_c = spanning treed graph of c

!T-path (each edge of G-path(e^T)) len $< N$, $c^G(\cdot) < N$ $\leftrightarrow \in \mathcal{Z}_N$ $\forall e^G$ in G-path(e^T), $e^G \in \tilde{Z}_c^N$

$\tilde{Z}_c^N := \{ e^G \in G_c : \text{!T-path}(e^G)$ len $< N$, $c^G(e^G) < N \}$

Properties: $Y = \bigcup \tilde{Y}_c^N$, $Z = \bigcup \tilde{Z}_c^N$. $\mathbb{C}(\tilde{Y}_c^N) \underset{g = \text{"!T-path"}}{\overset{f = \text{"G-path"}}{\rightleftarrows}} \mathbb{C}(\tilde{Z}_c^N)$ (i.e. $\text{id}_G \subseteq \underset{g}{\overset{f}{\circlearrowright}}$, $\Leftrightarrow 1 = g \circ f$)

(if $g(e^G) = \sum \otimes \cdot e^T_c$, e^T_c not in $\tilde{Y}_c^N$, just set its coeff $\otimes \in \mathbb{C} \to 0$)

$\mathrm{id}_G\ \mathbb{C}(\tilde{Y}^N_c) \underset{g=\text{"}!T\text{-path"} \text{ and then proj}}{\overset{f=\text{"}G\text{-path"}}{\rightleftarrows}} \mathbb{C}(Z^N_c)$

first $f=$"G-path", then $\mathrm{proj}_{\ker(g)^\perp}$

General fact Hilbert spaces
$H_1 \underset{\psi}{\overset{\varphi}{\rightleftarrows}} H_2$
$\psi\varphi$ = first φ, then $\mathrm{proj}_{\ker(\psi)^\perp}$ then ψ!

This is key magic of pf! ($\ker(g)$ is not combinatorial, it's linear algebra, fractional coeffs)
$\langle f g z, z\rangle$ not necc $\le \langle z,z\rangle$ ($fg(z)$)
but $\langle f\, \mathrm{proj}_{\ker g^\perp} g\, z, z\rangle \le \langle z,z\rangle$

b/c $\mathrm{proj}_{\ker(g)^\perp} + \mathrm{proj}_{\ker(g)} = 1$

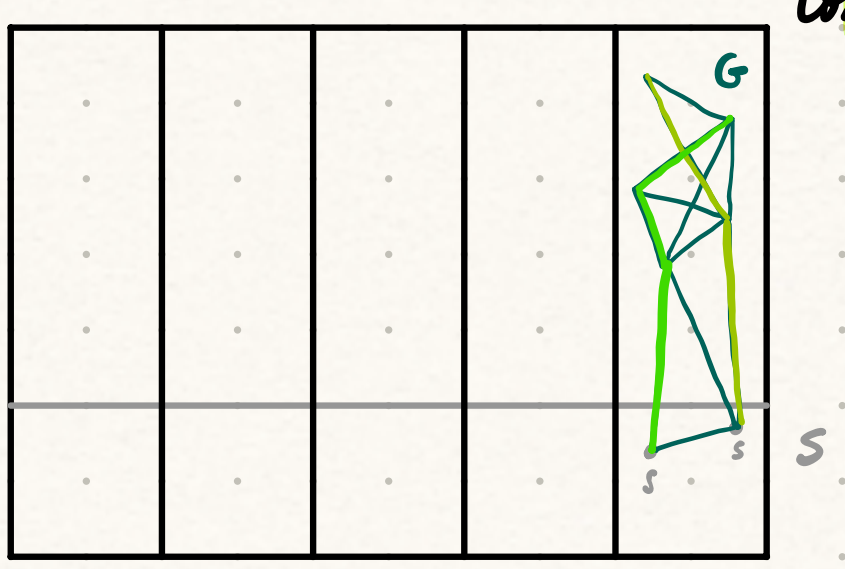
$\mathrm{id}\ G\ H_1 \underset{g}{\overset{f}{\rightleftarrows}} H_2$
f, g may not $= \mathrm{id}$,
but difference b/t $f\circ g$ and id is in $\ker(g) \subseteq H_2$. (i.e. $f\circ g = \mathrm{id} +$ error in $\ker(g)$).
$g(z - f(g(z))) = g(z) - \underbrace{g\circ f}_{=\mathrm{id}}(g(z)) = 0.$

→ also cheap to get (no sliding) $C_\mu(E) \le C_{\mu|_S}(E|_S) + \mu(X\setminus S)$ from below pf ($\mu(X\setminus S) = \int_S \deg_T(x)\, d\mu(x)$) (construction of T)

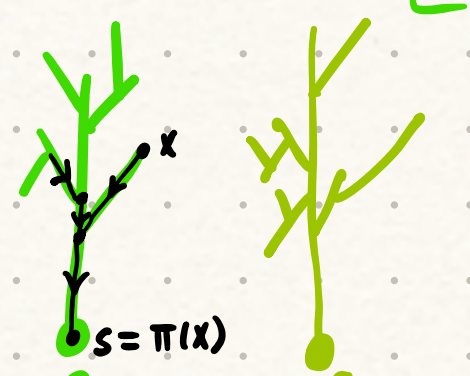
T:OE book. 1st nontrivial result:

cheap 21.1 $C_\mu(G) \ge \mu(X\setminus S)$, S complete section of CBER E on X, G any graphing of E.

Cor 21.3 a CBER ⇝ vanishing seq of markers. cheap 21.1 + limiting arg ⟹ $\phi_\mu(G) \ge \mu(X)$ (a CBER). 1st nontrivial thm (rule out ... graphings)



bright green tree. T treeing of subequiv rel $F \subseteq E$ which has S as Borel transversal.



S bright green, S yellow green

Addendum: used in 22.1 (ii)⇒(i): if pos msr set of $s \in S_n$ (P_n) s.t. tree $\mathcal{Y}_s$ is infinite, (detectable in Borel way so P_n is Borel) then König lemma ∃ ∞ branch, and shift P_n along branch gives ∞ many disjoint copies of P_n ⟹ $\mu(X) = \infty$. Thus, $\mu(X) < \infty$ ⟹ $\forall_\mu x$, T-comp of x $[x]_T$ is finite.

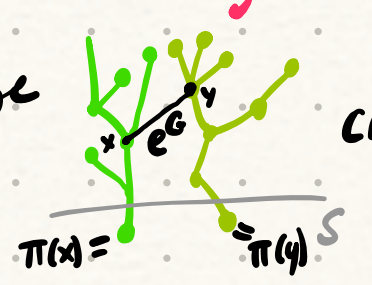
more gen. pmp setting ⇓ Borel transversal only on null set.

Full 21.1

we can give sharper lower bd (turns out to be equality) by analyzing all edges $e^G \in G \setminus T$. (union of trees $\mathcal{Y}_s$ above.)

Idea is called sliding.

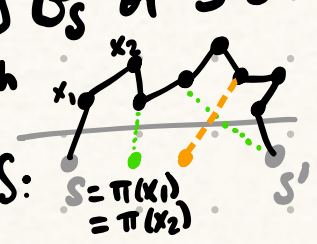
any edge e^G



can just slide down under the influence of gravity to be edge $(\pi(x), \pi(y)) \in E|_S$.

G will slide down to graphing G_S of S b/c

any path G b/t $s, s' \in S$:



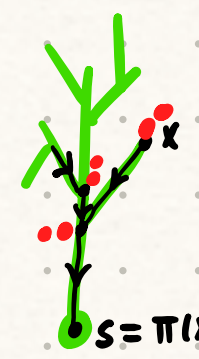
⇝ path in $E|_S$. ~~$(s, \pi(x_1)), (\pi(x_1), \pi(x_2))$~~ b/c $\pi(x_1) = s$ $(\pi(x_2), \pi(x_3))$ s' ...

⟹ $\phi_\mu(G) \ge \phi_{\mu|_S}(G_S) + \mu(X\setminus S)$.
⟹ $\phi_\mu(E) \ge \phi_{\mu|_S}(E|_S) + \mu(X\setminus S)$. Other dir ($\le$) is easy (cheap, see ↑) (start w any G, ⇝ T. Use T for)

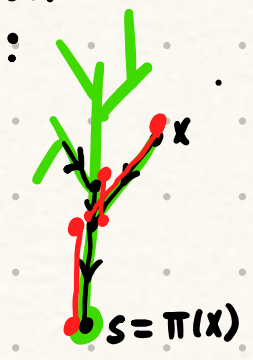
orient all edges in each tree toward root s.
(i.e. you're at x, walk in your tree to $s := \pi(x)$ — 1 unique path, so 1st step of your journey, make arrow marking)

Claim: $\mu(X\setminus S) =$ cost of above treeing (of subequiv rel of E)

Pf: have 2 dots at each $x \in X\setminus S$.



Have 1 dot at each x; walk along arrow: $W: X\setminus S \to X$ Borel inverse $W^{-1}: \mathrm{im}(W) \to X\setminus S$ is Borel producing exactly treeing T!



i.e. $2\mu(X\setminus S) = \int_X 1_{X\setminus S}(x) + 1_{W(X\setminus S)}$ (struck through)

$\deg_T(x) = [\#y \text{ s.t. } W(y) = x] + 1_{X\setminus S}(x)$

$1_{X\setminus S}(x) + 1_{X\setminus S}(x)$
apply W to 1 copy ↓
$1_{X\setminus S}(x) + [\#y : W(y) = x] = \deg_T(x)$ ☺

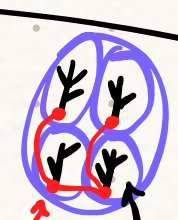
mass rearrangement equality (see proof above using injdoms (L-N!) showing indep of choice of lift $Y \xrightarrow{\text{Borel}} X \to X/E$)

22.1 ok, that's a nice start. Now, hfin aCBER.

query: "hfin cber pmp is treeable" Bowen slides

lem: hfin CBER is treeable. Pf: $E_1 \subseteq E_2 \dots \nearrow E$ meaning E_2-comps $= \bigsqcup E_1$-comps (finite) (finite)

line-able ⟺ graph of 1 Borel automorphism $X \to X$.



treeing of E_1-comp "basept"; each E_1-comp finite so has treeing; continue forever up.

in fact, can tree E_1-comp by line head—tail, min max. Then connect all lines in E_2-comp:

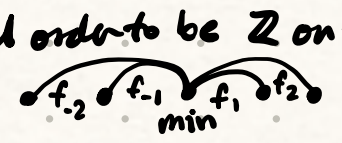


order heads.

Further more: also have Borel transversals S_n of E_n (each E_n-class fin. pick min elt, total Borel order on X.)

$S_1 \supseteq S_2 \dots$

May assume < on each comp has ∞ descending seq (o.w. that comp has minimum, and can change Borel order to be $\mathbb{Z}$ on that comp)



$\{\dots f_2, f_1, f_1, f_2 \dots\}$ Borel maps telling min where all its comp-mates are.

→ so that $\bigcap S_n = \emptyset$, vanishing seq of markers!

Have (any! such as line-ing above) treeing T of E $\supseteq T_n$ treeing of ~~E_n~~ F_n w/ Borel transversal S_n, so by above arg, $T \cap E_n$

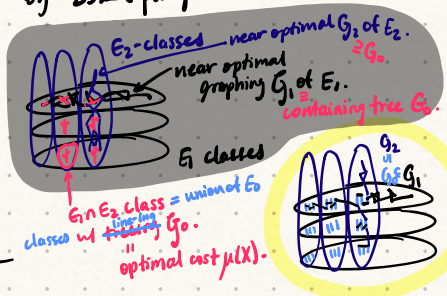
$F_n \subseteq T$ ... F_n classes, $\tilde{S}_n$

$\mu(X) \le \phi_\mu(E) \le \phi_\mu(T_n) = \mu(X\setminus \tilde{S}_n)$ ($\le \mu(X\setminus S_n)$). Take limit to conclude.

Normalized cost $\Phi_\mu := \Phi_\mu - \mu(X)$ (usually μ is prob msr).

23.4 Φ subadditivity (Φ subadd'y is trivial).
when $E_1 \cap E_2 \subseteq_{CBER}$.

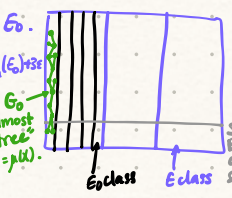
Pf: by 23.2 (pairpair construction), $\exists \subseteq_{CBER} E_0 \subseteq E_1 \cap E_2$ $\xrightarrow{22.1}$ $\exists$ ~~treeing~~ Line-ing G_0 of E_0 attaining cost $\mu(X)$.



$\Rightarrow (G_1 - G_0) \cup (G_2 - G_0) \cup G_0$ graphing of $E_1 \vee E_2$
cost $\leq \Phi_\mu(G_1) + \Phi_\mu(G_2) + \mu(X)$ as desired.

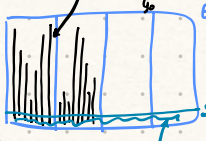
edit 3/31/26 but to Th
see below right for better picture

Need graphing extension lemma (23.1) $E_0 \subseteq E$, G_0 (near) optimal graphing of E_0.
$\subseteq_{CBER}$. $\exists G_0$ graphing E w/ $\Phi \leq \Phi_\mu(E) + \Phi_\mu(E_0) + 3\varepsilon$



do use more expensive half of 21.1
$\Phi_{\mu|_S}(E|_S) \leq \Phi_\mu(E) - \mu(X \setminus S)$.
$= \Phi_\mu(E) + \mu(S)$.
$< \varepsilon$.

3/31/26 **23.1** line-ing of E_0-classes



Pic Title: "Grass on Shallow Sea"

(23.1) $\exists$ graphing G_S of cost $\Phi(E) - 1 + O(\varepsilon)$
$\Rightarrow G_0 \cup G_S$ graph' of E, cost $\Phi(E) + O(\varepsilon)$
$(-\mu(X) + \Phi(G_0))$
" if G_0 is Line-ing of $\subseteq$ hfin E_0.

27.10 remark says every CBER has Borel fin graphing! That's nice, a general result better than complete graphing.

KJL 3.12

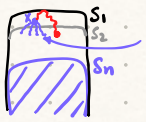
also 22.1 alternative pf uses "tail equiv'rel is hyperfinite". Query led to Slutsky notes 2.1.1 not smooth no finite inv msr
3.3.4 hyperfinite (by $\subseteq E_0$) reduction

uses std tools: Feldman Moore vanishing seq of markers — here we use strengthened version

Lemma: $\forall x$, $\exists$ short path from x to S_n using edges of small color ($\leq n^2$).
(R is any fixed graphing of E (could be split K$_{\aleph_0}$))
can strengthen VSM $\{S_n\}_n$ to: $R^{\leq n^2}$
$\leq n^2$

Pf: define $\check{S}_n := S_n \cup \{$all $x \in S_1 \setminus S_n$ s.t. $\nexists$ path len $\leq n$ using only colors $\leq n^2\}$.
(from x to any pt in lower layer)
i.e. all paths from x to lower layer either too long or too costly (uses too large color)





S_n split section, $\check{S}_n \supseteq S_n \Rightarrow \check{S}_n$ split section.

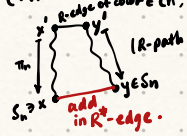
Check: $\bigcap \check{S}_n = \emptyset$. well, any x has path to pt in lower layer.
Just wait until $x \notin S_n$ & this path has len $\leq n$ using colors $\leq n^2$.
for n large enough

Check: suppose $x \notin \check{S}_n \leftrightarrow$ by def $\exists$ path len $\leq n$ using colors $\leq n^2$ from x to lower level.
At worst, $x \in S_1$, $x_1 \in S_2$ } $\leq n$, ..., $x_n \in S_n$ } $\leq n$
$\Rightarrow \exists$ len $\leq n^2$ path x to S_n using colors $\leq n^2$.

Using lemma we construct Borel fin graphing of E. $\subseteq_{CBER}$ which currently has complete graphing $R = K_{\aleph_0}$.

Borel fn $\pi_n(x) =$ endpt of lex least R-path x to S_n of len $\leq n^2$ color $\leq n^2$.
($\pi_n \equiv$ id on S_n).

Define graphing R^*:



(R-path len $\leq n^2$ color $\leq n^2$).

- Nice short Exc: induct on color of edge $(x',y') \in R$ to show $x' R y' \Rightarrow \exists R^*$ path x' to y'.

- R^* Borel fin. Also nice exc.

3/27/26 return to **22.1** other direction. (pg 71)
alternative pf .... Adams end theorems

Case 1: have treeing T of E w/ $\deg_T = 2$ (a.e.) restrict to full set on which $\deg_T \equiv 2$.

Prop: $\Rightarrow E$ hfin. (another common situation used (e.g. Schrier graph of Borel action $\mathbb{Z} \curvearrowright X$)).

Pf: same 2 tools:
- Feldman Moore: Borel N-color edges of T.
- vanishing seq of markers.
$S_n \searrow \emptyset$. (sidenote: just implies S_n is split section. S_n is C-infinite! ($\forall C \in X/E$))

blue < red (as colors $\in \mathbb{N}$)



so w is left nbr of x.
(possible that x is left nbr of x...)

Def "v is left of x" $\Leftrightarrow$ the unique T-path v to x goes thru left nbr of x.
v, x same T-comp

Subcase 1: if C s.t. $\exists^n x_0 \in S_n \cap C$ s.t. all other $y \in S_n \cap C$ are "right of x_0", then those C are "smooth part" of E (x_0 is Borel representative/selector of C) $\Rightarrow$ hyperfinite on that portion of X.
(or "left of x_0")

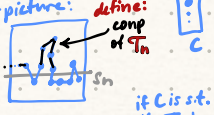
Else: $\forall n \forall C \in X/E$, $\forall x \in S_n \cap C$ $\exists^\omega y_m$ "on left/right of x"



recede as $n \to \infty$. Like low tide.

Define subtree $T_n \subseteq T$ by killing all edges $e^T \in T$ touching S_n
C-fin! "dry edges" if we think of S_n as tide at time n.

summarize. Each comp $C \in X/E$ is bamboo tree.
Add in S_n to our picture:



Subcase 1: if C is s.t. $\forall^n T_n$ has $|\text{comp}| = \infty$ in C, then $\exists^! x \in C$, Borel selector!
Else: T_n FBER $\subseteq T$. ✓

pruning
Case 2: (leaf-trim operation countably many times. lines = birays in T survive above pruning.

Borel flavored countable combinatorics.

- ctbl dots "class C"

things we can be given
- each dot has phonebook of all its classmates. (L-N thm)
  x knows $\{x, f_1(x), f_2(x), \dots\} = C$ (no repeats, say)

→ enough for Feldman-Moore (and strengthening to N-coloring of intersection graph of finite subsets)

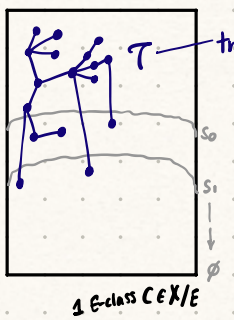
- ~~have order $(\subseteq \mathbb{Q})$, i.e. assignment $C \hookrightarrow \mathbb{Q}$.~~ $(\cong \zeta_0)$ Borel isothm
  Wait no good this is ctble coloring... OK, so just oracle $<(x,y) \to \{-1, 0, 1\}$ obeying total order axioms.
- ~~Borel~~ max'l indp set
  223D Lec21: $\chi_B(G) \le \aleph_0$ e.g. ⇒ pairpair construction ⇒ vanishing seq of markers.

I think maybe $C \hookrightarrow \mathbb{N}$ more accurate (to Borel Isothm)

Do still get ctbl basis U_n, and hard to get. simultaneous algo for N-coloring from this data.

- locfin ⇒ $\chi_B \le \aleph_0$ in 223D Lec19 uses ctble basis U_n
  - $\exists U_n \subseteq C$, w/ "basis property" $\forall x \neq y$ in C, $\exists U_n \ni x, U_n \not\ni y$, and $\forall U_n, U_m \exists U_k \subseteq U_n \cap U_m$.
    (oracle every $x \in C$ has access to, "U_n")
  - ~~alternatively $\exists C \subseteq \mathbb{R}^2$ as pictured in 223D arguments holds~~ Wait $C \hookrightarrow \mathbb{Q} \subseteq \mathbb{R}$ should suffice for this ctble basis stuff.

Sliding pf of 22.1 (ii) ⇒ (i) hfin.



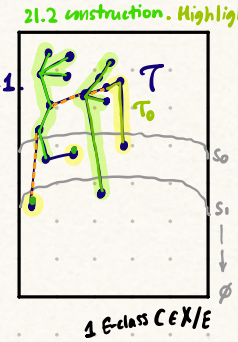
cost = 1 attained

tracing of E attaining cost = 1.

1 E-class $C \in X/E$

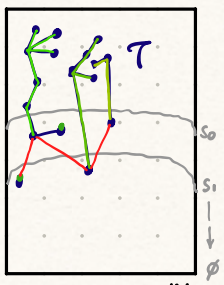
21.2 construction. Highlighted finite sets ≡ E_0-classes.

dotted edges indicated slide!



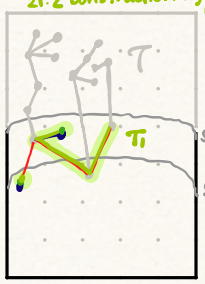
1 E-class $C \in X/E$

cost $\phi_\mu(T_0) = \mu(X \setminus S_0)$ from 21.1



1 E-class $C \in X/E$

21.2 construction again w/ "base" S_1.



1 E-class $C \in X/E$

$\phi_\mu(T_1) = \mu(S_0 \setminus S_1)$.

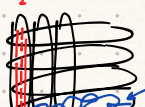
$T_0 \cup T_1$ comps = E_1-classes. = unions of E_0-classes.

Edit 9/13/26: This interjection on 24.8 is 3/28/26; I remember thinking about this while at Gin West, Saturday 3/28.

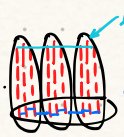
hyperfinite ⊆ CBER

L_2 a Line-ing of $F_2 \subseteq E_2$.

24.8



24.7 "i.m. stitches condition" ↪ $\exists G_1 \subseteq G_1$ s.t. w/ mrr $< \varepsilon$.



"enough horizontal movement that w/ help of Line-ings, graph $E_1 \vee F_2$.



Dually, $\exists$ mrr $G_2 \subseteq G_2$ connect each E_2 class. So $\frac{1}{\varepsilon}$ graph nbh of $E_1 \vee E_2$.

$E_0 \subseteq E_1 \subseteq \dots \subseteq E$ all FBERs and $\phi_\mu(E) = \mu(X) = \mu(X \setminus S_0) + \mu(S_0 \setminus S_1) + \dots$

⇒ a.e. $\bigcup E_n = E$ (not just ⊆).

25.1



$[F:E] = n < \infty$.

$E \subseteq F$ a CBERs.

Claim: $\phi_\mu(F) \le \phi_\mu(E)$

Intuition:



need only add a few edges to stitch E-graphings together ↪ F-graphing



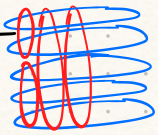
red edges $(n-1)$ per $x \in S$ increase cost $< \frac{\varepsilon}{2}$.

for all $x \in S$, knows how to get to all F-classmates, so goes thru and finds $x_1 := x, x_2, \dots, x_n$ in different E-classes. in Borel way

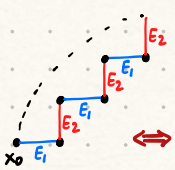
↪ Borel fns $f_1 = id, f_2, \dots, f_n : S \to S$

25.3 $\phi_\mu(E) \le n\, \phi_\mu(F)$. Cor, $\Phi_\mu(E) \le n\, \Phi_\mu(F)$. by applying to

non commuting:

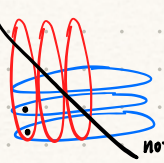


27 Def $E_1 \perp E_2 :\iff$ no alternating cycles



$\iff$ unique path

alternating



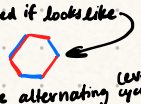
not good picture here.

actually in context of below notations, no



remark:  i.e. can work w/ only even cycles.

$E_1 \perp_{E_3} E_2 \iff E_1 \perp_\Lambda E_2$ still means but allowed if looks like



no



but OK

E_3 has to be $= E_1 \cap E_2$

disallow pure alternating (even) cycles (impure allowed)

27.4 R_1 hfin! R_2 join to $R := R_1 \vee R_2$

Suppose: R_1 blue $\cap$ red $= R_3$ — $\mathbb{I}_1$ blue edges, $\mathbb{I}_2$ red edges, disjoint union is graphing of R.

Then, $\exists$ $\mathbb{I}_1 \sqcup \mathbb{I}_2$ disj union is graphing of R_1 — disj union is treeing of subequiv rel of R_3

Claim 1: $R_3 \cap R_1' = R_3 \cap R_2'$ (*) $\Rightarrow R_1' = R_1$, $R_2' = R_2$.

Pf: s.t.s.o.c. $R_1' \subsetneq R_1$.



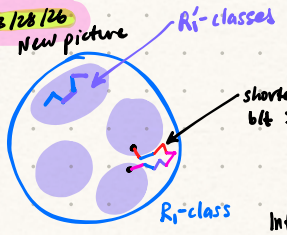


may not $\exists$ blue path b/t 2 black dots b/c $\mathbb{I}_1$ not nec graphing of R_1

do exist br paths, possibly bper paths. (R_1, R_2)

Choose shortest such path

2/28/26 New picture



for some reason book calls base space Y.

$\Rightarrow$ start and end w/ red or pink.

Infact, I've drawn it WRONG b/c shortest path b/t 2 purple islands in 1 big blue lake is path made entirely of



But then $(x,y) \in R_1$ and $\in R_2$ (i.e. $\in R_2'$ w/ (*)), i.e. $\in R_3$!

$\Rightarrow (x,y) \in R_3 \cap R_2' = R_3 \cap R_1'$ (*)

$\Rightarrow (x,y) \in R_1'$, $\dot{\lightning}$.

By R_3 hfin, $= \bigcup_{\text{FBERs}} R_3^n$. R_3^0 = singleton ER.

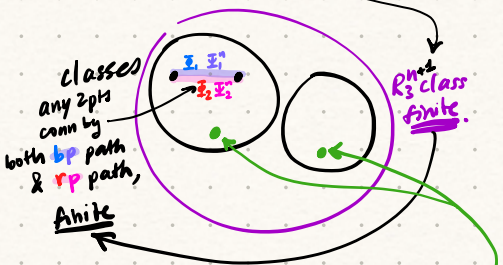
Induction hypothesis: $(x,y) \in R_3^n \Rightarrow$

(i) $[x \sim y \text{ in } \mathbb{I}_1^n \sqcup \mathbb{I}_2^n] \Rightarrow [x \sim y \text{ in } \mathbb{I}_1^n \sqcup \mathbb{I}_1$ and $\mathbb{I}_2 \sqcup \mathbb{I}_2^n]$

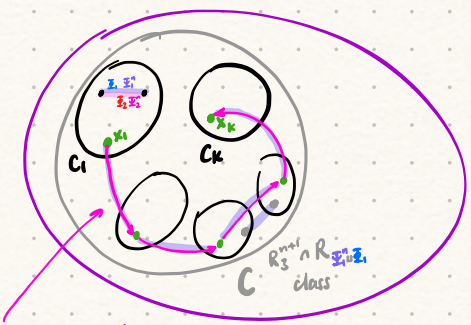
$x \sim y$ in $\dfrac{\mathbb{I}_1^n \sqcup \mathbb{I}_1}{\mathbb{I}_2 \sqcup \mathbb{I}_2^n} \overset{\text{(2) } n \text{ even}}{\underset{\text{(3) } n \text{ odd}}{\rightleftharpoons}} \text{ in } \dfrac{\mathbb{I}_2 \sqcup \mathbb{I}_2^n}{\mathbb{I}_1^n \sqcup \mathbb{I}_1}$ "back and forth argument".

Base $\mathbb{I}_1^0, \mathbb{I}_1^0 = \emptyset$.

wlog $n+1$ is even.



so Borel way of picking representative in each black class.



new edges in $\mathbb{I}_2^{n+1}$

$\mathbb{I}_1^{n+1} := \mathbb{I}_1^n$ stay the same.

27.6. E_i sunflower (pairwise $\cap$ = common $\cap =: F$)

top of pg 90 !! E_0 hfin has line-ing $\mathbb{I}_0 = \{\phi_0\}$

finite and each "near optimal" (for the relation it generates)

cost of difference $\leq \frac{1}{n^2}$

$\subseteq \mathbb{I}_1^1 \subseteq \mathbb{I}_2^1 \ldots \nearrow \mathbb{I}^1$ graphing of E_1

$\subseteq \mathbb{I}_1^2 \ldots \ldots \nearrow \mathbb{I}^2$ // E_2

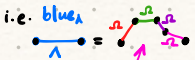
$\mathbb{I}^{\leq n} =$ finite sense $= \{\phi_1 \ldots \phi_{k_n}\}$ — disj union of or equivalently whole upper left.

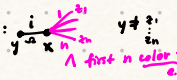
join E

27.2 pg 88 "L-graphing" $\Omega := \Omega_1^{E_1} \sqcup \Omega_2^{E_2}$

Λ near optimal graphing of E $\quad \lambda \in \Lambda$ = composition of w_n^i.

I think I can think of as graphing G w/ N-coloring of edges $e^E \in G$ (sure, in my framework, all elts of Ω, Λ are involutions)

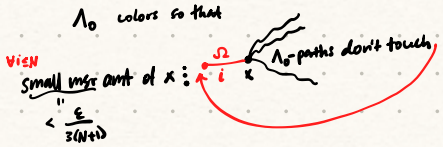
i.e. blue$_\lambda$ = 

$w_n^i := \{$ all x:  $\}$

Λ first n color types e.g.

N fixed.

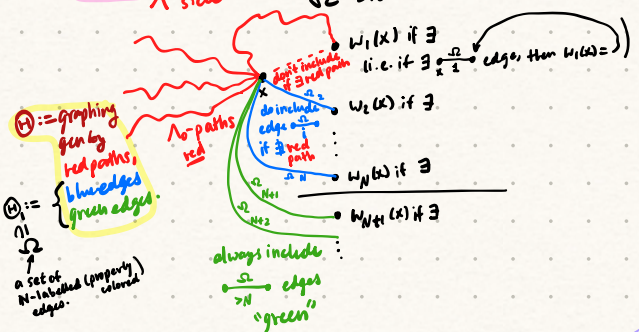
$\forall i \le N$, Λ_0 so that

Λ_0 colors so that 

small msr ant of x: $< \frac{\varepsilon}{3(N+1)}$

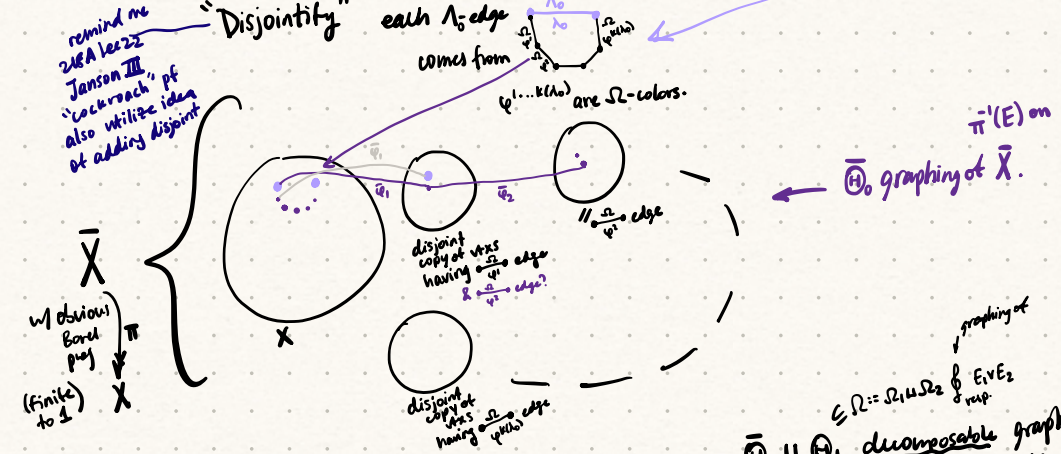
3/19/26

Λ side $\qquad$ Ω side



remind me 216A Lec 22 Janson III "cocursach" pf also utilize idea of adding disjoint

"Disjointify" each Λ_0-edge comes from $\varphi^{1 \dots k(\lambda_0)}$ are Ω-colors.



$\subseteq \Omega := \Omega_1 \sqcup \Omega_2$ graphing of $E_1 \vee E_2$ resp.

$\bar{\Theta}_0 \sqcup \Theta_1$ <u>countable</u> graphing of $\pi^{-1}(E) = \pi^{-1}(E_1) \vee \pi^{-1}(E_2)$

so can apply 27.4.

27.6 is $C_\mu^F \ge 0$? "monotonicity" $C_\mu^F\left(\bigstar_F^{i \in I} E_i\right) \ge C_\mu^F\left(\bigstar_F^{i \in I' \subseteq I} E_i\right)$ right? (Just like ctbly superadd of msrs follows from monotonicity & finite superadditivity)

ineq would follow from ... and implies, assuming ...

→ some ergodic theory needed

Farrell Vadharajan unit ergodicdecomp ← Ronnie Chen has 3pg pdf uses Carathéodory ext thm @ 1 small step

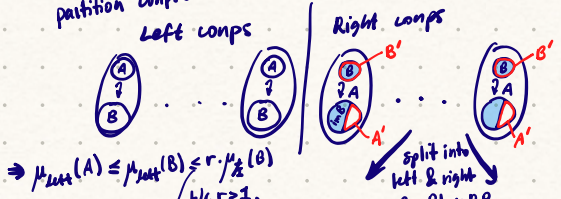
easy (1pg) Tao's "metric space ext" proof suffices.

finite $\mu^*(X)$ case: pseudometric $\mu^*(A \triangle B)$

elems = nbhds. $A \mapsto \mu^*(A)$ Lipschitz ctn by subadd

his lemma 2 = bonkers induction on easy Lemma 1 (see diary 2026)

partition comps of $X = L \sqcup R$



$\Rightarrow \mu_{Left}(A) \le \mu_{Left}(B) \le r \cdot \mu_R(B)$ b/c $r \ge 1$.

induction

$\mu_{RL}(A') \le (r-1) \cdot \mu_{RL}(B)$ $\qquad$ $\mu_{RR}(B) \le (r-1) \cdot \mu_{RR}(A')$

Then $\mu_{LURL}(A) = \mu_L(A) + \mu_{RL}(A) \le r \cdot \mu_{LURL}(B)$ ☺. Similary, $\mu_{RR}(B) \le r \cdot \mu_{RR}(A)$

$\le r \cdot \mu_L(B)$ $\quad = \mu_{RL}(B) + \mu_{RL}(A') \le (r-1) \cdot \mu_{RL}(B)$