# ZFC

1/6/26 my curation of the axioms (this list is equivalent to standard ZFC)

1 Equality/extensionality:

$$\forall A, B(A = B \leftrightarrow \forall x(x \in A \leftrightarrow x \in B))$$

3 Combinatorial axioms: (inexact/overshoot) pairing

$$\forall a, b(\exists P : a \in P \wedge b \in P),$$

union

$$\forall \mathscr{F}, \exists U : \forall x \forall F(x \in F \in \mathscr{F} \to x \in U),$$

power set

$$\forall X, \exists \mathscr{P} : \forall S(S \subseteq X \to S \in \mathscr{P})$$

of course $A \subseteq B \ :\iff \ \forall x(x \in A \to x \in B)$.

2 Large cardinal axioms (invoking existence of things unreachable from below): Empty Set:

$$\exists \varnothing : \forall x(x \notin \varnothing)$$

Infinity:

$$\exists w : (\varnothing \in w \wedge (\forall x, x \in w \to S(x) \in w))$$

of course $S(x) \coloneqq x \cup \{x\}$

1 Choice:

$$\forall \mathscr{F}, \underbrace{\mathscr{F} \text{ is family of non-empty pairwise disjoint sets}}_{\forall A,B,(A,B\in\mathscr{F}\to(A\neq\varnothing\wedge(A\cap B\neq\varnothing\to A=B)))} \to \exists Z : \forall A(A \in \mathscr{F} \to \exists! z : z \in Z \wedge z \in Y)$$

where of course things like "$S \neq \varnothing$" is shorthand for $\exists x(x \in S)$ and "$A \cap B \neq \varnothing$" is shorthand for $\exists x(x \in A \wedge x \in B)$. And $\exists! z : \varphi(z)$ is $\exists z(\varphi(z) \wedge \forall z'(\varphi(z') \to z = z'))$.

1 Induction/Foundation: in context of other 9 standard axioms (key thing being $\mathbb{N}$-recursive definition, to get transitive closure), https://math.stackexchange.com/questions/4494302/why-is-the-axiom-of-foundation-not-presented-in-terms-of-its-arguably-more-intu, https://math.stackexchange.com/questions/1757287/axiom-of-foundation-regularity-implies-epsilon-induction

$\in$-induction schema is: for 1 var formulas $\varphi_1(\mathsf{x}), \ldots$ (can have computer print them out, unhaltingly)

$$\Big(\forall A, \big(\forall a, a \in A \to \varphi_1(a)\big) \to \varphi_1(A)\Big) \to \forall x, \varphi_1(x)$$

$$\vdots$$

2 Schema: comprehension/separation/(definable) subset/cutting out Axiom. https://cms-cdn.lmu.de/media/10_phil/mcmp/downloads/elementary_set_theory.pdf#page=13 For 1+$n$-free var+param

formulas $\varphi_1(\mathsf{x}, \mathsf{p}_1, \ldots, \mathsf{p}_n), \ldots$ (can have computer print them out, unhaltingly)

$$\forall p_1, \ldots, p_n,\ \forall U \exists S : \forall x (x \in S \leftrightarrow \big(x \in U \wedge \varphi_1(x, p_1, \ldots, p_n)\big))$$

$$\vdots$$

notation is p for parameter, U for universe, S for subset

Replacement/Image/Codomain or Collection https://en.wikipedia.org/wiki/Axiom_schema_of_replacement#Collection For $2{+}n$-free var+param formulas $\varphi_1(\mathsf{x}, \mathsf{y}, \mathsf{p}_1, \ldots, \mathsf{p}_n), \ldots$

$$\forall p_1, \ldots, p_n,\ \forall D, \exists C : \forall x \Big( \big(x \in D \wedge \exists y : \varphi_1(x, y, \vec{p})\big) \to \exists y \in C : \varphi_1(x, y, \vec{p}) \Big)$$

p for parameter, D for domain, C for codomain

But replacement is $\iff$ principle of transfinite recursion https://jdh.hamkins.org/transfinite-recursion-as-a-fundamental-principle-in-set-theory/

1/7/25 $[\![\bullet \in \bullet]\!]$ is class function $V^B \times V^B \to B$. Defined using https://spot.colorado.edu/~szendrei/ST_S21/lec-ordinals.pdf

$$\text{“}u \in V^B\text{”} \coloneqq \mathrm{InVB}(u, B) \qquad \text{“}[\![u \in v]\!] = b\text{”} \coloneqq \mathrm{BvaluedIn}(u, v, b, B) \checkmark$$

query "transfinite recursion class function on well ordered class"

Statement of transfinite recursion theorem: $C$ is a class (so correspond to 1st order formula $x\mathrm{isInC}$ — Szendrei uses $\psi(x)$), $R$ is a class relation that is well-founded and set-like on $C$, i.e. $\mathrm{inR}(x, y)$ or $R$-precedes $x\mathrm{R prec} y$ — Szendrei uses $\varphi(x, y)$

well-foundedness is ($X$ is non-empty subset of $C$) — Szendrei display (1)

$$\forall x, y (\mathrm{inR}(x, y) \to x, y \mathrm{isInC}) \wedge \Big( \forall X \big( \underbrace{\varnothing \neq X \subseteq C}_{X \neq \varnothing \wedge \forall x \in X (x \mathrm{isInC})} \big) \to \exists x \in X : \underbrace{\forall y \in X, \neg \mathrm{inR}(y, x)}_{\text{“}x \text{ is } R\text{-minimal in } X\text{”}} \Big)$$

set-likeness is

$$\forall x, y (\mathrm{inR}(x, y) \to x, y \mathrm{isInC}) \wedge \forall x \big( x \mathrm{isInC} \to \underbrace{\exists S : \forall z (z \in S \leftrightarrow \mathrm{inR}(z, x))}_{\text{“}\mathrm{prec}_{R,C}(x) \coloneqq \{z \in C : zRx\} \text{ is a set } S \in V\text{”}} \big)$$

Recursive formula is given by class function $G : C \times V \to V$ correspond to formula $\theta_G$

Want to build formula for class function $F(c) = G(c, F|_{\mathrm{prec}_{R,C}(c)})$, called $\theta_F(x, y)$ (or Szendrei calls $\theta(x, y)$), again key source is pg. 11 of https://spot.colorado.edu/~szendrei/ST_S21/lec-ordinals.pdf

Formula $f\mathrm{IsApproxToF}$ is: $f$ isFunction AND for all $a \in \mathrm{dom}(f)$ we want $f(a) = G(a, f|_{\mathrm{prec}_{R,C}(a)})$ so in particular we need $\mathrm{prec}_{R,C}(a) \subseteq \mathrm{dom}(f)$. The bit after the “AND” is formalized ($f'$ is the symbol representing $f|_{\mathrm{prec}_{R,C}(a)}$)

$$\underbrace{\psi(f', f, a)}_{f' = f|_{\mathrm{prec}_{R,C}(a)}} \coloneqq f'\mathrm{isFunction} \wedge \underbrace{(\mathrm{dom}(f') = \mathrm{prec}_{R,C}(a))}_{\forall x (x \mathrm{InDomOf} f' \leftrightarrow x \mathrm{R prec} a)} \wedge \forall a \big( a \mathrm{InDomOf} f' \to \underbrace{f'(a) = f(a)}_{\exists y : y \mathrm{IsEval}(f', a) \wedge y \mathrm{IsEval}(f, a)} \big)$$

$$f\mathrm{IsApproxToF} \coloneqq f\mathrm{IsFunction} \wedge \forall a \big( a \mathrm{InDomOf} f \to \exists f' \big( \underbrace{\psi(f', f, a)}_{f' = f|_{\mathrm{prec}_{R,C}(a)}} \wedge \underbrace{\theta_G(a, f', f(a))}_{\exists z : z \mathrm{IsEval}(f, a) \wedge \theta_G(a, f', z)} \big) \big)$$

$$\theta_F(x,y) \coloneqq \exists f(f\text{IsApproxToF} \wedge y\text{IsEval}(f,x))$$

And $[\![\varphi]\!]$ expand out (in my ontology, I have $\varphi$ as a string I wrote down in front of me, and I use the syntax manipulation rules moving $[\![,]\!]$, logical symbols to reduce to something involving only BvaluedIn, BvaluedEqual, InVB).

query "bell boolean forcing pdf" led to PDF https://math.i-learn.unito.it/pluginfile.php/97222/mod_resource/content/2/dispenseTI2014.pdf which does cover Boolean soundness theorem

## 0.1 Useful Formulas

$$\text{“}\exists! x\ \varphi(x)\text{”} \coloneqq \exists x(\varphi(x) \wedge \forall z(\varphi(z) \to x = z))$$

1/12/26 Also note bounded formula theorem https://dn790009.ca.archive.org/0/items/settheoryboolean00bell/settheoryboolean00bell.pdf 1.23(v), helpful for “isN”, “IsSubsetOfN”

I had other ideas for Pair (hasCard2, etc.) but https://math.stackexchange.com/questions/3961921/defining-ordered-pairs-using-only-first-order-logic has better method

(bounded/restricted quantifier formulas) $p = \{\{a\},\{a,b\}\}$, forall components $c \in p$,...

$$p\text{IsPair}(a,b) \coloneqq \forall c \in p(\underbrace{p = \{a\}}_{a\in p \wedge \forall x \in p(x=a)} \vee \underbrace{p = \{a,b\}}_{a,b\in p \wedge \forall x\in p(x=a\vee x=b)})$$

$$p\text{IsPair} \coloneqq \exists c \in p \exists a,b \in c : p\text{IsPair}(a,b)$$

$$p\text{IsPairFrom}(D,C) \coloneqq \exists a \in D \exists b \in C : p\text{IsPair}(a,b)$$

$$z\text{IsTr} \coloneqq \forall y \in z \forall x \in y(x \in z)$$

$$f\text{IsFn}(D,C) \coloneqq \forall p \in f\ p\text{IsPairFrom}(D,C) \wedge \forall p,q \in f(\underbrace{\text{same 1st coord}}_{\forall c\in p \forall d\in q \exists a\in c: a\in d} \to p = q)$$

https://spot.colorado.edu/~szendrei/ST_S21/lec-ordinals.pdf

$$\alpha\text{IsOrd} \coloneqq \alpha\text{IsTr} \wedge \forall \beta \in \alpha(\beta\text{IsTr})$$

$$S\text{isSucc}(x) \coloneqq \forall z \in S(z = x \vee z \in x) \wedge \Big(x \in S \wedge \forall z \in x(z \in S)\Big)$$

https://math.stackexchange.com/questions/228789/whats-the-definition-of-omega Michael G. definition

$$n\text{IsNat} \coloneqq n\text{IsOrd} \wedge \forall m \in n(\underbrace{m\text{IsOrd}}_{\text{automatic from } n\text{IsOrd}} \wedge \exists l : m\text{IsSucc}l)$$

$$S\text{IsSubsetOfN} \coloneqq \forall n \in S(n\text{IsNat})$$

$$\mathcal{S}\text{IsSubsetOfPowN} \coloneqq \forall S \in \mathcal{S}(S\text{IsSubsetOfN})$$

$$e\text{IsEmpty} \coloneqq \forall x \in e(x \neq x)$$

$$w\text{ContainsN} \coloneqq \exists e \in w(e\text{IsEmpty}) \wedge \forall x \in s \exists y \in s(y\text{IsSucc}x)$$