ZFC ⊬ CH CH: $\forall A \subseteq \mathbb{R}$ ($A \subseteq \mathbb{R}$ / $\mathrm{Pow}\mathbb{N}$) $A \hookrightarrow \mathbb{N} \vee \mathbb{R} \twoheadrightarrow A$.

$\neg$CH: $|\mathbb{N}| < |M| < |\mathbb{R}|$

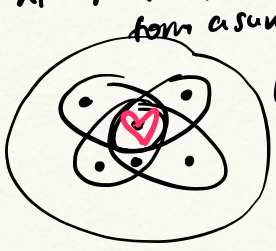
$\nexists h_1 : \mathbb{N} \twoheadrightarrow M$, $\nexists h_2 : M \twoheadrightarrow \mathbb{R}$.

$\exists f : \mathbb{R} \hookrightarrow S \in \mathrm{Pow}\mathbb{N}$

$\neg$CH: $\exists K\ \exists S\ \exists f\ \forall h_1\ \forall h_2$ (S is Subset Of $\mathrm{Pow}\mathbb{N}$ ∧ N is Set of Nats ∧ f is Fn (K, S) ∧ h_1 is Fn (N, M) ∧ h_2 is Fn (M, K) ∧ f is Inj ∧ $\neg h_1$ is Surj ∧ $\neg h_2$ is Surj)

Part C combinatorics.

Sunflower: $(X_i)_{i \in I}$ (I: finset, unc) then $\exists$ u.m. X_i that together form a sunflower



$U := \bigcup_{i \in I} X_i$

① unc X_i ctbly sized $0, 1, 2 \ldots$ ⇒ unc X_i of same size $n \in \mathbb{N}$.

② Induct on n.

($n=0$) $X_i = \emptyset$ $\heartsuit = \emptyset$

③ ($n \to n+1$) $|X_i| = n+1$

Case1: $\exists \bullet \in U$ in unc X_i.

Remove $\bullet$. Ind Hyp $\heartsuit_n$

$\heartsuit_{n+1} := \heartsuit_n \cup \{\bullet\}$

④ **Case2**: o.w $\forall \bullet \in U$ has $\leq$ c.m $X_i \ni \bullet$

$\forall X_i \Rightarrow \leq$ c.m $X_j \cap X_i$.

construct unc fam:

X_0.

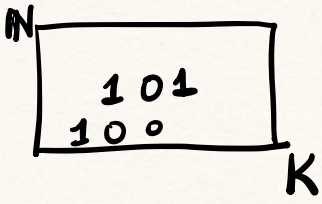
X_i

unc fam disjoint sets.

K any set.

open subsets of $2^{K \times \mathbb{N}}$

$\downarrow$ elts 01 tables

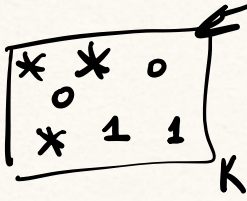


* notation

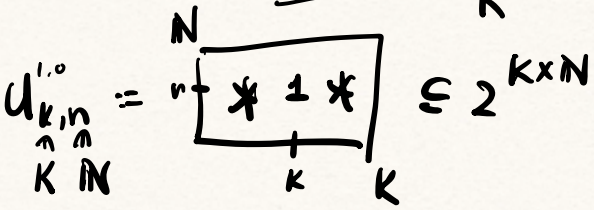
$0* = \{00, 01\} \subseteq 2^2$

findet sets (subset of $2^{K \times \mathbb{N}}$)

$p: K \times \mathbb{N} \to 2 = \{0,1\}$ findom $\to$ $U_p := \{$set all $f: K \times \mathbb{N} \to 2$ extending $p\}$.



$\subseteq 2^{K \times \mathbb{N}}$



$\subseteq 2^{K \times \mathbb{N}}$

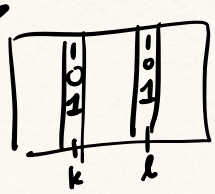
We define open subsets $2^{K \times \mathbb{N}}$ = arb $\bigcup$ of findet sets

closed $\leftarrow$ arb $\bigcap$ of finite unions of findet

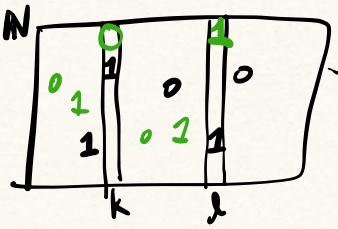
e.g. Fix $k \neq l$ in K

$\emptyset \neq \{$01 tables $\mid$ col k = col l$\}$ $\subseteq 2^{K \times \mathbb{N}}$ = $\bigcap_n$ [1*1 at k, l] $\cup$ [0*0 at k, l] $\downarrow$ closed

findet



Pf: consider arb findet set



countable disj open

Claim: $(U_i)_{i\in I}$ unc fam, then some pair not disj

opens in $2^{K\times\mathbb{N}}$

$U_{p_i}\ U_{p_j}$

(fam pairdisj open $\Rightarrow$ fam countable

thm: open sets of $\mathbb{R}$

Pf: U_i open

$\forall$

U_{p_i} findom $p_i : K\times\mathbb{N}\to 2$

$U_{p_i}\cap U_{p_j}=\emptyset \iff$ no function extends both $\operatorname{dom}(p_i)\cap\operatorname{dom}(p_j)$

$p_i(\cdot)\neq p_j(\cdot)$

$(\operatorname{dom}(p_i))_{i\in I}$ fin $\Rightarrow$ sun unc p_i

doms pairwise $\cap = \heartsuit$ fin $\subseteq K\times\mathbb{N}$



unc p_i $\Rightarrow$ unc p_i whose restr to $\heartsuit$ is the same

fin map $\heartsuit\to 2$

$\downarrow$

all those p_i will have common extension

Part B Boolean alg.

$B := \operatorname{Pow}(S)$ (S: space) $\subseteq$, $0_B = \emptyset$, $1_B = S$, $b^\neg := S\setminus b$.

inf = glb = $\bigcap$

sup = lub = $\bigcup$

complete Bool alg.

$\forall b\in B$ $b\vee b^\neg = 1_B$, $b\wedge b^\neg = 0_B$

distr. $a\cap(b\cup c) = a\cap b \cup a\cap c$.

$b\Rightarrow b' := b^\neg \vee b'$

Fact: $(b\Rightarrow b') = 1_B$ iff $b\subseteq b'$.

$B := RO(S)$ (reg opens)

picture $\mathbb{R}$, $\mathbb{R}^2$

set U (open) reg op

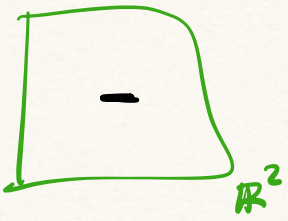
$U^{\circ} = U$

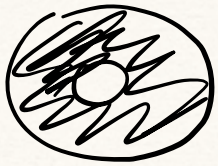
$\subseteq$, 0_B, 1_B

$0_B = \emptyset$, $1_B = S$

ex. $\overline{(1,2) \cup (2,3)} \subseteq_{op} \mathbb{R}$

$[1,3]^{\circ} = (1,3)$





Thm $\bigwedge b_i := (\bigcap b_i)^{-\circ}$ is the glb of b_i.

Pf: suppose $U \in B$ (reg op) lowbd of b_i

$U \subseteq b_i$ $\forall i$

$U = U^{-\circ} \subseteq (\bigcap b_i)^{-\circ}$

Part S: some set theory

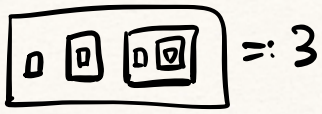
Ex of sets we care:

$\emptyset = \square =: 0$

$\{\emptyset\} = \boxed{\square} =: 1$

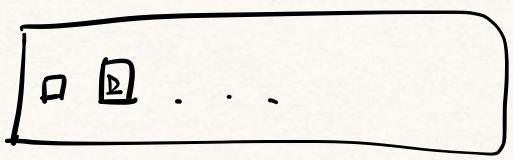
S ↓

$\{\emptyset, \{\emptyset\}\}$ $\boxed{\square\boxed{\square}}$ $=: 2$



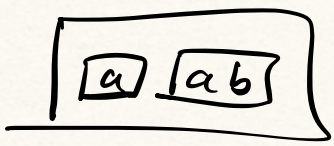
$\mathbb{N}$

$\omega = \{0, 1, 2, 3 \ldots\}$



ordered pair

$(a,b) := \{\{a\}, \{a,b\}\}$



$(a,b) = (c,d) \leftrightarrow a=c \wedge b=d$.

$pIsPair(a,b) := \forall c \in p \; (c = \{a\} \vee c = \{a,b\})$

$a \in C \wedge \forall x \in C \; (x = a)$

ext thm of ZFC: $\forall p,q \; \forall a,b,c,d \; (p \text{ is Pair } (a,b) \wedge q \text{ is Pair } (c,d)) \leftrightarrow a=c \wedge b=d$

$pIsPair := \exists a,b \; (p \; IsPair(a,b))$.

$pIsPair := \exists c \in p \; \exists a,b \in c \; pisPair(a,b)$.

$pIsPairFrom(D,C) := \exists a \in D \exists b \in C \; p \text{ is Pair } (a,b)$

$f \; isFn(D,C) := \forall p \in f \; p \; isPairFrom(D,C)$

$\wedge \forall p,q \; ($ have 1st coord $\rightarrow p=q)$

$\forall \exists \in =$

Fix B complete Boolean

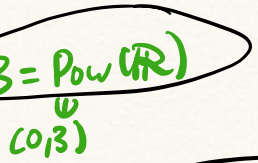
B-tagged set (Bset)

$\emptyset = \square$

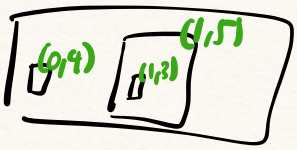


$\{(\emptyset, b)\}$.

Example: $B = Pow(\mathbb{R})$



$u :=$ 

$v =$ 

Location $s \in S = \mathbb{R}$

eval at s.

$u' := \square = 1$

$u'^{0} := \square = 0$

$u \langle s \in (0,4) \rangle = 1 =$

$u \langle s \notin (0,4) \rangle = 0$

$\{(\emptyset, (0,4)), (\{(\emptyset,(1,3)\}, (1,5))\}$

$(0,4) \in B$

B-sets : functions B-sets $\to B$.

$v \langle s \in (1,3) \rangle = 2 =$ 

$v \langle s \in (0,1] \cup [3,5) \rangle = 1$

$v \langle s \notin (0,5) \rangle = 0$

pointwise.

Def for $B = Pow(S)$

$B\text{-set}$

$B \ni [u \in v] = \{s \in S : u^s \in v^s\} = (1,3) \cup [4,5)$

$[u = v] = \ldots = [3,4) \cup (0,5)$

both 1 both 0.

"functions"

$(B\text{-sets})^2 \to B$

u is TaggedSet$(B) :=$ ~~~~

(u, v) InWhen$(b) :=$ "$[u \in v] = b$"

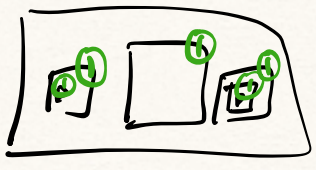
Bell

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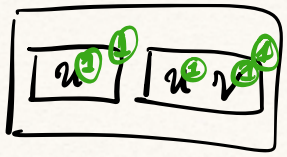
set



u

$\check{u} :=$ 

— "location invariant" deterministic

B-sets u, v $\quad Y(u, v) =$  $= \mathcal{P}$

Ex: p IsPair(a,b) =

$$[p \text{ IsPair}(u,v)] = \Big[\forall c \in p \; \big(\underbrace{c = \{a\}}_{a \in c \,\wedge\, \forall x \in c\,(x=a)} \vee \underbrace{c = \{a,b\}}_{\cdots} \big) \Big] = 1$$

$$= \bigwedge_{c \in \mathrm{dom}(p)} \Big([u \in c \wedge \forall x \in c\,(x=u)] \vee \cdots \Big) \in B.$$

$c \in \mathrm{dom}(p)$ — B set.

$[u \in c \wedge \forall x \in c\,(x=u)] = 1$

$[u \in c] = 1$ (for $c = \{u\}$)

$[\forall x \in c\,(x=u)] = \bigwedge_{x \in \mathrm{dom}\, c} [x=u] = 1$ (for $x = u$, $[x=u] = 1_B$)

$c = \{u\}$

$c = \{u,v\}$

Thm: φ (sentence in $\mathcal{L}_{set}$) is axiom of ZFC then $[\varphi] = 1_B$

Soundness Thm: φ can be deduced from axioms by logical (modus ponens) then $[\varphi] = 1_B$.

$[\varphi] = 1 \Rightarrow [\psi] = 1$

$[\varphi \to \psi] = 1$

$[\varphi] \Rightarrow [\psi]$

$[\varphi]^{\neg} \vee [\psi]$

φ is bdd quantifiers

Thm: $\varphi(x_1, \ldots x_n)$ holds

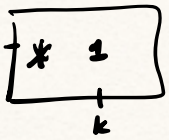
$$[\varphi(\check{x}_1, \ldots \check{x}_n)] = 1$$

Part A: (second!)

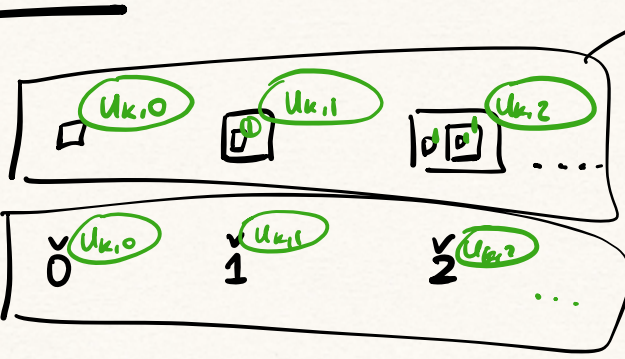
$K \hookrightarrow PN$

$\mathbb{B} := Pow(2^{K \times \mathbb{N}})$ (any set.)

$\mathbb{U}$ S = 01 tables

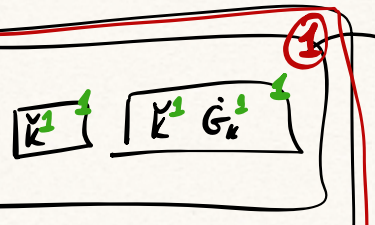
$U_{k,n} :=$ 

define $\mathbb{B}$-set $k \in K$

$\dot{G}_k :=$ 

$\{\dot{G}_k \text{ is Subset of } \mathbb{N}\} = 1$

$=$

define pairs $(\check{k}, \dot{G}_k) =$ 

$\{(\check{k}, \dot{G}_k) \text{ is Pair}\} = 1.$

define $f :=$ 

$= \{((\check{k}, \dot{G}_k), 1) \mid k \in K\}.$

$\{f \text{ is Fn}(\check{K}, S) \wedge S \text{ is Subset of } Pow\,\mathbb{N}\} = 1$

Now Claim: Not $\{f \text{ is Inj}(\check{K}, S)\} = 0$

$= \{\exists \check{k}, \check{\ell} \in \check{K} \; (\check{k} \neq \check{\ell} \wedge f(\check{k}) = f(\check{\ell}))\}$

For K very large (think $Pow\,Pow\,\mathbb{N}$)

$= \bigvee_{\substack{\check{k},\check{\ell} \in \check{K} \\ \check{k} \neq \check{\ell}}} \{\dot{G}_k = \dot{G}_\ell\}$ $= \{01 \text{ tables } \exists k \neq \ell \; col\,k = col\,\ell\}.$ $= 1_{\mathbb{B}}.$

" $\{01 \text{ tables } \mathbb{N} \times k \mid col\,k = col\,\ell\}$

Denow trick: $\mathbb{B} = RO(2^{K \times \mathbb{N}})$

$1_{\mathbb{B}} = (\bigcap)^{-0}$

$$\bigwedge_{\check{n}\in N}\left([\check{n}\in G_k]\wedge[\check{n}\in G_\ell]\right)\vee(\ldots\notin\ldots\wedge\ldots\notin\ldots)=0.$$

($[\check{n}\in G_k]=U_{k,n}$, $[\check{n}\in G_\ell]=U_{\ell,n}$)

$$\bigvee_{\ldots} O_\beta = O_\beta$$

Finale nonsurj. Fix sets $\check{Z}$, $\check{A}$ in V^B with card $|A\times N|<|Z|$.

WTS $[\exists f\ (f \text{ is } Fn(\check{A},\check{Z})\wedge f \text{ is Surj})]=0$.

$$\bigvee_{f\ B\text{-tagged}}[f \text{ is } Fn(\check{A},\check{Z})]\wedge[f \text{ is Surj}]$$

Goal: for all f (Fix f, B-set) $=0$

Thm of ZFC $\left[\forall D,C,\ \forall f,x,y,z\ \left(f \text{ is } Fn(D,C)\wedge f(x)=y\wedge f(x)=z\to y=z\right)\right]=1$

(y IsEval (f,x).)

$$\bigwedge_{\check{A},\check{Z},\ f,\check{a},\check{z}_1,\check{z}_2\ (B\text{-tagged sets})}\{\ \}=1$$

for any $a\in A$, $z_1\neq z_2\in Z$:

$$\underbrace{[f \text{ is } Fn(\check{A},\check{Z})]}_{=:b\in B}\wedge[f(\check{a})=\check{z}_1]\wedge[f(\check{a})=\check{z}_2]\leq[\check{z}_1=\check{z}_2]=0$$

$$(b\wedge[f(\check{a})=\check{z}_1])\wedge(b\wedge[f(\check{a})=\check{z}_2])=0.$$

$\in B = RO(\mathbb{P}\times\mathbb{P})$

$\Rightarrow \forall a\in A,\quad B_a := \{\underbrace{b\wedge[\check{f}(\check{a})=\check{z}]}_{\text{open}} : z\in Z\} \subseteq B$

pairwise disj

↑ Step 1

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② use Thm Part C $\quad \forall a\in A$ B_a is ctblc.

③ Claim $\forall a\in A$ only ctblc many $z\in Z$ $=: Z_a \subseteq Z$ s.t. $b\wedge[\check{f}(\check{a})=\check{z}]\neq 0$.

Pf: fix $a\in A$. use $u.m$ $z\in Z$.

pigeonhole: $u.m\ z\in Z$. $\Rightarrow$ $u.m\ z\in Z$ have same $b\wedge[\check{f}(\check{a})=\check{z}]\neq 0$ $=b\neq 0$.
ctblc B_a

④ union $\bigcup_{a\in A} Z_a \subseteq Z$. size $\leq |\mathbb{N}\times A| < |Z|$.

$\Rightarrow \exists \underline{z}\in Z$ outside union.

by def $\forall a\in A$, $b\wedge[\check{f}(\check{a})=\check{\underline{z}}]=0$

⑤ $[\![f \text{ is Surj } (\check{A},\check{Z})]\!] = \{\forall y\in Z\ \exists x\in A\ f(x)=y\}$

$$= \bigwedge_{z\in Z}\bigvee_{a\in A}[\check{f}(\check{a})=\check{z}] \leq \bigvee_{a\in A}[\check{f}(\check{a})=\check{\underline{z}}]$$

$$[\![f \text{ is Fn } (\check{A},\check{Z})]\!]\wedge[\![f \text{ is Surj } (\check{A},\check{Z})]\!] \leq b\wedge\bigvee_{a\in A}[\check{f}(\check{a})=\check{\underline{z}}] = \bigvee_{a\in A} b\wedge[\check{f}(\check{a})=\check{\underline{z}}] = \bigvee_{a\in A} 0 = 0.$$

$b := [\![f \text{ is Fn } (\check{A},\check{Z})]\!]$

Finale $[\![\neg CH]\!]=1 \iff [\![CH]\!]=0$ so ZFC $\not\vdash$ CH

contrapositive of Soundness