Edge-checkable-labeling problems Π on virtually free (f.g.) groups Γ

ATC talk

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UCLA

12/3/25

1. Birds-eye view
2. Definitions/Set-up
3. Fillability
4. Decidability
5. Future work

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Roadmap

- **Topic:** (inclusion relationships between) classes of LCL problems Π on a fixed finitely generated group Γ.

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- Many inclusion relationships already known, in the cases of $\Gamma = \mathbb{F}_d$ (free), $\mathbb{Z}^d$ (grid).

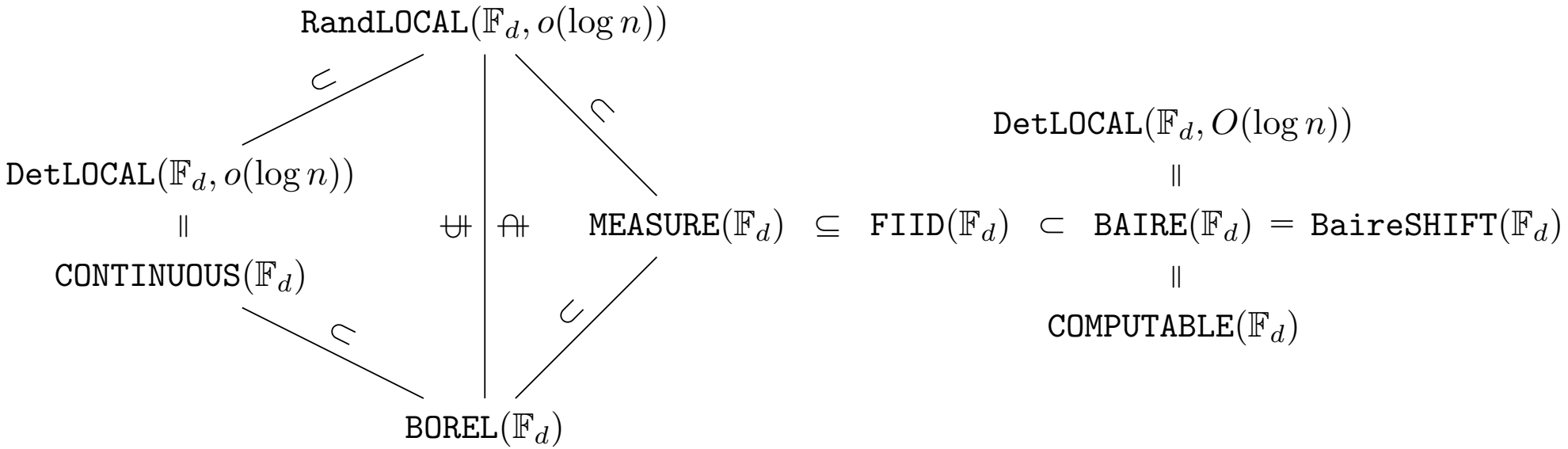
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- **Topic:** (inclusion relationships between) classes of LCL problems Π on a fixed finitely generated group Γ.
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- Many inclusion relationships already known, in the cases of $\Gamma = \mathbb{F}_d$ (free), $\mathbb{Z}^d$ (grid).
- We study classes of Π on virtually free groups Γ.

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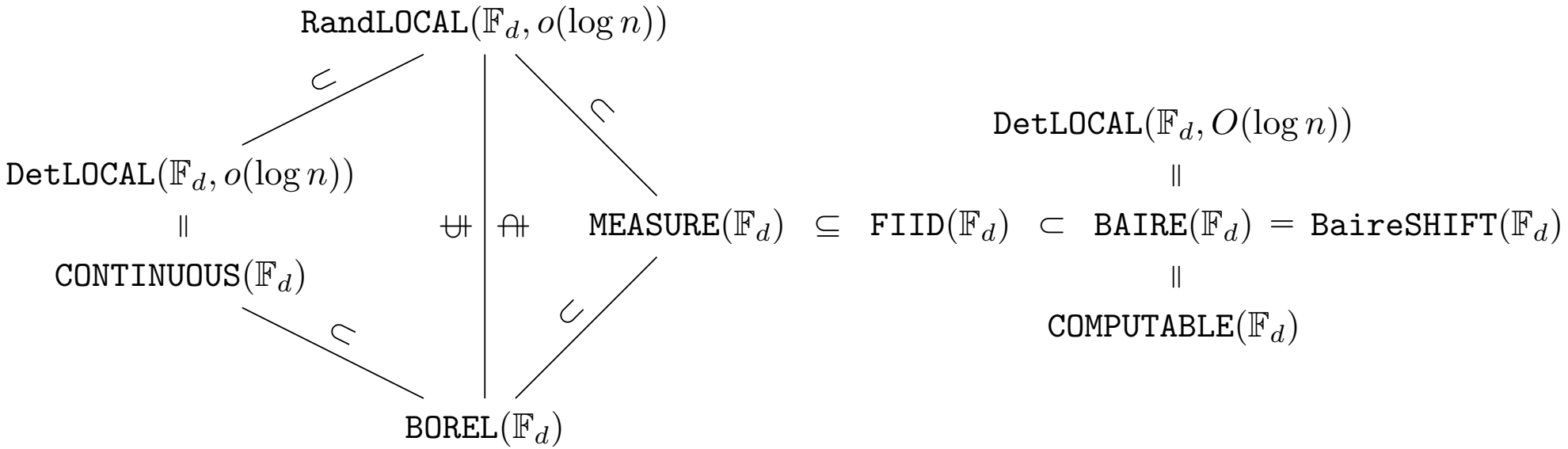
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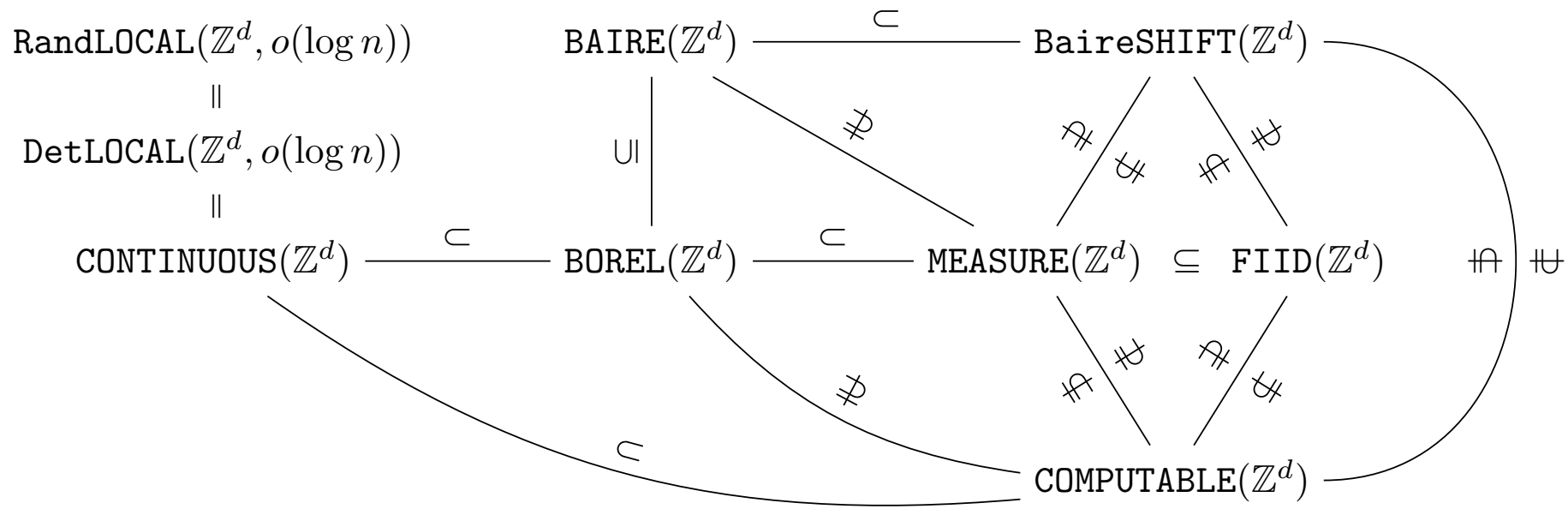
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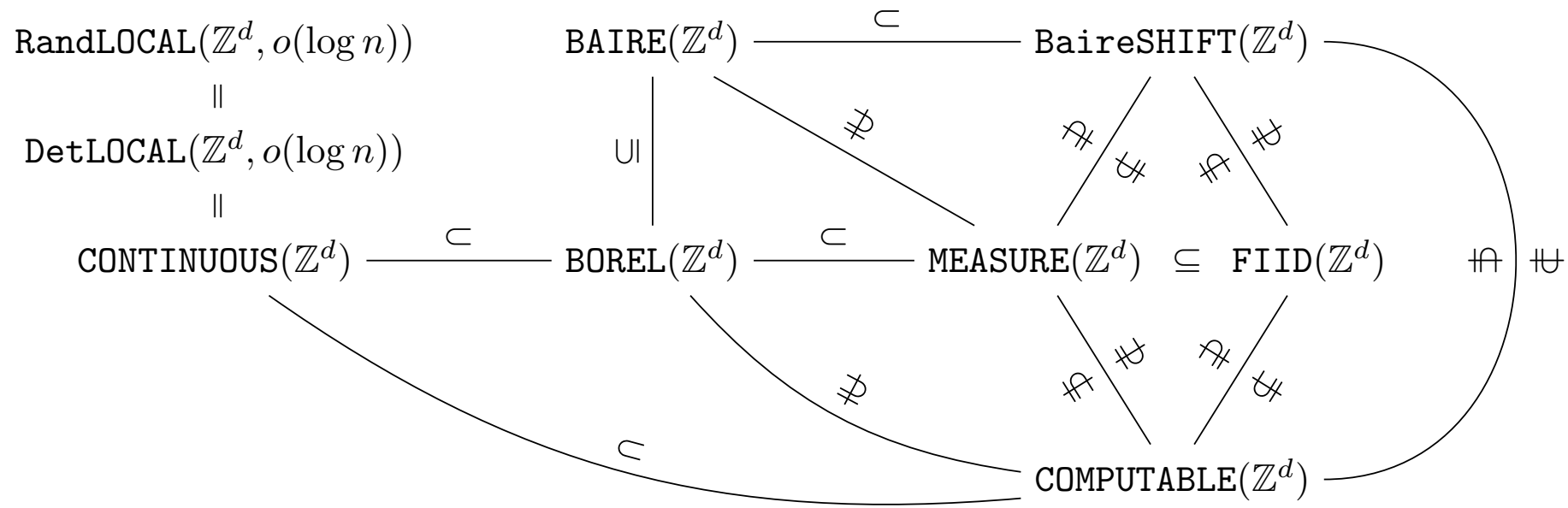
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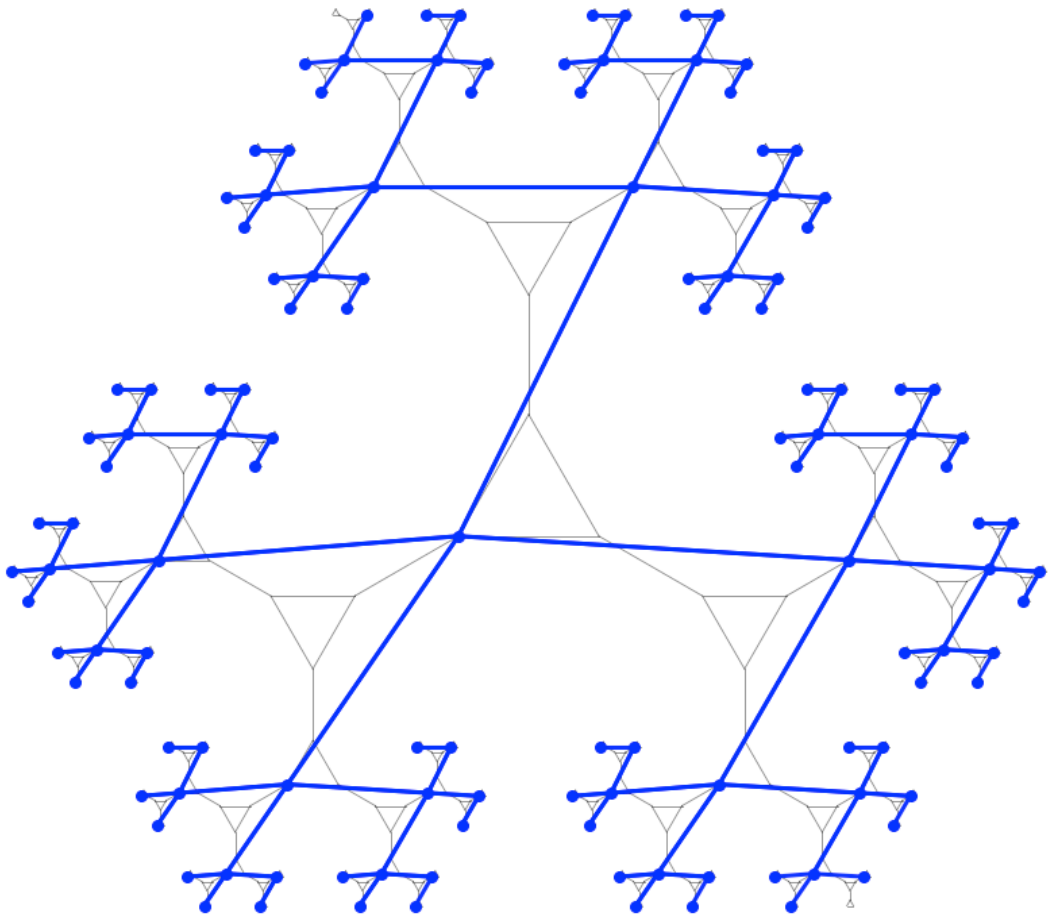
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[Grebík–Rozhoň'21]: for $\Gamma = \mathbb{Z}$, all 9 of the above classes become the same.

A Driving Question

What about for Γ a virtually free, finitely generated group?



ECL vs. LCL

This talk: we fix $(\Gamma, \textbf{gens})$, to reformulate LCL problems on Γ as ECL problems on $(\Gamma, \textbf{gens})$.

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This makes no difference to the class inclusion relationships.

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Example of $\Gamma = \mathbb{Z}$ and $\Pi \in \texttt{ECL}(\Gamma, \textbf{gens})$

$\Gamma = \mathbb{Z}, \textbf{gens} = \{R\}$.

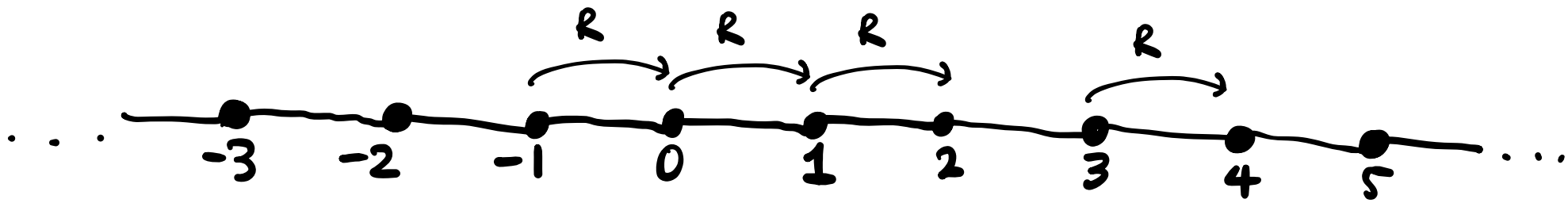
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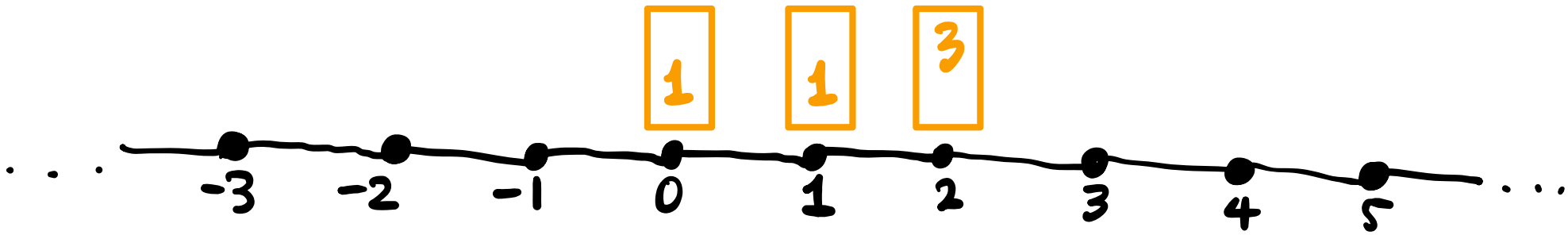
edges=$\left\{\{\gamma, \mathsf{gen} \cdot \gamma\} : \begin{smallmatrix} \mathsf{gen} \in \textbf{gens} \\ \gamma \in \Gamma \end{smallmatrix}\right\}$



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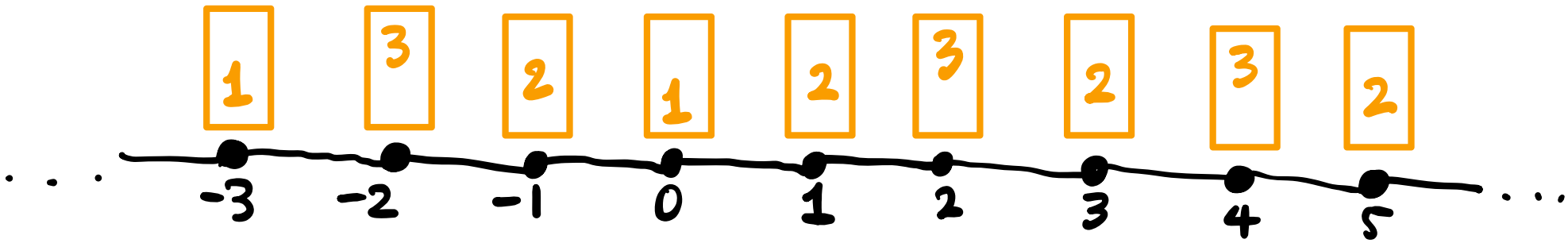
ECL problem Example:
$\Pi_{3\text{-lab}}$ on $(\mathbb{Z}, \{R\})$ is the data:
$\Lambda = \{1, 2, 3\}$ (label set)
$\mathscr{A}_R = \{\substack{(1,2),(1,3),(2,3) \\ (2,1),(3,1),(3,2)}\} = \Lambda^2 - \mathrm{diag}$



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Def.: A Π-labeling on Γ is a function $f : \Gamma \to \Lambda$ s.t.
$(f(\gamma), f(\mathsf{gen} \cdot \gamma)) \in \mathscr{A}_{\mathsf{gen}}$.
If Π has a solution f on Γ, we say $\Pi \in \texttt{SOLV}(\Gamma, \textbf{gens})$.

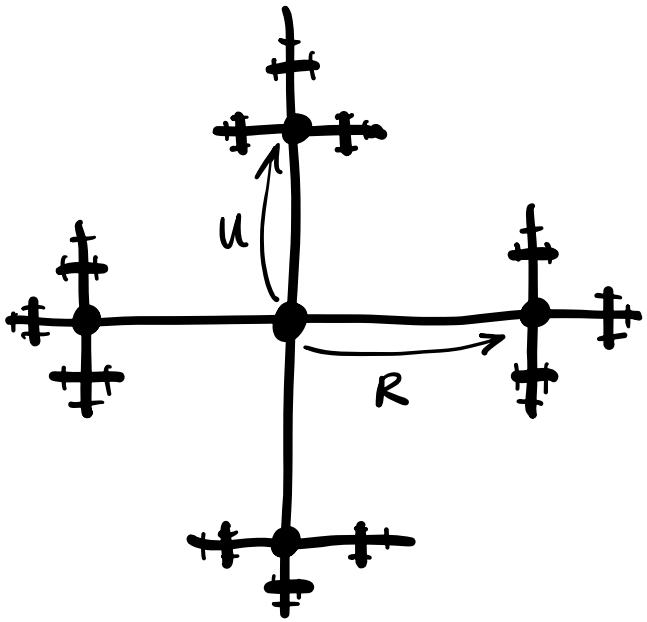
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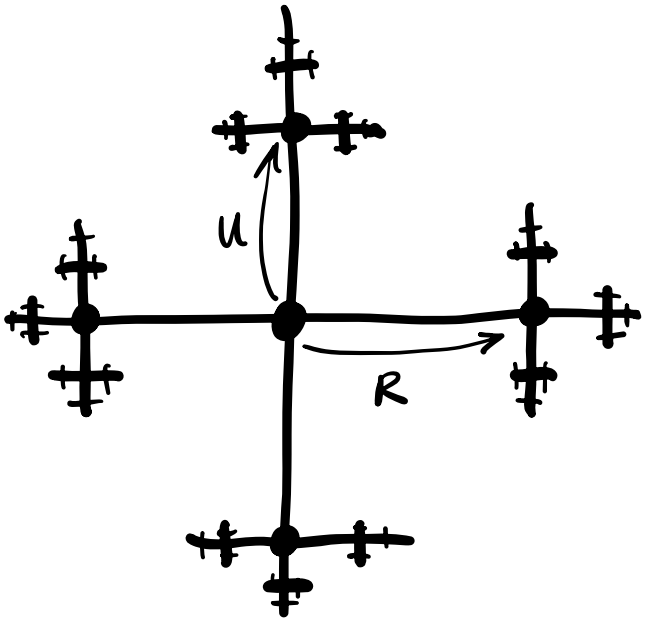
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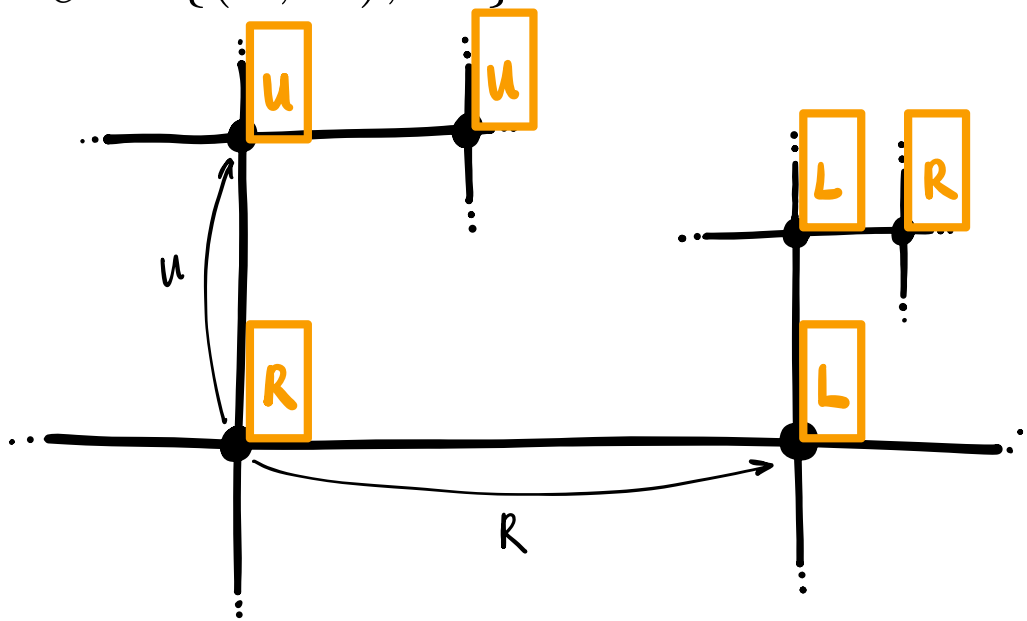


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ECL problem Ex. 2:
Π_{match} on $(\mathbb{F}_2, \{R, U\})$ is data:
$\Lambda = \{R, L, U, D\}$ (label set)
$\mathscr{A}_R = \{(R, L), (L, R), \ldots (U, U) \ldots\}$
$\mathscr{A}_U = \{(U, D), \ldots\}$



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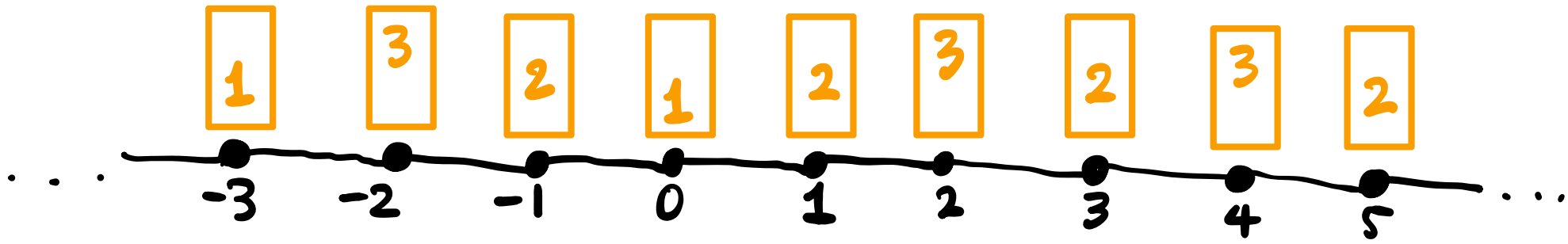
Where are the complexity classes for $\Pi \in \mathtt{ECL}(\Gamma, \mathbf{gens})$?

Need $\Gamma \curvearrowright X$ instead of $\Gamma \curvearrowright \Gamma$.

Example of $\Gamma = \mathbb{Z}$ and $\Pi \in \texttt{ECL}(\Gamma, \textbf{gens})$

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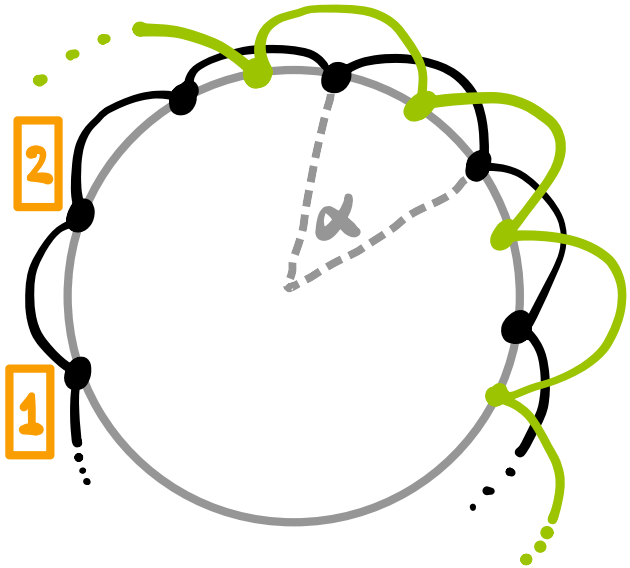
$\Gamma = \mathbb{Z}, \mathbf{gens} = \{R\}$.

Schreier graph $\mathrm{Schr}((\Gamma, \mathbf{gens}) \curvearrowright \mathbb{T}^1)$:

vertices$=\mathbb{T}^1 =: X$

edges$=\{\{x, \mathsf{gen} \cdot x\} : \substack{\mathsf{gen} \in \mathbf{gens} \\ x \in X}\}$.

Action: $Rx := x \cdot e^{-i\alpha}$ for $\alpha \notin \mathbb{Q}\pi$

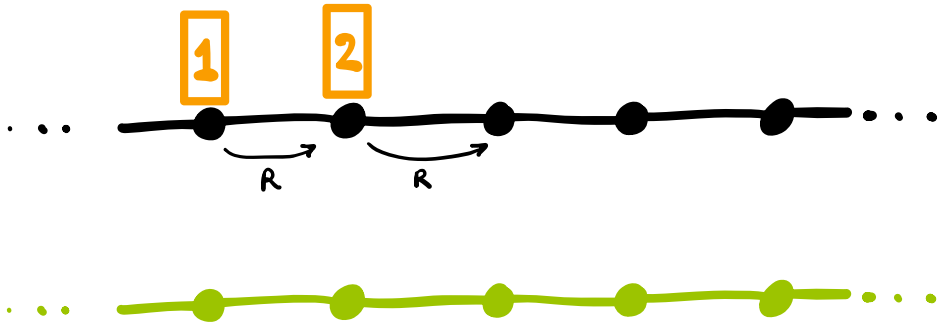


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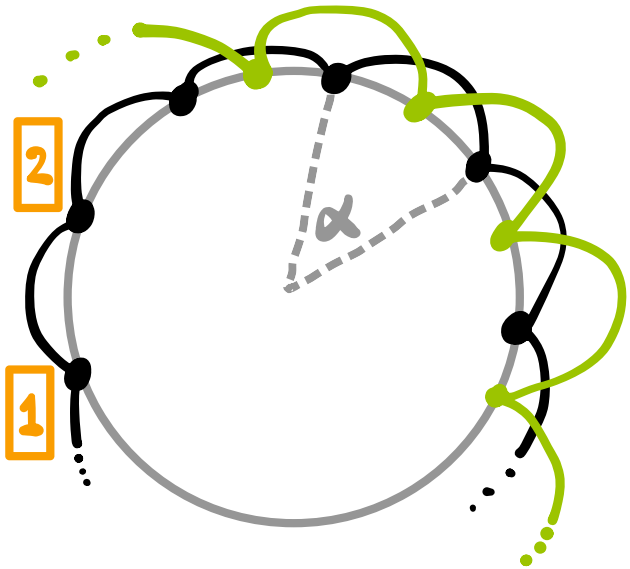
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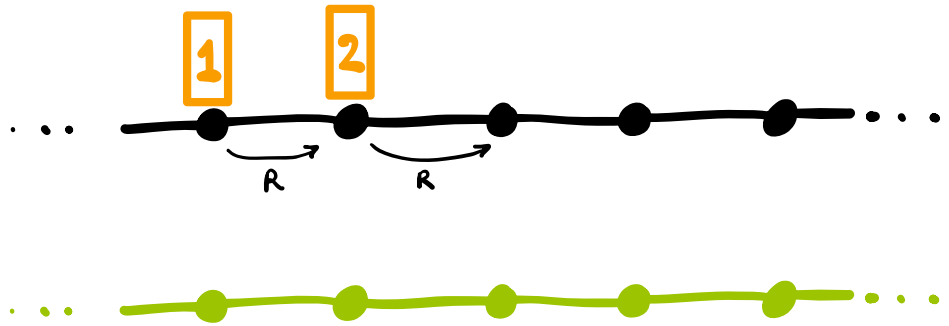


Def.: A Π-labeling on $X \curvearrowleft (\Gamma, \mathbf{gens})$ is a function $f : X \to \Lambda$ s.t. $(f(x), f(\mathsf{gen} \cdot x)) \in \mathscr{A}_{\mathsf{gen}}$.

Such an f might be "regular" in some way: Borel, measurable, etc.

Regularity/complexity of Π-labelings $f : X \to \Lambda$





Intuition for a Π-labeling $f : X \to \Lambda$ being "regular": we can find a Π-labeling on *all* copies of $\mathrm{Cay}(\Gamma, \textbf{gens})$ in a "uniform"/"regular" way.

Example: $\Pi_{2\text{-lab}} \in \mathtt{ECL}(\mathbb{Z}, \{R\})$ is solvable on $\mathrm{Schr}(\mathbb{Z} \overset{\alpha}{\curvearrowright} \mathbb{T}^1)$ (use Axiom of Choice to select a starting point in each conp. But that's not "regular").

But: no Lebesgue-measurable $f : \mathbb{T}^1 \to \{1, 2\}$ satisfies $\Pi_{2\text{-lab}}$

Intuition on Borel, (Baire-)measurable labelings

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Think of measurable f (or Baire-measurable f) as: on a "large" Borel subset $D \subseteq X$ — conull (or comeager) — $f : D \to \Lambda$ is a Borel function.
(Outside of D, we may be forced to use the Axiom of Choice)

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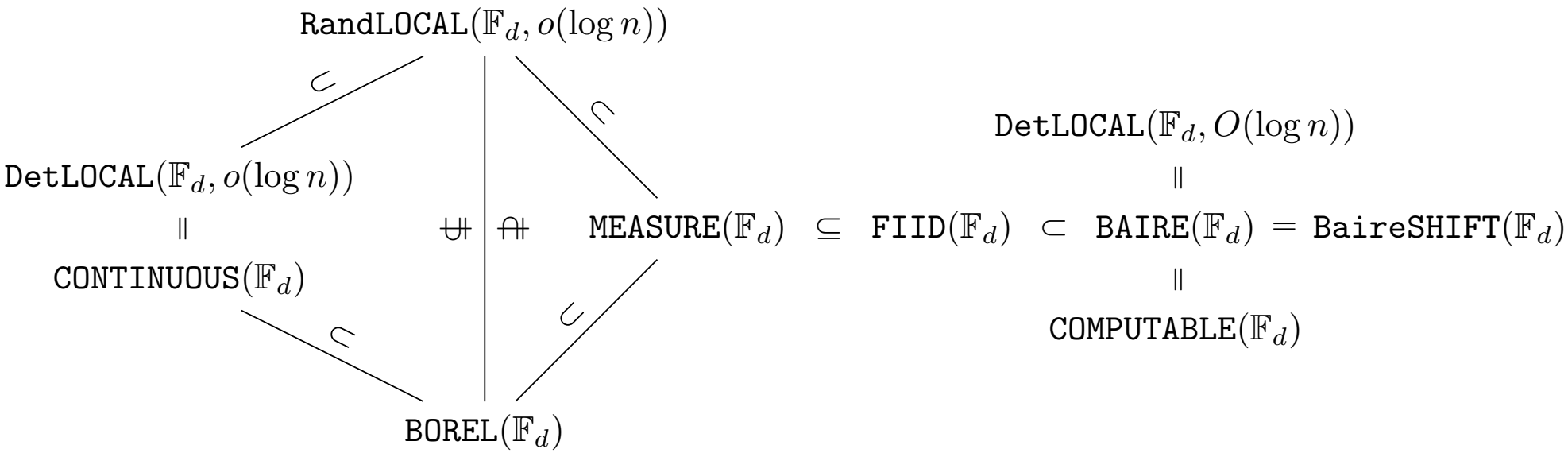
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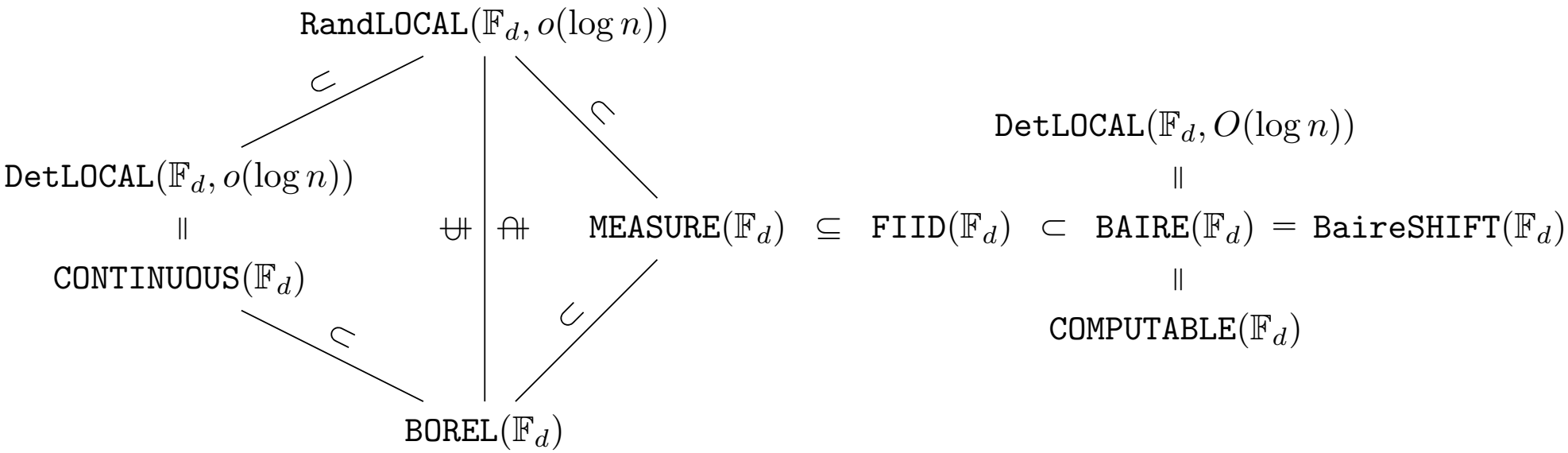


Focus on the inclusions:

$$\mathtt{MEAS}(\mathbb{F}_d, \mathbf{gens}) \longrightarrow \mathtt{FIID}(\mathbb{F}_d, \mathbf{gens}) \longrightarrow \mathtt{BAIRE}(\mathbb{F}_d, \mathbf{gens}) = \mathtt{BaireSHIFT}(\mathbb{F}_d, \mathbf{gens})$$

Inclusions we focus on

THEOREM 5.2. *For the free group $\mathbb{F}_d$ with $d \geqslant 2$, we have:*



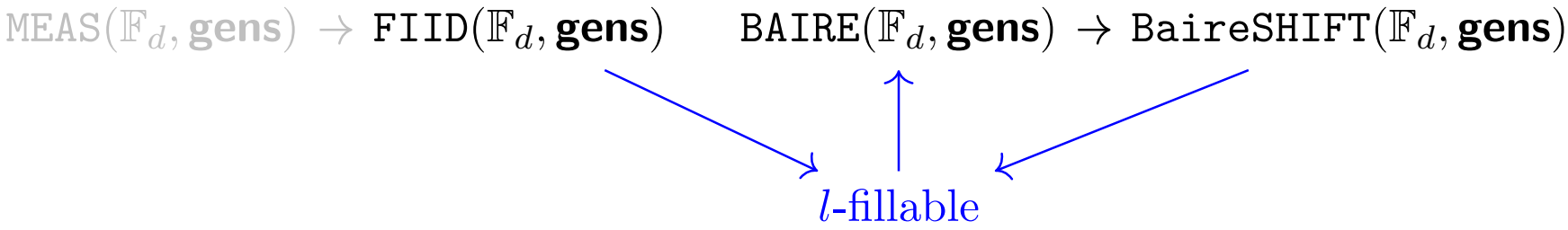
Focus on the inclusions:

$$\texttt{MEAS}(\mathbb{F}_d, \textbf{gens}) \longrightarrow \texttt{FIID}(\mathbb{F}_d, \textbf{gens}) \longrightarrow \texttt{BAIRE}(\mathbb{F}_d, \textbf{gens}) = \texttt{BaireSHIFT}(\mathbb{F}_d, \textbf{gens})$$

The proofs of these inclusions first appeared in Brandt–Chang–Grebík–Grunau–Rozhoň–Vidnyánszky (2021) as sketches based on private communications from Bernshteyn.

Old implications

The sketch in Brandt–Chang–Grebík–Grunau–Rozhoň–Vidnyánszky (2021) has **Key:** "combinatorial condition" called l-fullness or l-fillability.

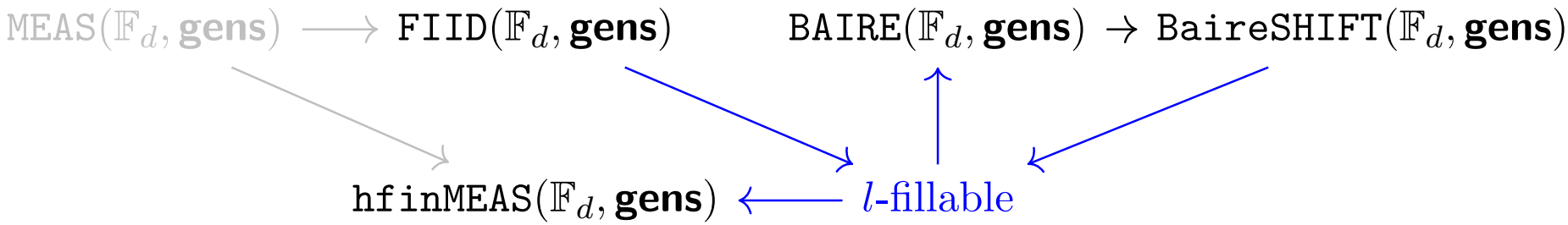


The blue implications prove the (black) relationships

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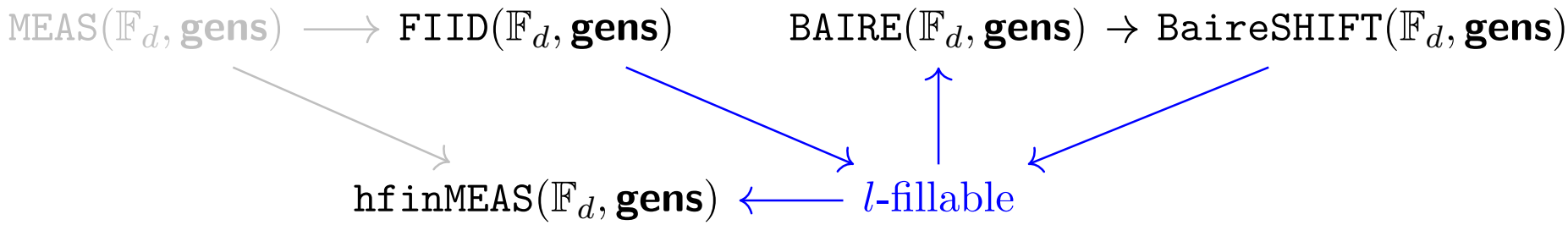
My work

- more details on the arguments in BCGGRV21 (using "path decomposition" instead of "toast")
- added one implication to a "new" class:



My work

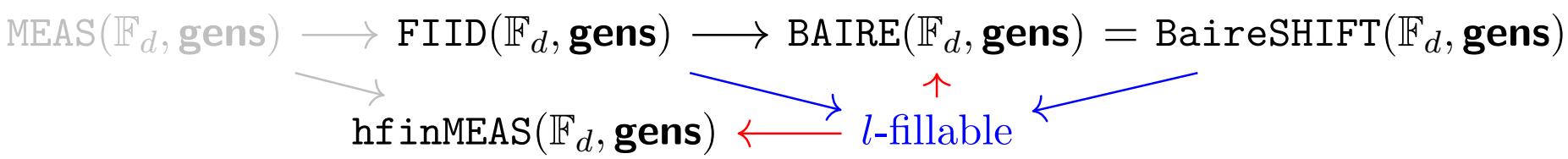
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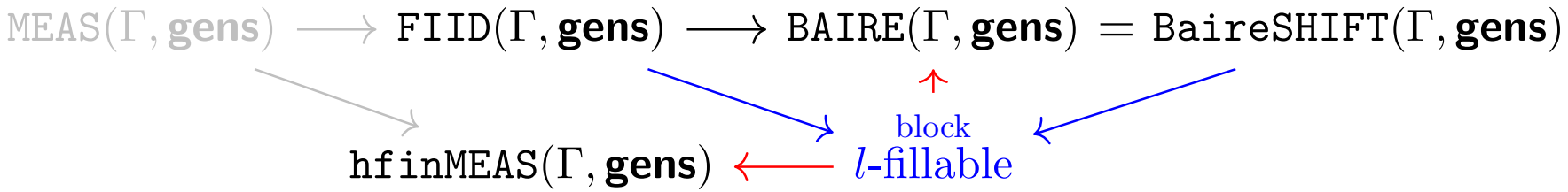
- More importantly, proved the same (blue) implications for ECL problems Π on **virtually free (v.f.)** Γ in place of $\mathbb{F}_d$, and "block l-fillable" in place of "l-fillable".

This talk

Goal 1/3: sketch red implications.



Goal 2/3: emphasize new ideas needed for red implications for **virtually** free (v.f.) Γ:



The remaining blue implications are more routine, **true for all** Γ.

1 Birds-eye view

2 Definitions/Set-up

3 Fillability

4 Decidability

5 Future work

Roadmap

Fix $(\Gamma, \textbf{gens})$, fix $\Pi \in \texttt{ECL}(\Gamma, \textbf{gens})$.

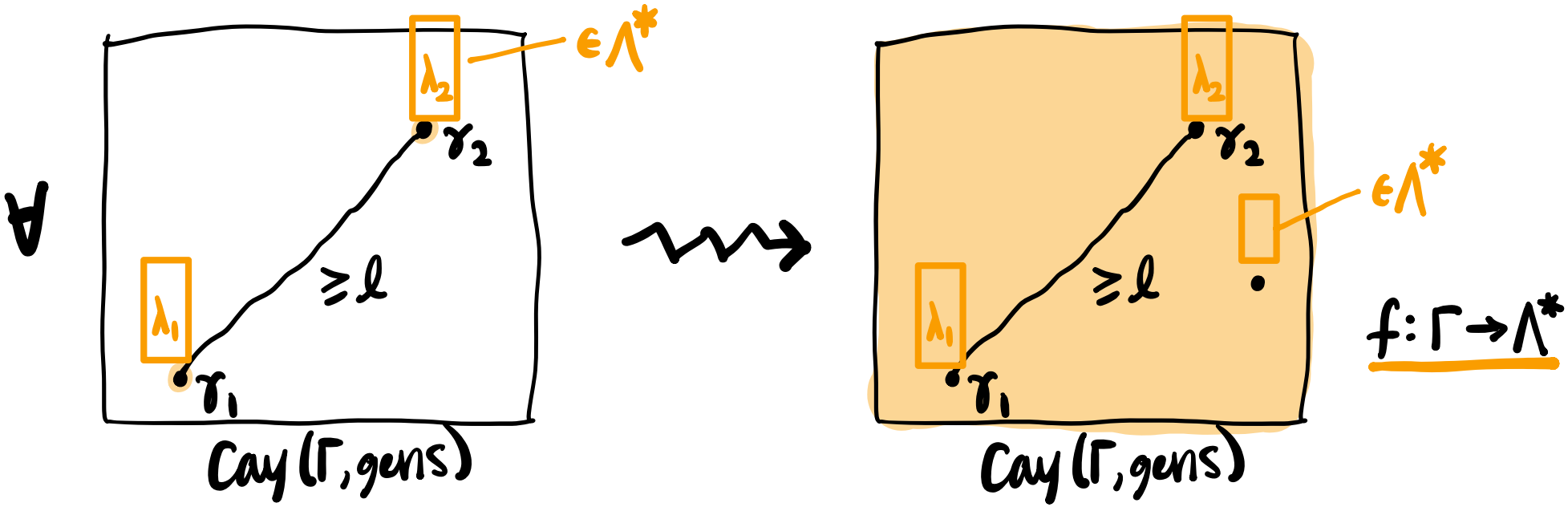
This section:

- define "Π is l-fillable"
- explain how to use l-fillable, in conjunction with "path decompositions" of **trees** (Cayley graphs of free groups), to build solutions $f : X \to \Lambda$
- discuss tweaks for Γ virtually free.

Definition of l-fillable (w.r.t. $\Lambda^\star \subseteq \Lambda$)

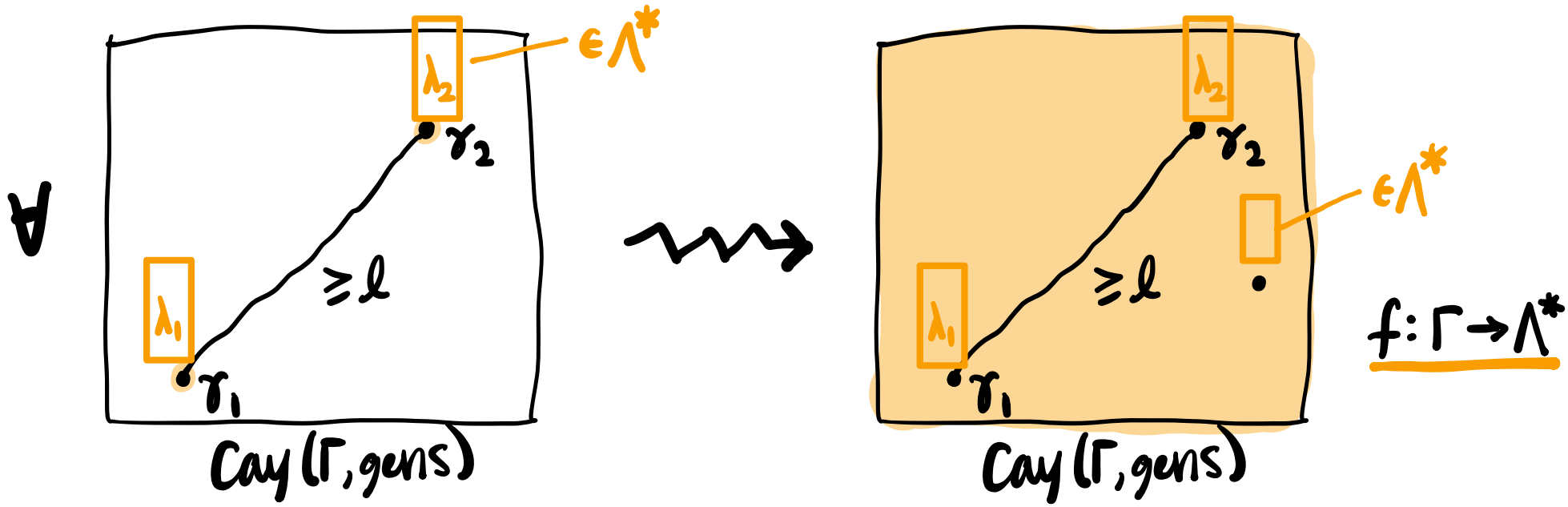
Fix a f.g. group $(\Gamma, \textbf{gens})$, a problem $\Pi \in \texttt{ECL}(\Gamma, \textbf{gens})$, and $\ell \in \mathbb{N}^+$.

Def.: We say $\Pi = (\Lambda, \mathscr{A})$ is ℓ-fillable w.r.t $\Lambda^\star \subseteq \Lambda$, if



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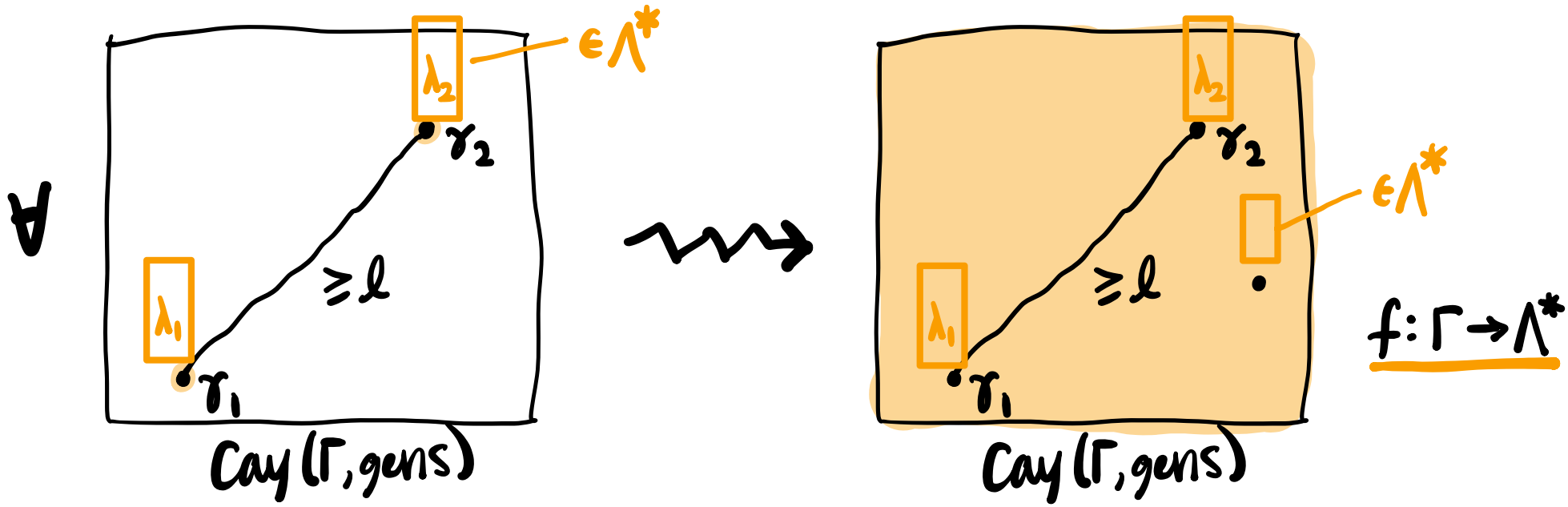
(So $f : \Gamma \to \Lambda$ is both Π-admissible and $\Lambda^\star$-admissible!)
We say Π is l-fillable, if $\exists \ell \in \mathbb{N}^+, \Lambda^\star \subseteq \Lambda$ s.t. Π is ℓ-fillable w.r.t $\Lambda^\star$.

Example: $\Pi_{3\text{-lab}} = (\{1,2,3\}, \mathcal{A}_R) \in \mathtt{ECL}(\mathbb{Z}, \{R\})$ is ℓ-fillable w.r.t. $\Lambda^\star = \Lambda := \{1,2,3\}$, for all $\ell \geqslant 2$.



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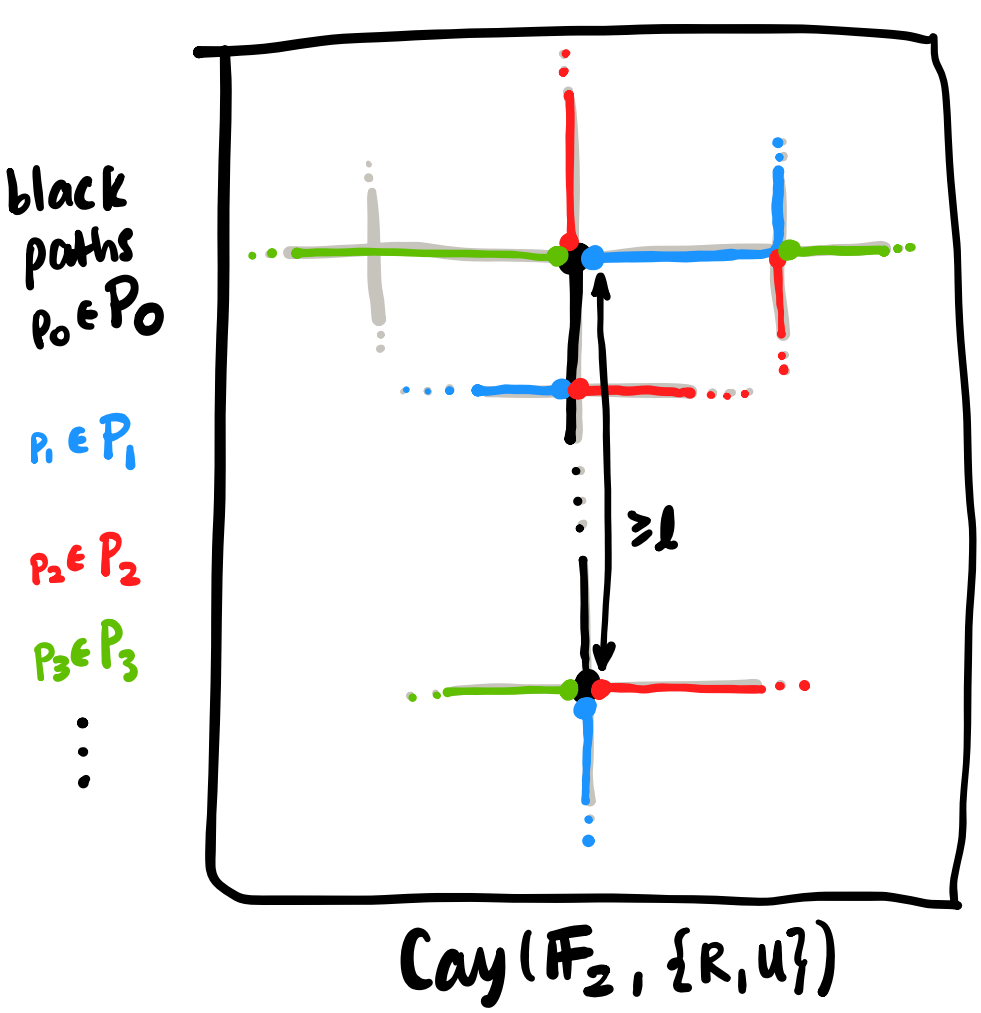
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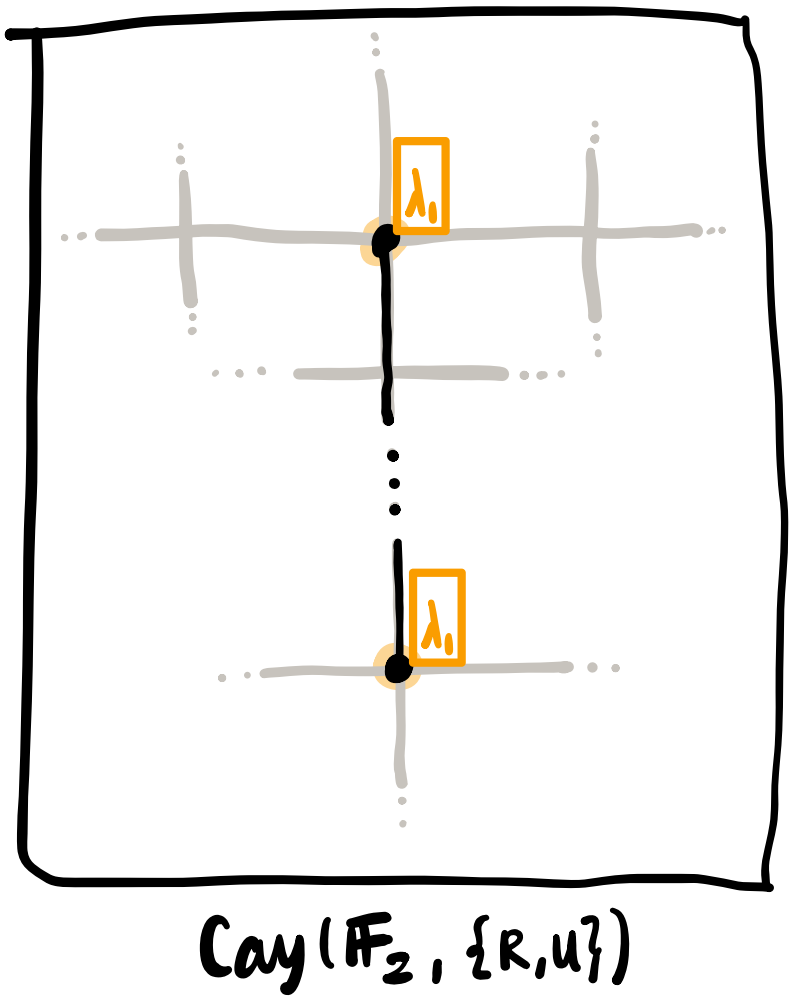
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[Γ free] Going from ℓ-fillable to solution $f : X \to \Lambda^\star$: 1 conp view

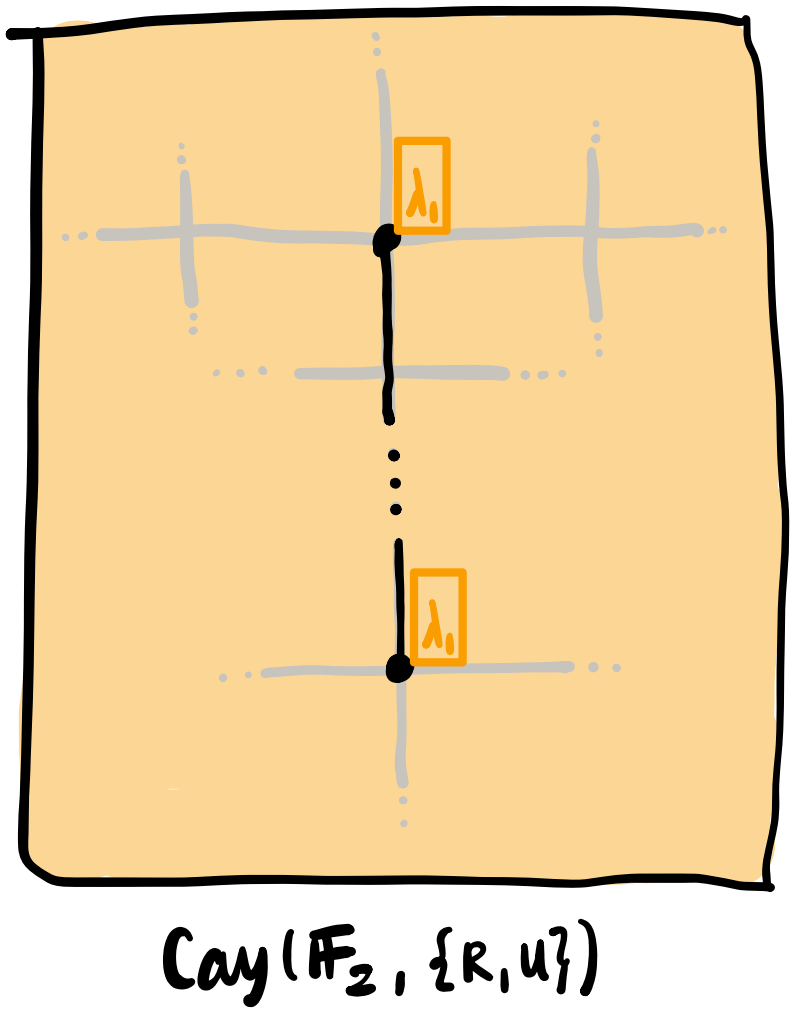
Suppose we had a ℓ-path decomposition of the tree Cay(Γ, **gens**):

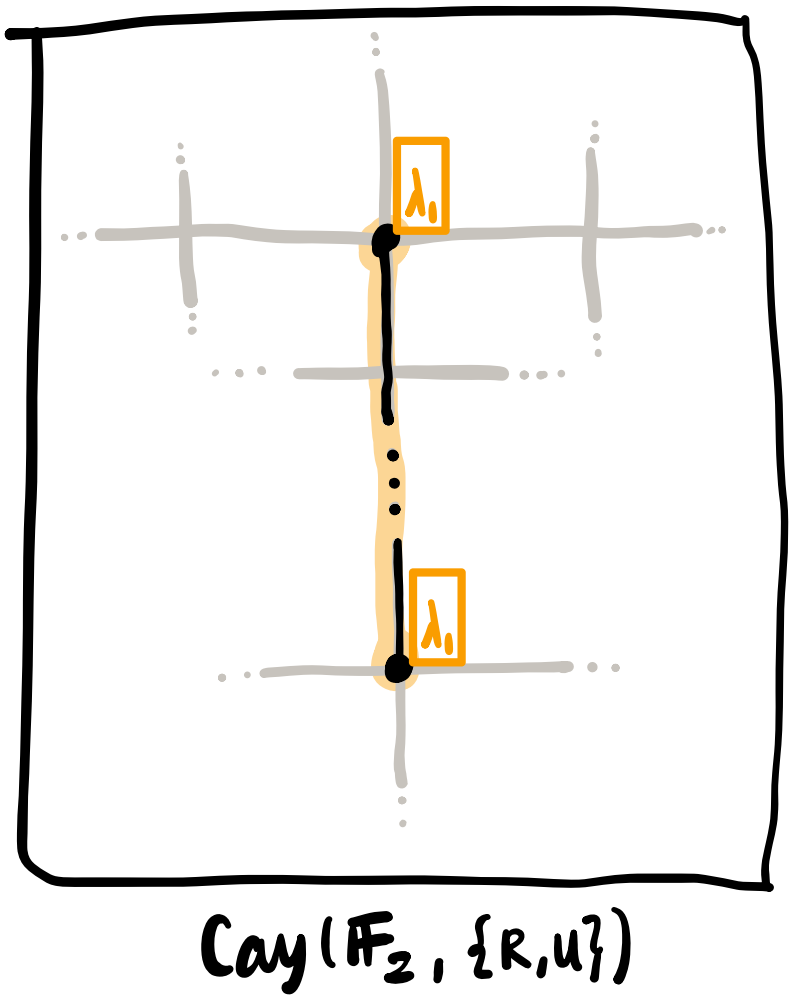




[Γ free] "Flood and restrict to path"

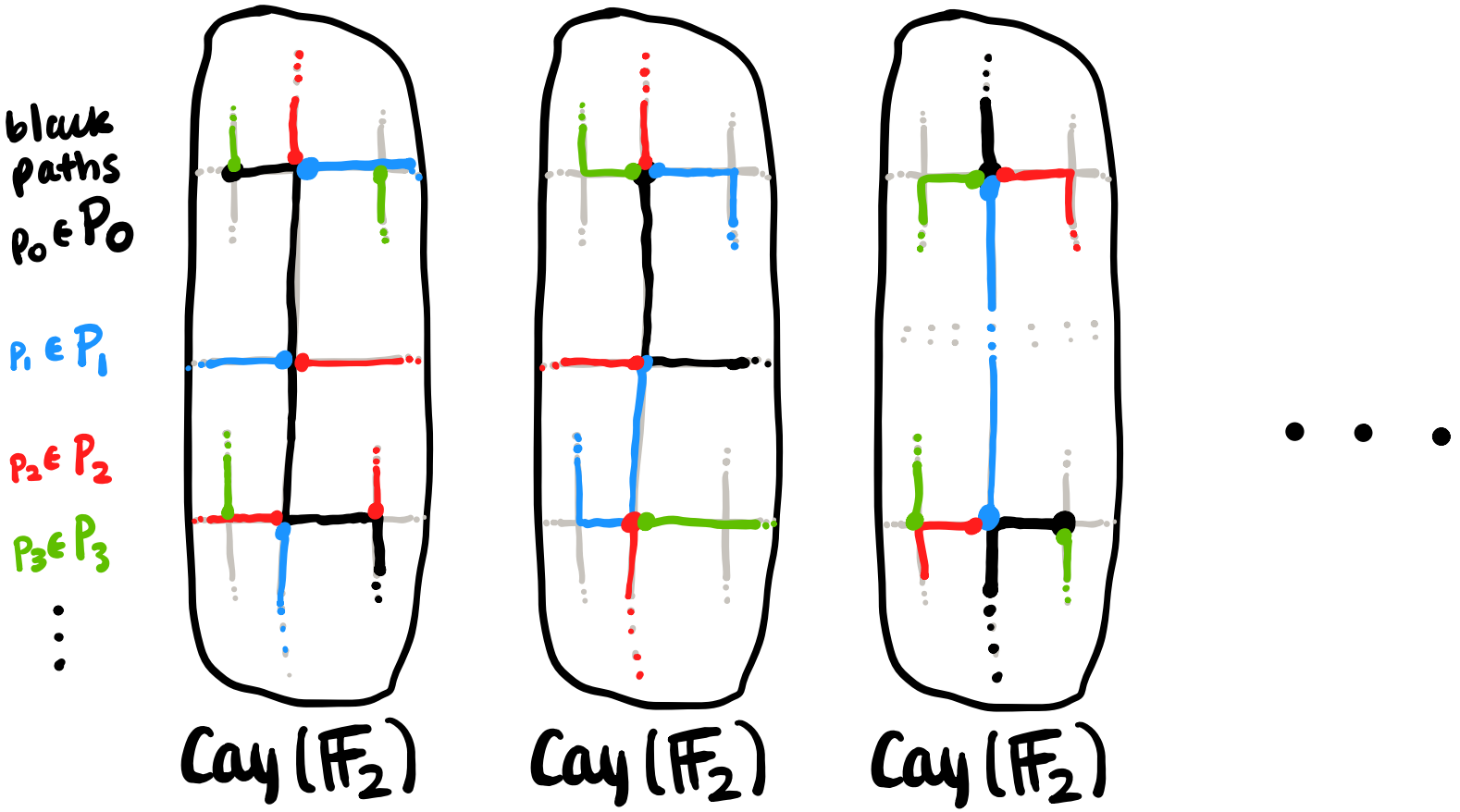
Intuitive idea: for each path, apply ℓ-fillability to fill the whole picture, then keep only labels on the path.





[Γ free] Going from ℓ-fillable to solution $f : X \to \Lambda^\star$

Intuitive idea: try the same on every conp "in parallel". Need path decomp. for Schreier graph (each conp is a tree)



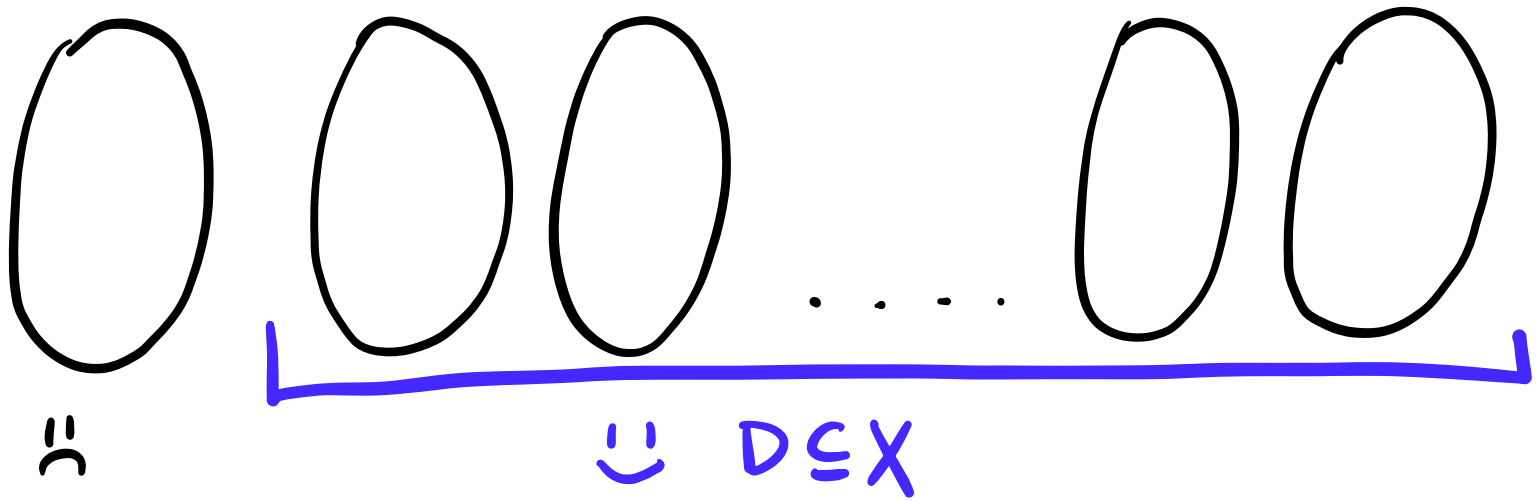
[Γ free] When can we get a path decomposition?

In general for our $(X, G) := \mathrm{Schr}(\mathbb{F}_d \overset{a}{\curvearrowright} X)$, we can not get a Borel path decomposition.

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However, Conley-Marks-Unger (2020) identified 2 cases in which we get a Borel path decomposition on a "large", union-of-conps, $D \subseteq X$
(i.e. Borel path decomp. on "most" conps. of (X, G)):



[Γ free] 2 "sources" of path decomposition on large subset $D \subseteq X$

$(X, G) := \mathrm{Schr}(\mathbb{F}_d \overset{a}{\curvearrowright} X)$. Conley-Marks-Unger (2020):

- $\exists$ a comeager union-of-conps $D \subseteq X$ s.t. $(D, G\restriction_D)$ has Borel path decomposition
- if μ is a Borel prob. measure on X s.t. $\exists$ a μ-conull union-of-conps set D on which $(D, G\restriction_D)$ is **"hyperfinite"**, then $(D, G\restriction_D)$ has Borel path decomposition

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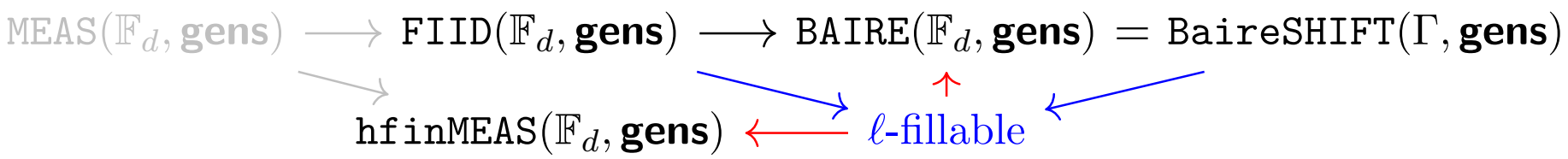
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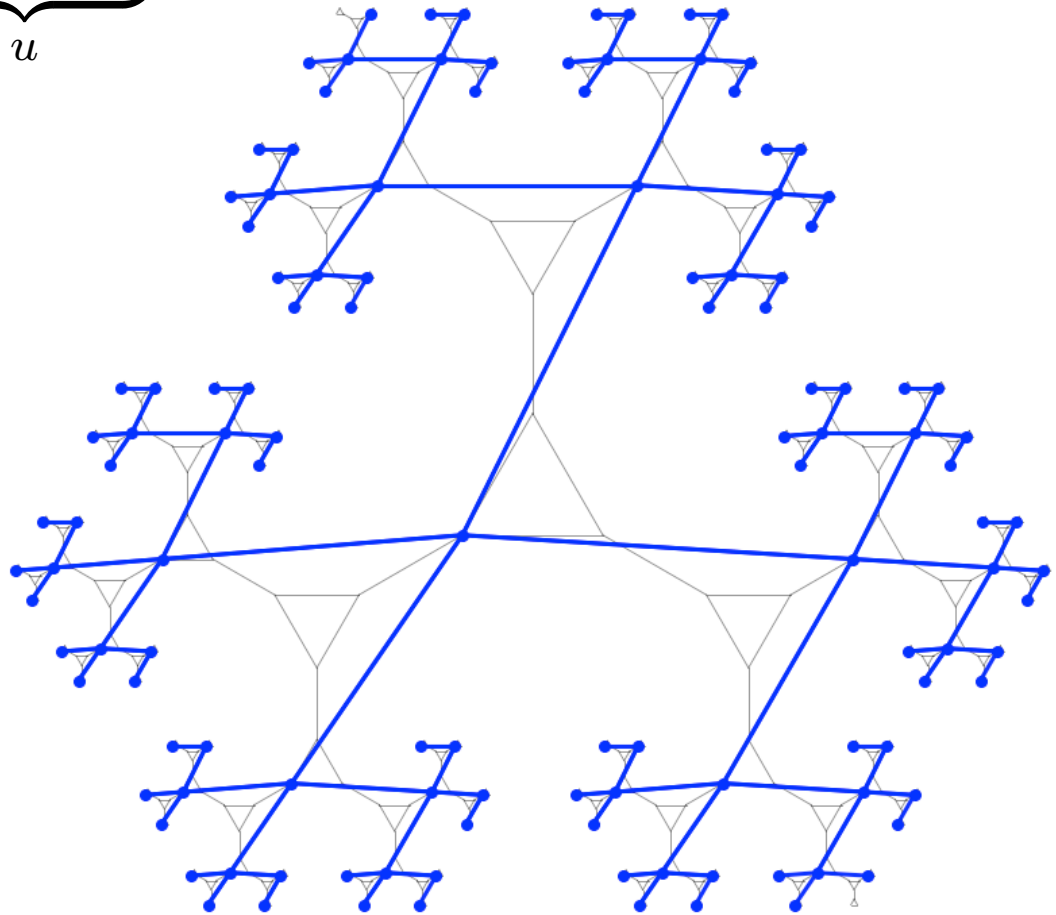
[Γ v.f.] Example and picture

Def.: Γ is virtually free if $\exists$ a finite index free subgroup $\Phi \leqslant \Gamma$.

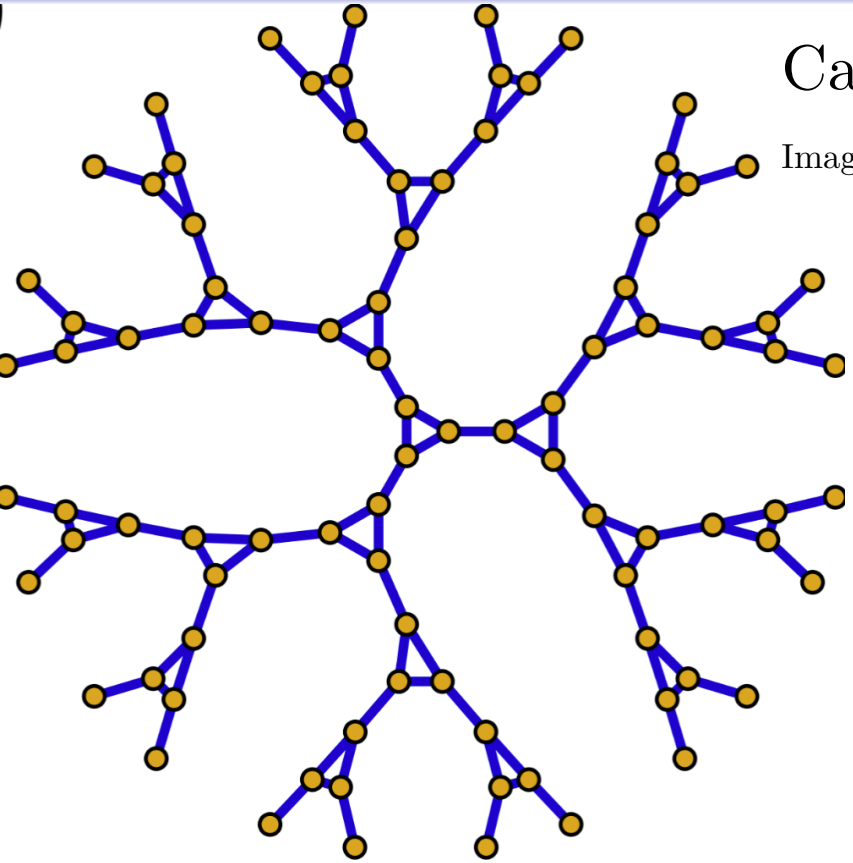
Example:

$\mathrm{PSL}(2,\mathbb{Z}) \cong (\mathbb{Z}/2)*(\mathbb{Z}/3) = \langle a, b | a^2 = b^3 = 1 \rangle$ $(a, b \mapsto \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right))$.

$\Phi := \langle \underbrace{abab}_{r}, \underbrace{ab^{-1}ab^{-1}}_{u} \rangle \leqslant \Gamma := \langle a, b | a^2, b^3 \rangle$ is a index-6 free subgroup.



[Γ v.f.] Examples and pictures emphasizing "structure tree"

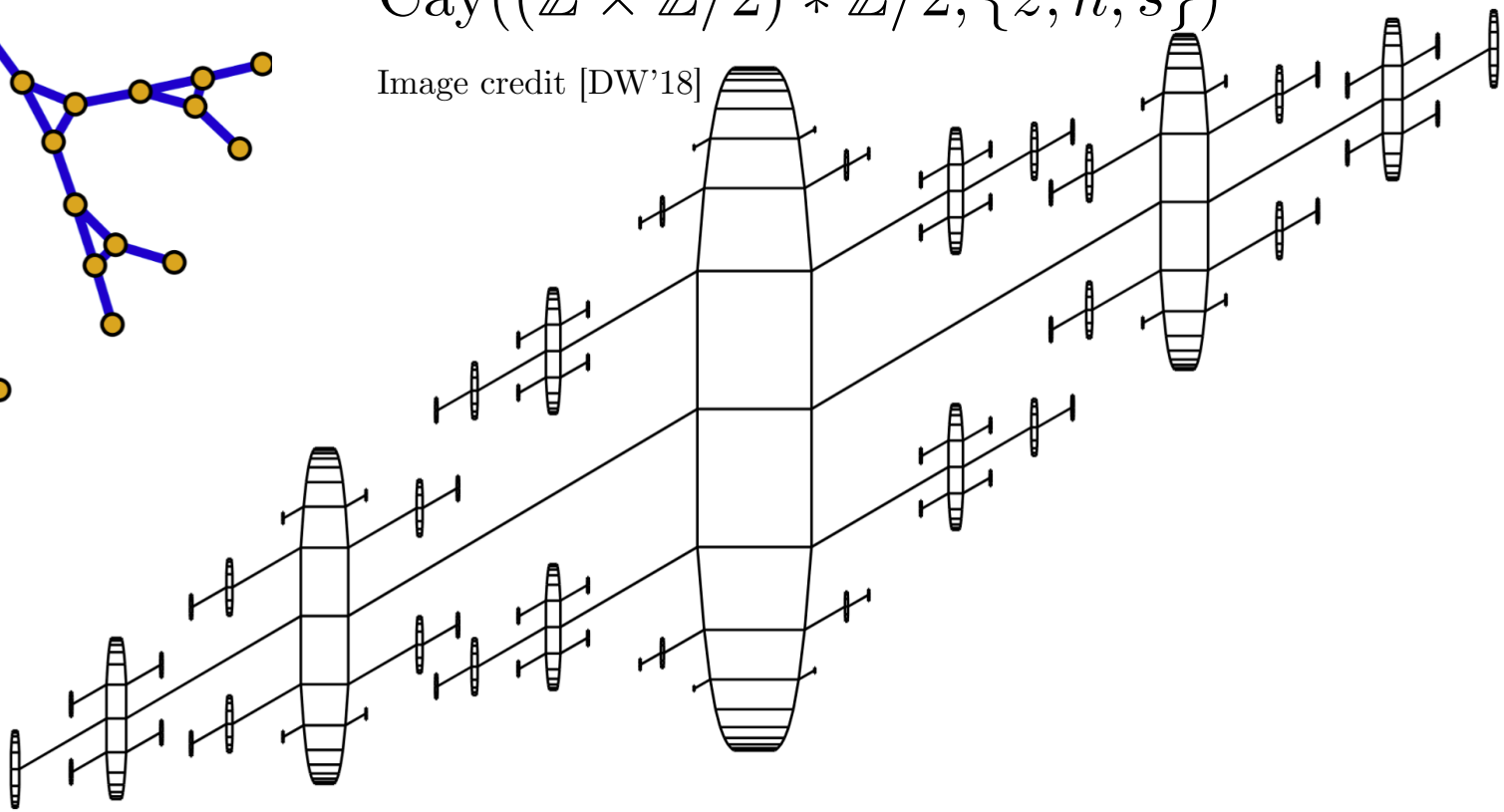


$$\mathrm{Cay}(\langle a, b \mid a^2 = b^3 = 1 \rangle)$$

Image credit [KHS'19]

$$\mathrm{Cay}((\mathbb{Z} \times \mathbb{Z}/2) * \mathbb{Z}/2, \{z, h, s\})$$

Image credit [DW'18]



[Γ v.f.] Intuition leading to block ℓ-fillability

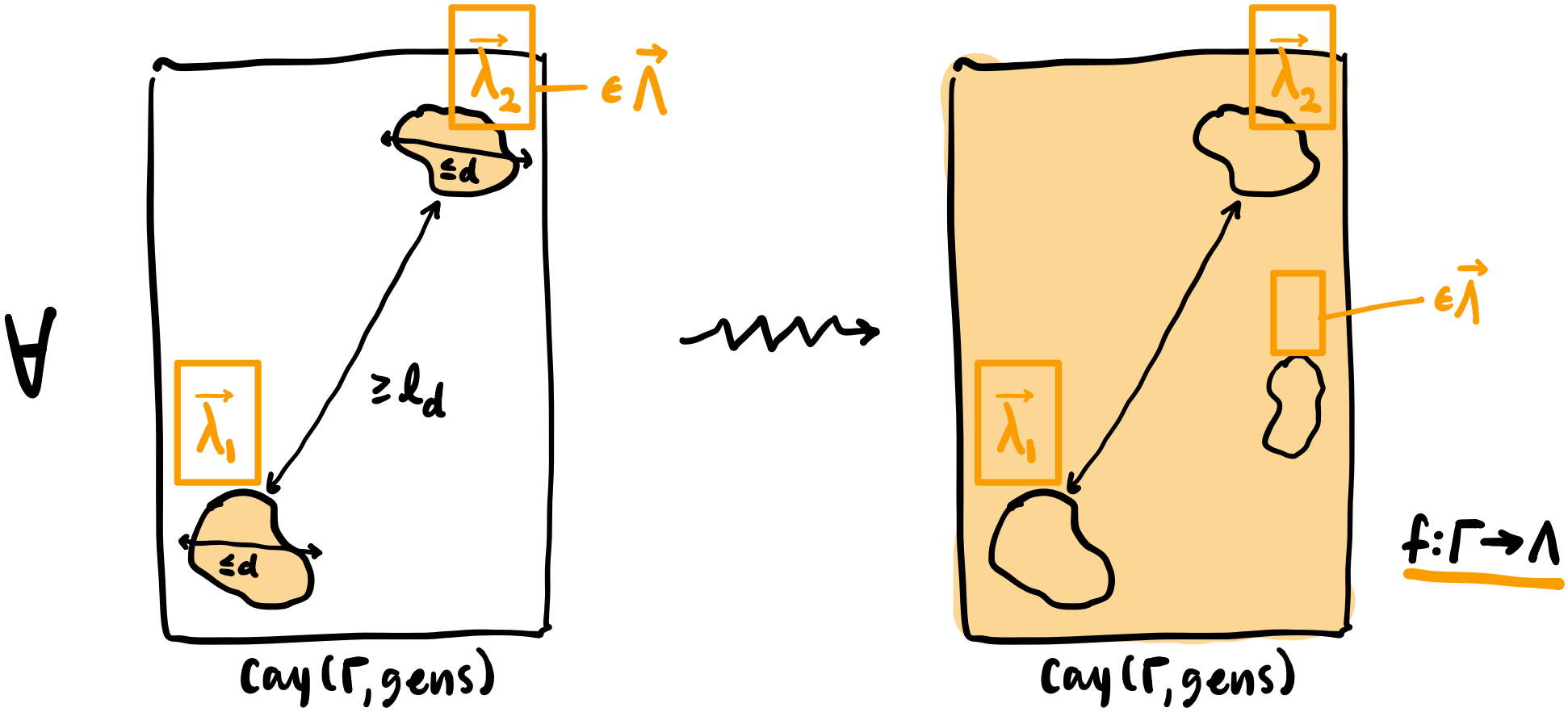
Intuition: the Cayley graph of a v.f. Γ looks like a tree made of "blocks".

Try to adapt previous strategy:

- "blockpath decomposition"
- "flood and restrict"

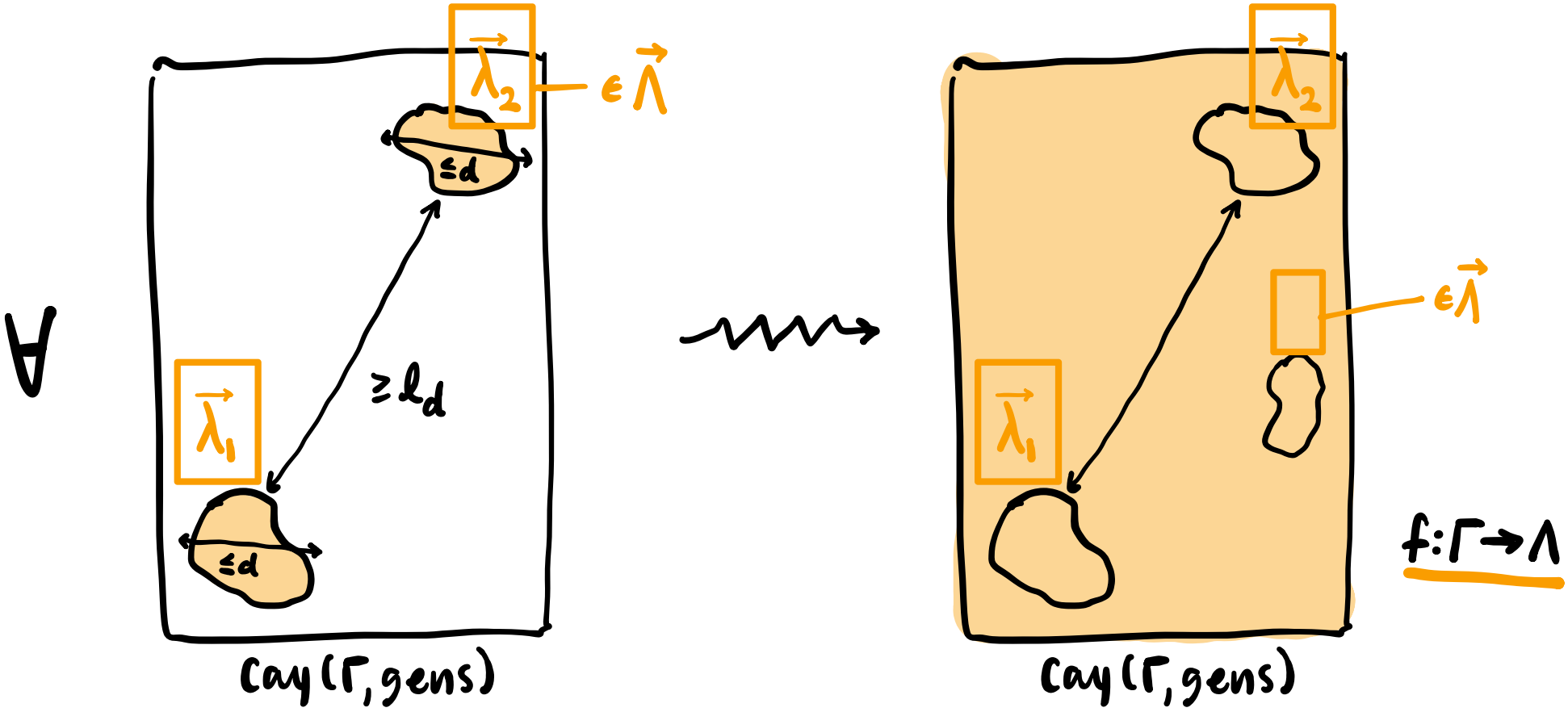
[Γ v.f.] Definition of block l-fillable

Def.: We say $\Pi = (\Lambda, \mathscr{A})$ is block l-fillable w.r.t $\vec{\Lambda}$, if $\forall d \in \mathbb{N}^+$, $\exists \ell_d \in \mathbb{N}^+$ s.t. Π is "d-block ℓ_d-fillable w.r.t $\vec{\Lambda}$":



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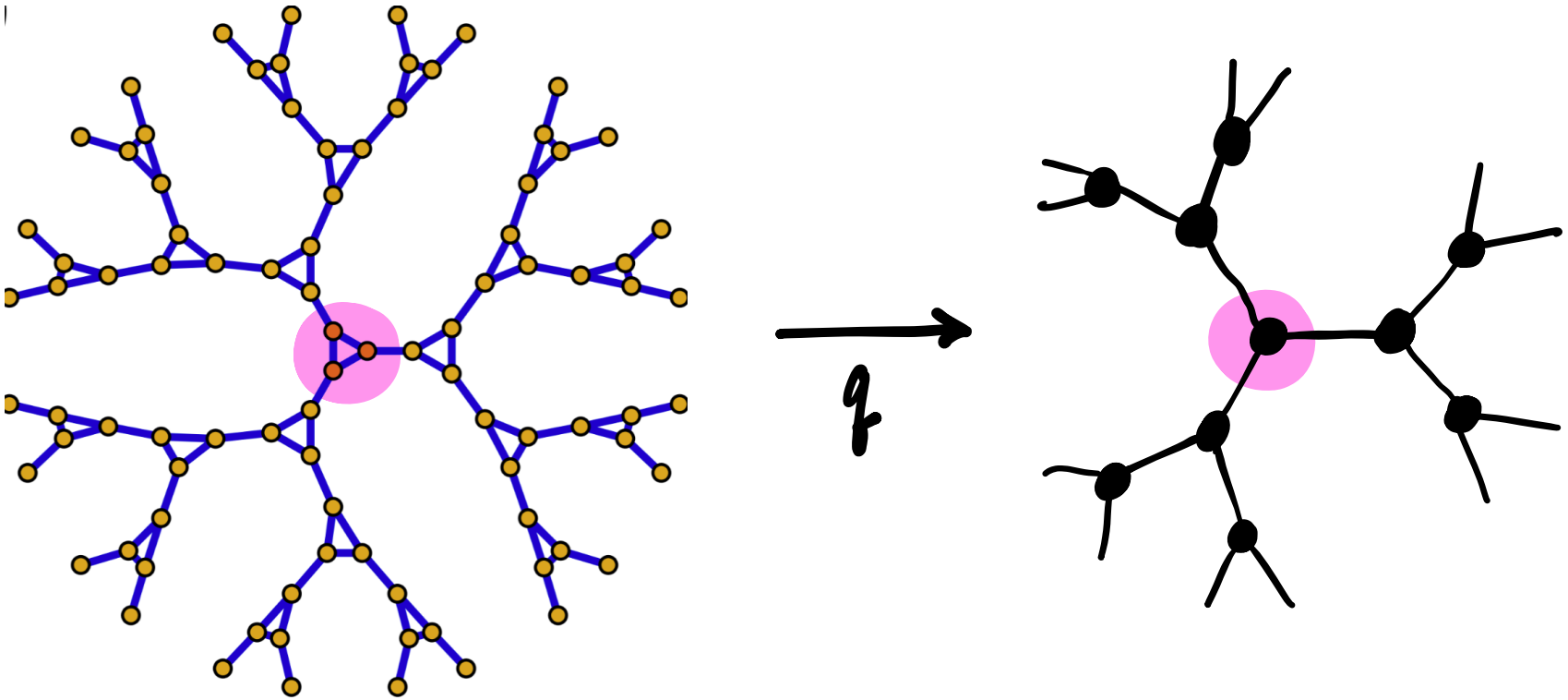
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(So $f : \Gamma \to \Lambda$ is both Π-admissible and $\vec{\Lambda}$-admissible!)

[Γ v.f.] Our hope for a blockpath decomp. by pulling back

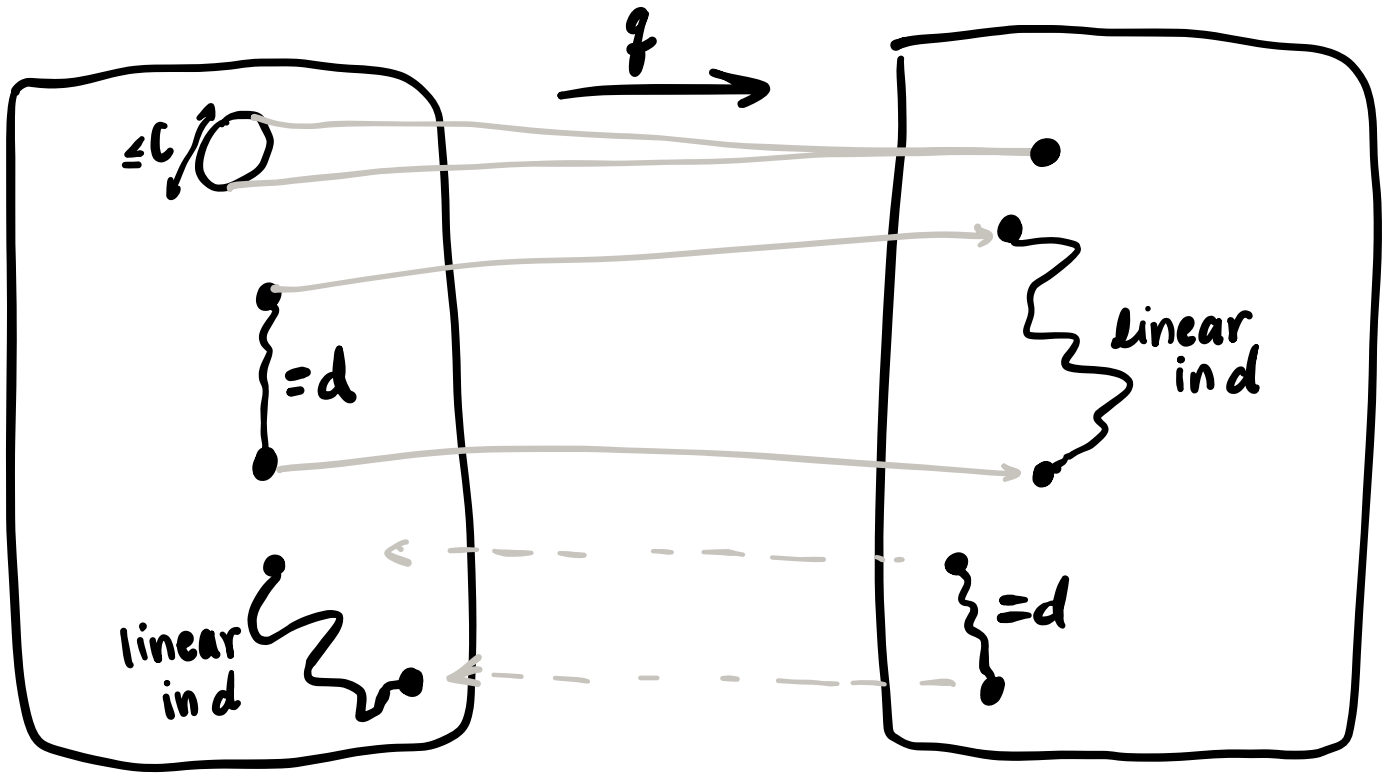
Suppose we had a Borel map $\mathfrak{q} : (X, G) \to (Y, F)$ a Borel forest:



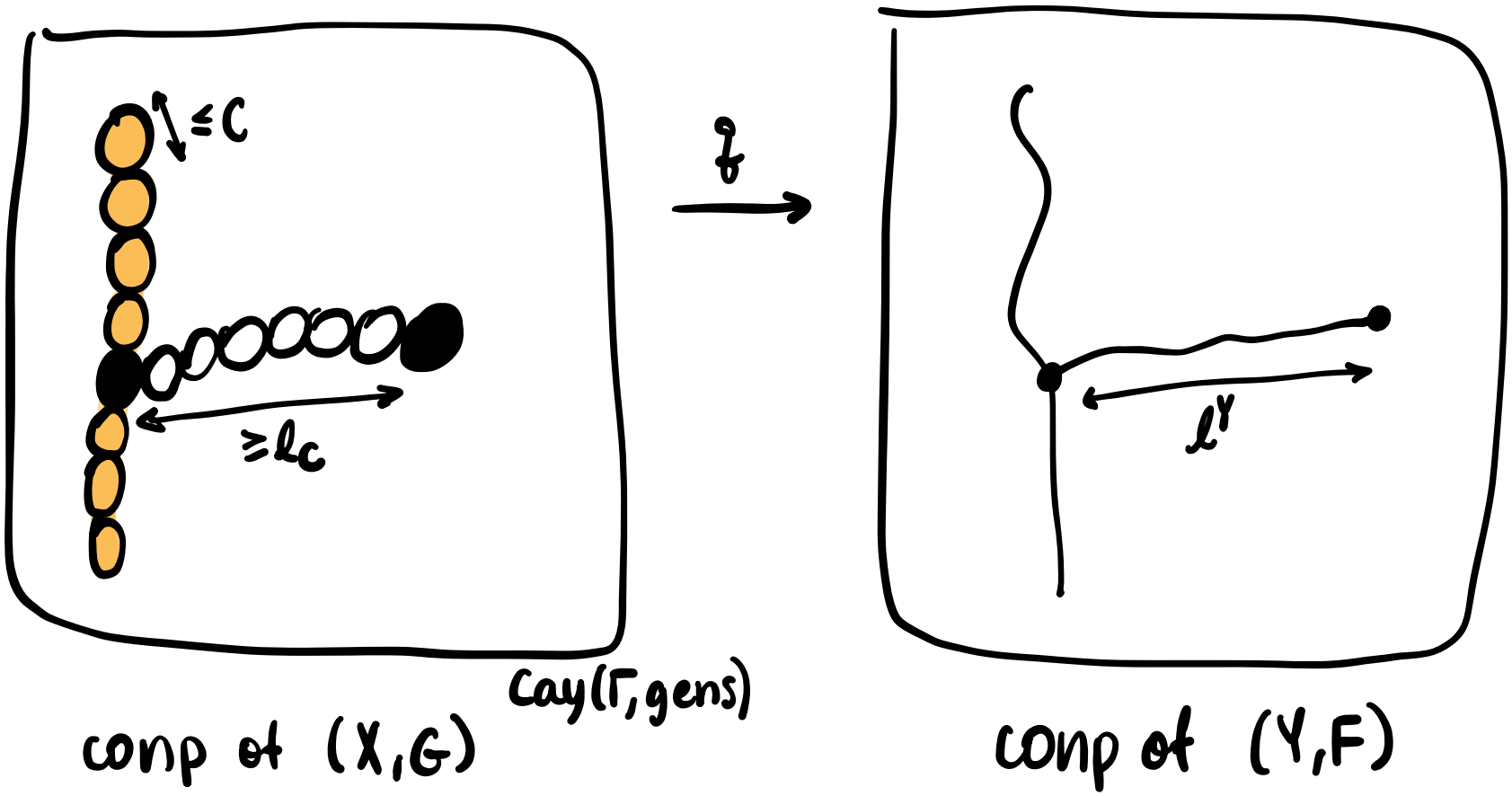
Then we could pull back a Borel path decomp. on (Y, F), to get a blockpath decomp. on (X, G), and apply block l-fillability.

[Γ v.f.] Wish granted, by [CPTT'25] "structure tree theorem"

[Chen-Poulin-Tao-Tserunyan'25]: For our Schreier graphs (X, G), there exists a Borel forest (every conp a tree) (Y, F) and a Borel **quasi-isometry** $\mathfrak{q} : X \to Y$:

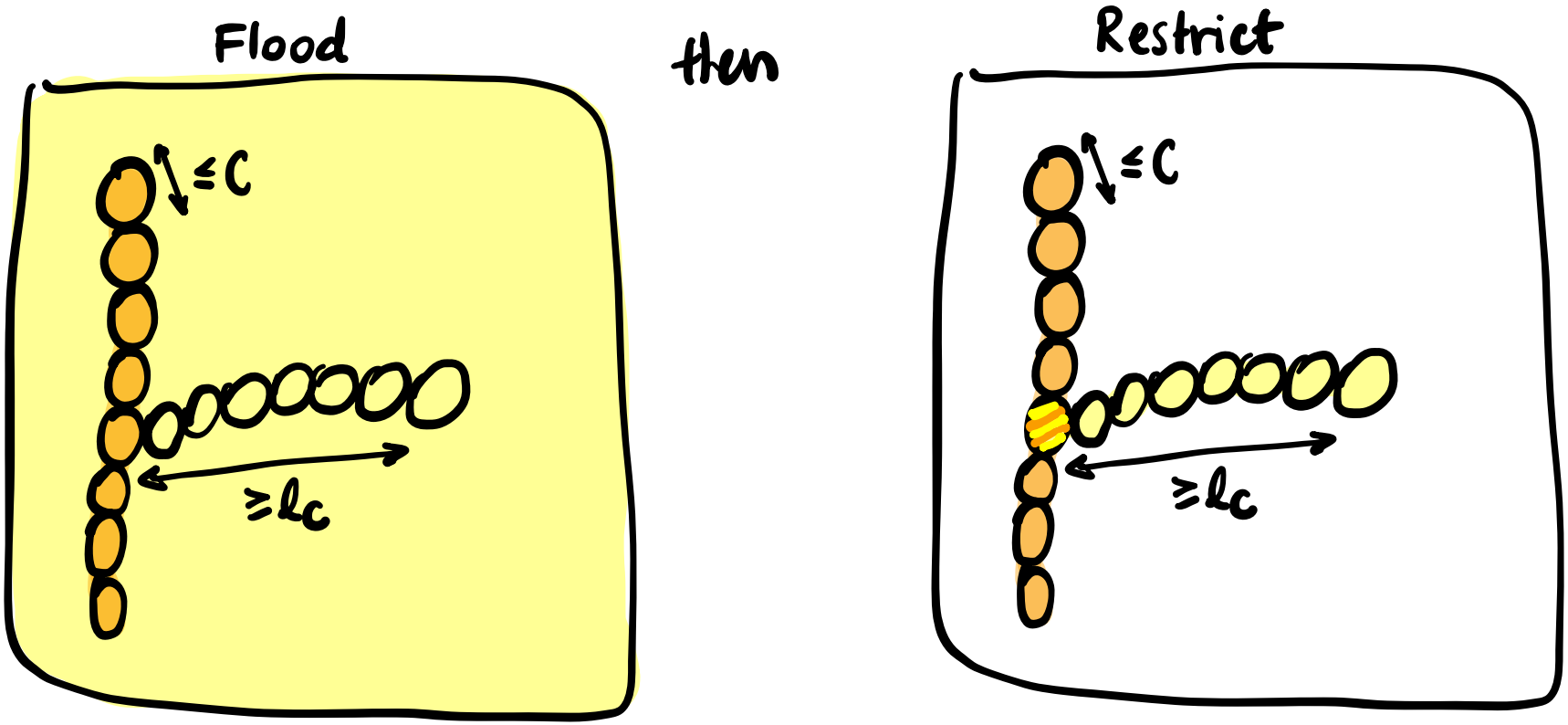


[Γ v.f.] Pulling back path decomp. on (Y, F)



[Γ v.f.] "Flood and restrict to blockpath"

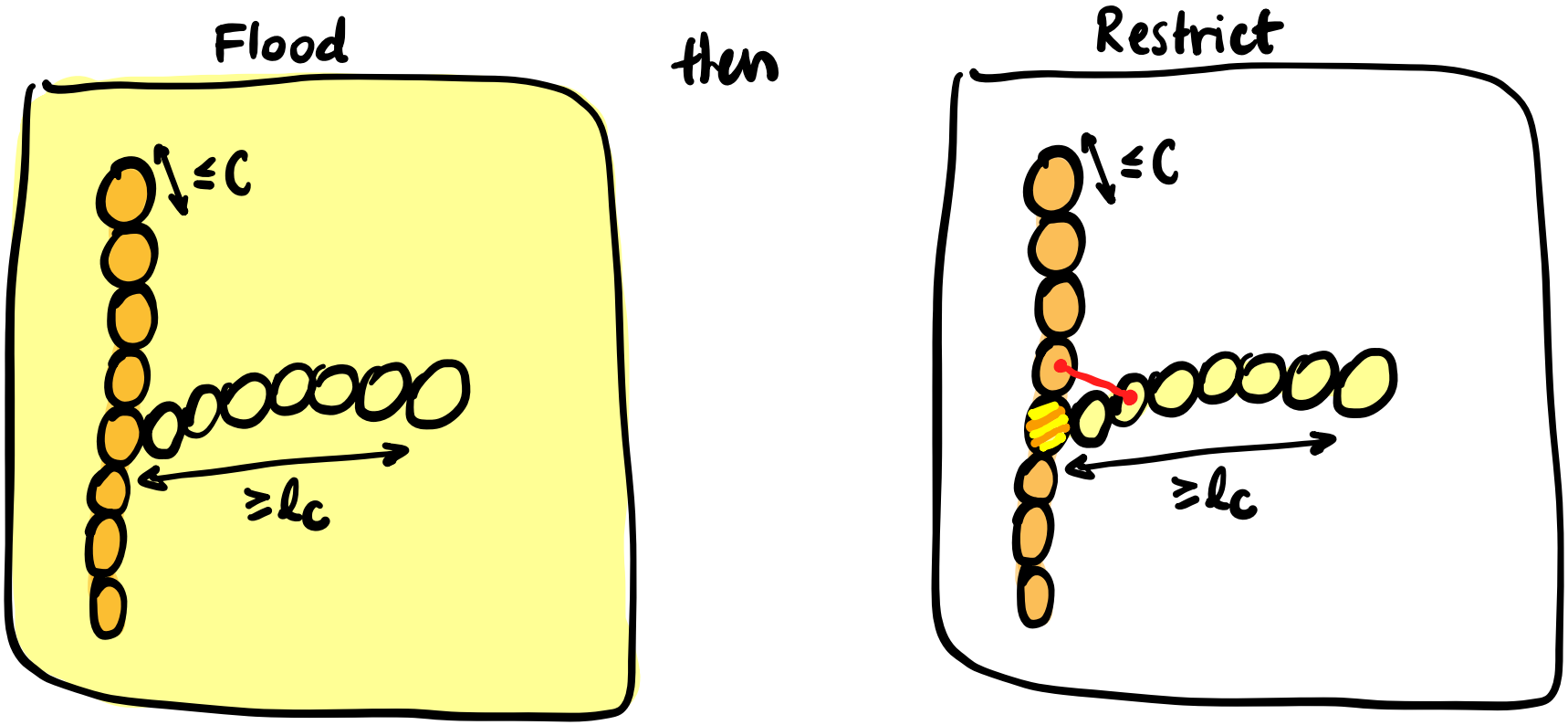
Intuitive idea: for each blockpath, apply C-block ℓ_C-fillability to fill the whole picture, then keep only labels on the blockpath.



Issue 1: red edge between pure orange and pure yellow labels.

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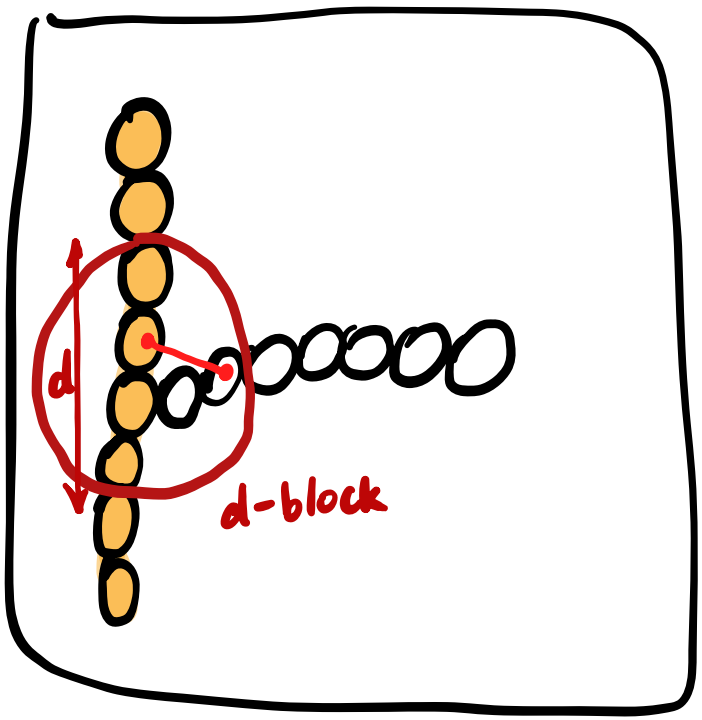
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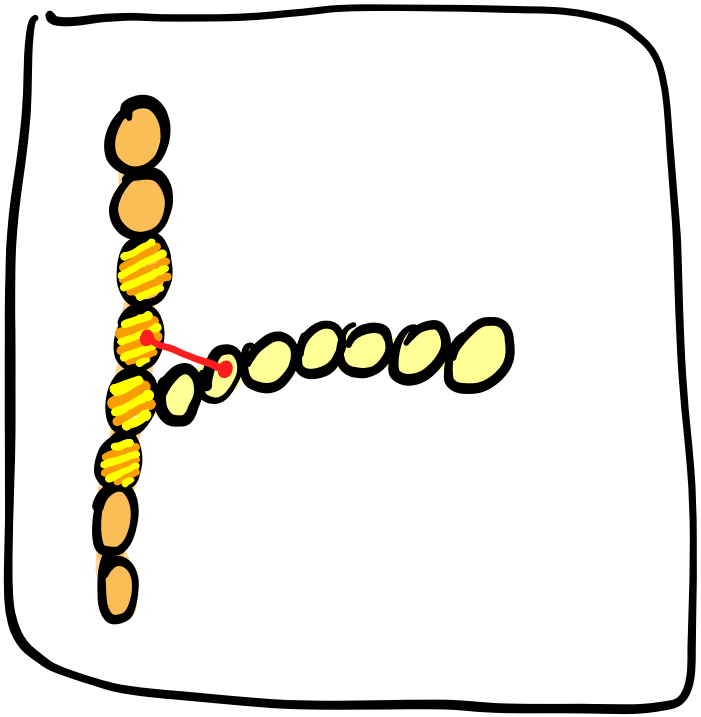


Issue 1: red edge between pure orange and pure yellow labels.

[Γ v.f.] Fix to Issue 1

Fix: choose the 2 starting blocks to be much larger — d-blocks instead of C-blocks.





[Γ v.f.] Issue 2 and fix 2

Issue 2: recall Conley-Marks-Unger (2020) produce a Borel path decomp. on a Borel comeager subset $D^Y \subseteq Y$.

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Fix: have to open the [CMU'20] black box slightly.

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For the other source of Borel path decomp. in [CMU'20], "**hyperfiniteness**", no such issue arises: $D^Y \subseteq Y$ being $\mathfrak{q}_*\mu$-conull means $\mathfrak{q}^{-1}(D^Y) \subseteq X$ is μ-conull.

However, still a little check that (Y, F) is still $\mathfrak{q}_*\mu$-hyperfinite.

[Γ v.f.] Conclusion

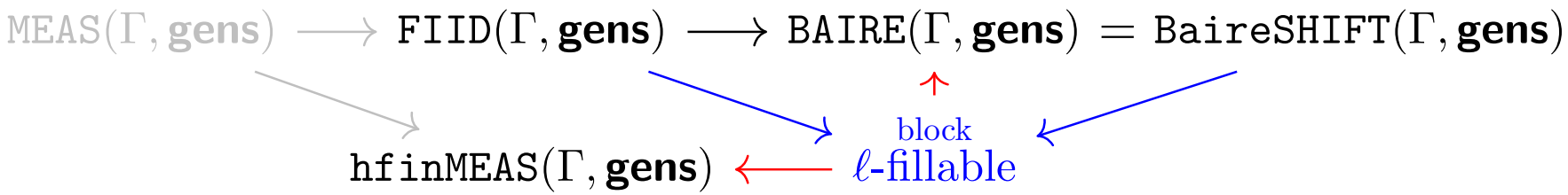
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Showed [Π is (block) l-fillable] $\iff$ [$\Pi \in \texttt{BAIRE}(\Gamma, \textbf{gens})$].

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(Note: "(block) l-fillability" takes place on just 1 copy of $\mathrm{Cay}(\Gamma, \textbf{gens})$. No more Borel/regularity stuff!)

Answer: Yes, for Γ free [Bernshteyn, personal communication], and for Γ virtually free [R., in progress].

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l-fillability w.r.t. $\Lambda^\star$ ...

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The latter is algorithmically decidable (classical FSM theory)

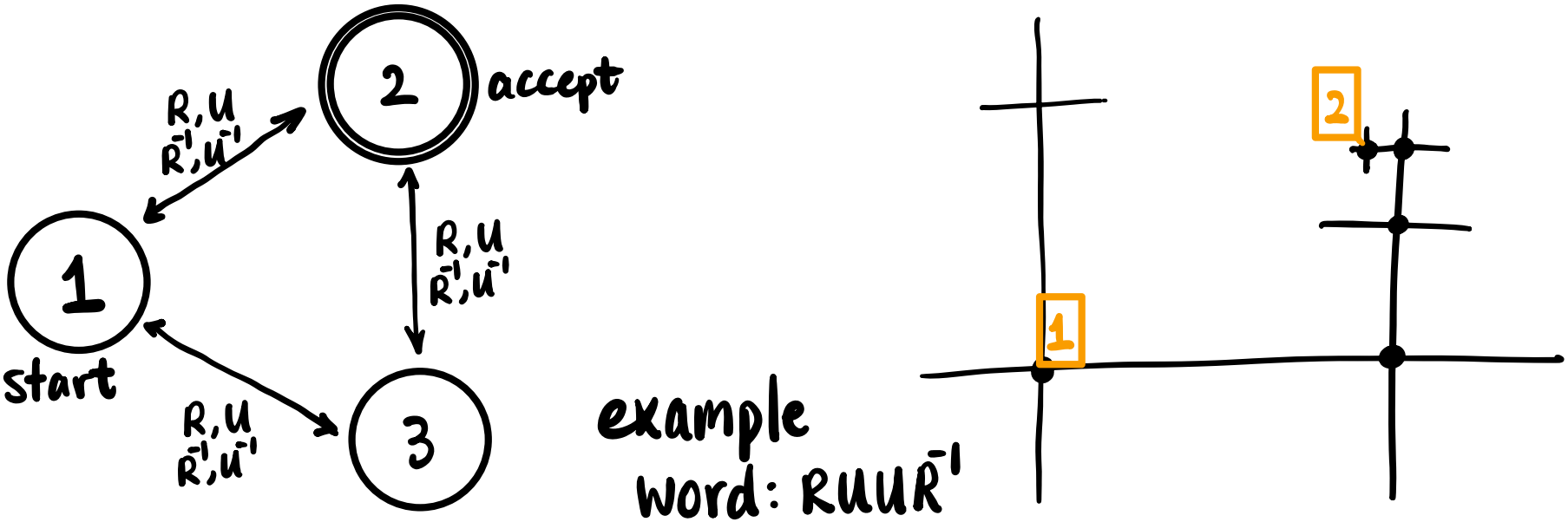
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Example: $\Pi_{\text{3-lab}} = (\{1,2,3\}, \{\mathcal{A}_R, \mathcal{A}_U\})$ on $(\mathbb{F}_2, \{R, U\})$.



Words with no "$\text{gen} \cdot \text{gen}^{-1}$" $\quad \leftrightarrow \quad$ paths in the tree $\text{Cay}(\Gamma, \textbf{gens})$

Conclude Γ free, turn to Γ v.f.

What about v.f. Γ?

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What about v.f. Γ?

Above reasoning relied on

- tree structure (paths)
- tight relationship between paths in the tree $\mathrm{Cay}(\Gamma, \mathbf{gens})$, and walks in the NDFA — the "edges" of both are named after generators $g \in \mathbf{gens}$.

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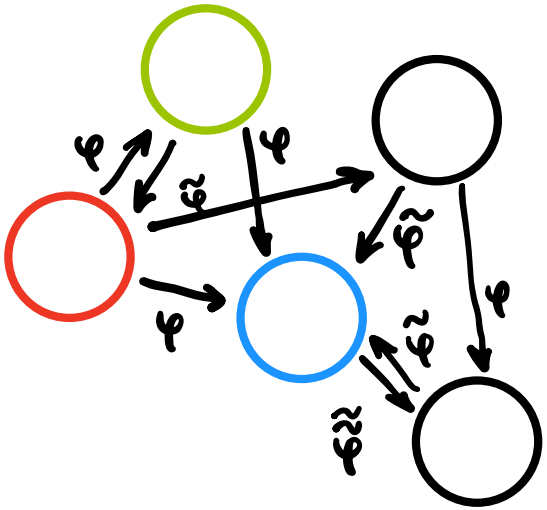
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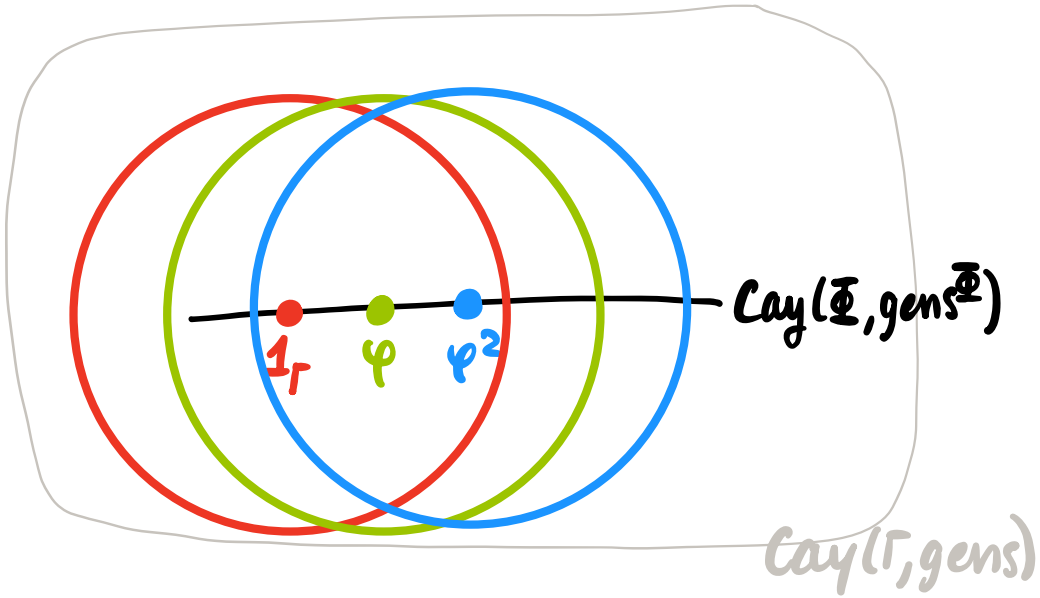
Hard to understand the edges of the more abstract structure tree from CPTT'25, but in this non-Borel setting, we have an extremely concrete structure tree:

the Cayley graph of the finite-index free subgroup $\Phi \leqslant \Gamma$!

[Γ v.f.] FSM hints

The FSM for v.f. Γ will have arrows named by generators $\varphi \in \mathbf{gens}^{\Phi}$, and states that are labelings $\vec{\sigma} : B(1_\Gamma, D) \to \Lambda$.





No more details, but **Key Point:** in this non-Borel setting of just the countable graph $\mathrm{Cay}(\Gamma, \mathbf{gens})$, we can just use the easily traversable structure tree given by $\mathrm{Cay}(\Phi, \mathbf{gens}^{\Phi}) \hookrightarrow \mathrm{Cay}(\Gamma, \mathbf{gens})$.

1 Birds-eye view

2 Definitions/Set-up

3 Fillability

4 Decidability

5 Future work

Open questions I will think about

- [Gao-Jackson-Krohne-Seward'23] show that determining membership of LCL problems Π in $\texttt{CONT}(\mathbb{Z}^2)$ is undecidable, even for LCL problems Π using 4 labels ($|\Lambda| = 4$).
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- My work for v.f. Γ so far only covers a couple of the classes from the big diagrams of Bernshteyn shown at the beginning. Do the remaining classes for v.f. Γ have exactly the same relationships as for free Γ? Is there a trichotomy: the diagram must either look like the one for $\mathbb{F}_d$, the one for $\mathbb{Z}^d$, or the one for $\mathbb{Z}$?

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  - Another question: if the Borel quasi-forest we start with is "very structured" (like a Schreier graph of a free Borel action), then are the resulting structure trees also "very structured"?
- There are other "combinatorial structure theorems" besides path decomposition, like toast, or ASI, that I would like to learn more about and perhaps apply to prove similar results.

Thanks!