4/1/24 Monica Visan 257B, KdV equation.

Conventions: (Fourier transform): $f \in S(\mathbb{R}, \mathbb{C})$

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} f(x)\, dx \quad \text{so that} \quad \text{FIF} \quad f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi} \hat{f}(\xi)\, d\xi.$$

and Plancherel has $\|f\|_2 = \|\hat{f}\|_2$.

(mostly homog, but define inhomog too)

often use Sobolev spaces: $H^s(\mathbb{R}) :=$ completion of $S(\mathbb{R})$ wrt. norm

$$\|f\|_{H^s} := \| \langle \xi \rangle^s \hat{f}(\xi) \|_{L^2_\xi}$$

"up to s derivatives in L^2"

defined on Fourier side

$\langle x \rangle := (1 + |x|^2)^{1/2}$ (linear growth at ∞, const at 0)

homog: $\|f\|_{\dot{H}^s} := \| |\xi|^s \hat{f}(\xi) \|_{L^2_\xi}$

Consider init-val problem for Airy eq: $\begin{cases} u_t + u_{xxx} = 0 \\ u(0,x) := u_0(x) \in S(\mathbb{R}) \end{cases}$

Here (physical considerations), only interested in real-valued fns. $u: \mathbb{R}_t \times \mathbb{R}_x \to \mathbb{R}$

How to solve? Take FT wrt. space variable!:

$$\begin{cases} \partial_t \hat{u}(t,\xi) - i\xi^3 \hat{u}(t,\xi) = 0 \\ \hat{u}(0,\xi) = \hat{u}_0(\xi) \end{cases}$$

Now an ODE in t w parameter ξ, $\Rightarrow \quad \hat{u}(t,\xi) = e^{it\xi^3} \hat{u}_0(\xi)$ (1)

unique sol to above init val problem.

Deduce immediately: from (1)

1) $u_0 \in S(\mathbb{R}) \Rightarrow \hat{u}_0 \in S(\mathbb{R}) \Rightarrow e^{it\xi^3} \hat{u}_0(\xi) \in S(\mathbb{R}_\xi) \quad \forall t \in \mathbb{R}$

$\Downarrow$ inverse FT

$u(t,\xi) \in S(\mathbb{R}_\xi) \quad \forall t \in \mathbb{R}$

2) ∞-many conservation laws.

$$\|u(t)\|_{H^s} := \|\langle\xi\rangle^s \hat{u}(t,\xi)\|_2 \underset{(1)}{=} \|\langle\xi\rangle^s \hat{u}_0(\xi)\|_2 =: \|u_0\|_{H^s} \quad \forall t \in \mathbb{R}, \ s \in \mathbb{R}$$

(1) unimodular factor, goes away

Note also that

$$\int u(t,x)\,dx = \int u_0(x)\,dx \quad \forall t \in \mathbb{R}$$ by taking ∂_t and seeing it is 0:

$$\partial_t \int u(t,x)\,dx = -\int \partial_x^3 u(t,x)\,dx = 0 \quad (\text{b/c } \lim_{x\to\infty} u_{xx} = 0)$$

integral of derivative of nice fn, FTC

u is Schwartz, discussed above

Taking inverse FT of (1),

$$u(t,x) = \frac{1}{\sqrt{2\pi}} \int e^{ix\xi} \hat{u}(t,\xi)\,d\xi = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi} e^{it\xi^3} \hat{u}_0(\xi)\,d\xi = \frac{1}{\sqrt{2\pi}} \int e^{i\xi(x+t\xi^2)} \hat{u}_0(\xi)\,d\xi.$$

compare to $u_0(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\xi x} \hat{u}_0(\xi)\,d\xi$

superposition of cosines given by $\hat{u}_0(\xi)$

superposition of wave packet travel to left b/c $+t\xi^3 > 0$.

(wave packet of wave vector ξ travels to the left w/ velocity ξ^2)

b/c high frequencies escape faster,

expect

- decay in time
- local smoothing effects (∀ cpct region of spacetime, after some time, high freq all escape, leaving behind low freq, which are smoother)

defining feature of dispersive eq: different frequencies travel at different velocities

(ex Airy eq & dispersive eq).

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we use the following notation for sol of the Airy eq:

$$u(t,x) = \left[e^{-t\partial_x^3} u_0\right](x)$$ (taking FT of both sides get (1)).

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we write $\left[e^{-t\partial_x^3} u_0\right](x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{it\xi^3 + i\xi x} \hat{u}_0(\xi)\,d\xi.$

want to get representation in terms of u_0, not $\hat{u}_0$.

Simply replace $\hat{u}_0$ w/ def:

$$= \frac{1}{2\pi} \int_{\mathbb{R}} e^{it\xi^3 + i\xi x} \int_{\mathbb{R}} e^{-iy\xi} u_0(y)\, dy\, d\xi.$$

illegal b/c cont. integrate in ξ, no decay in ξ!

$$= \text{"Fubini"} \quad \frac{1}{2\pi} \iint e^{it\xi^3 + i\xi(x-y)} u_0(y)\, d\xi\, dy$$

to justify Fubini, ξ-of-room w/ $e^{-\varepsilon\xi^2}$.

$$\int e^{it\xi^3 + ix\xi} e^{-\varepsilon\xi^2} \hat{u_0}(\xi)\, d\xi$$

$\downarrow$ DCT $\varepsilon \downarrow 0$ (dominated by just $|\hat{u_0}|\in L^1$ b/c it's Schwartz)

$$\int e^{it\xi^3 + ix\xi} \hat{u_0}(\xi)\, d\xi$$

and we understand

$$Ai(x) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\pi} \int e^{i\frac{\xi^3}{3} + i\xi x} e^{-\varepsilon\xi^2}\, d\xi$$

(or improper integral at $\varepsilon = 0$)

$$= \frac{1}{2\pi} \iint e^{i\frac{\eta^3}{3} + i\eta\frac{x-y}{(3t)^{1/3}}} u_0(y)\, dy\, (3t)^{-1/3}\, d\eta$$

$t\xi^3 = \frac{\eta^3}{3}$

$\xi = (3t)^{-1/3}\eta$

$$= (3t)^{-1/3} \int Ai\left(\frac{x-y}{(3t)^{1/3}}\right) u_0(y)\, dy \quad (3)$$

convolution of dirty fn.

$Ai(x) := \frac{1}{2\pi} \int e^{i\frac{\xi^3}{3} + i\xi x}\, d\xi.$

(must justify why well-defined, later)

observe that $\overline{Ai(x)} = Ai(x)$ $\leftrightarrow$ ξ change var to $-\xi$

$\Rightarrow Ai(x)$ is real,

$$= \mathrm{Re}\, \frac{1}{2\pi} \int e^{i\frac{\xi^3}{3} + i\xi x}\, d\xi$$

$$= \frac{1}{\pi} \int \cos\left(\frac{\xi^3}{3} + \xi x\right) d\xi$$

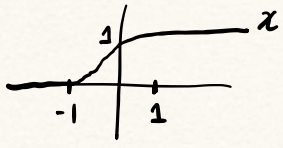
We'll show that $Ai(x)$ is uniformly bad. Then (3) yields

$$\| e^{t\partial_x^3} u_0 \|_{L^\infty_x} \le (3|t|)^{-1/3} \|Ai\|_{L^\infty} \|u_0\|_{L^1_x} \lesssim |t|^{-1/3} \|u_0\|_{L^1_x} \quad \forall t \neq 0.$$

This is dispersive estimate (already see smoothing: instantaneously in L^∞)

Lemma: $\sup_{x\in\mathbb{R}} |Ai(x)| \lesssim 1$. (will actually show $Ai(x) \to 0$ as $|x| \to \infty$)

Pf: split integral for Airy fn by $\chi \in C^\infty(\mathbb{R})$,



$$\chi(x) = \begin{cases} 1 & x \ge 1 \\ 0 & x \le -1 \end{cases}$$

and moreover $\chi(x) + \chi(-x) \equiv 1$.

$$\int e^{i\frac{\xi^3}{3} + i\xi x}\, dx = \int e^{i\frac{\xi^3}{3} + i\xi x} \chi(\xi)\, d\xi + \underbrace{\int e^{i\frac{\xi^3}{3} + i\xi x} \chi(-\xi)\, d\xi}_{= \int e^{-(i\frac{\xi^3}{3} + i\xi x)} \chi(\xi)\, d\xi}$$

change var $\xi \mapsto -\xi$

$$= 2\,\mathrm{Re} \int e^{i\frac{\xi^3}{3} + i\xi x} \chi(\xi)\, d\xi.$$

oscillatory integral. — understand stationary pts & phase

what derivatives (if any) uniformly bdd below

phase $\phi(\xi) = \frac{\xi^3}{3} + \xi x \Rightarrow \phi'(\xi) = \xi^2 + x$

$\phi''(\xi) = 2\xi$

$\phi'''(\xi) = 2$.

For $x \geq 1$, phase does not have stationary pts which means we do non stationary phase, Integrate By Parts:

$$e^{i\phi(\xi)} = \frac{1}{i\phi'(\xi)} \partial_\xi e^{i\phi(\xi)} = \frac{1}{i(\xi^2+x)} \partial_\xi e^{i\phi(\xi)}$$

$$\Rightarrow \int e^{i\phi(\xi)} \chi(\xi)\, d\xi = \int (\partial_\xi e^{i\phi\xi}) \cdot \frac{1}{i(\xi^2+x)} \chi(\xi)\, d\xi$$

$$= \int e^{i\phi(\xi)} \partial_\xi \left[\frac{1}{i(\xi^2+x)} \chi(\xi) \right] d\xi$$

what happens at bdry?

left bdry, 0 b/c $\chi \equiv 0$ at $-\infty$

right bdry, 0 b/c $\frac{1}{\xi^2} \to 0$ as $\xi \to \infty$.

So bdry contribution is 0.

$$= -\int e^{i\phi(\xi)} \underbrace{\partial_\xi \frac{1}{\xi^2+x} \cdot \partial_\xi \frac{1}{\xi^2+x} \cdot \chi(\xi)}_{\text{compose } \partial_\xi \text{ and mult-by-} \frac{1}{\xi^2+x} \text{ operators.}} d\xi$$

So $$\left| \int e^{i\phi(\xi)} \chi(\xi)\, d\xi \right| \lesssim \left\| \frac{\chi''}{(\xi^2+x)^2} \right\|_{L^1} + \left\| \frac{\chi' \cdot \xi}{(\xi^2+x)^3} \right\|_{L^1} + \left\| \frac{\chi \cdot \xi^2}{(\xi^2+x)^4} \right\|_1$$

both derivatives hit χ — 1 derivative hit χ — both derivatives miss χ.

χ' and χ'' cpctly supported on $[-1,1]$, both bdd by $\langle x \rangle^{-2}$

how to integrate $\int_{-1}^{\infty}$, ξ^2 in num, cancel w/ 1 factor in denom, integrate in ξ takes another ξ from denom, still end up w/ $\lesssim \langle x \rangle^{-2}$.

$$\lesssim \langle x \rangle^{-2}$$

For $x < 1$: we may use van der Corput on $|\phi'''(x)| \geq 2$

Lemma (van der Corput): For $k \geq 1$ and assume $|\phi^{(k)}(x)| \geq \delta > 0 \quad \forall x \in [a,b]$.

If $k = 1$, we assume additionally that ϕ' is monotone.

Then, $$\left| \int_a^b e^{i\phi(x)} \psi(x)\, dx \right| \lesssim \delta^{-\frac{1}{k}} \left[|\psi(b)| + \|\psi'\|_1 \right]$$

$\lesssim$ only depends on k, (not on δ, ψ, ϕ, and importantly not on a, b (so can take $b \to \infty$))

$$\left| \int e^{i\phi(\xi)} \chi(\xi)\, d\xi \right| \lesssim \|\chi\|_\infty + \|\chi'\|_1 \lesssim 1.$$

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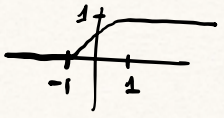
Recall last time, sol'n to $\begin{cases} u_t + u_{xxx} = 0 \\ u(0) =: u_0 \in \mathcal{S}(\mathbb{R}) \end{cases}$ is

$$u(t,x) = [e^{-t\partial_x^3} u_0](x) = \frac{1}{\sqrt{2\pi}} \int e^{it\xi^3 + ix\xi} \hat{u}_0(\xi)\, d\xi$$

$$= (3t)^{-1/3} \int \mathrm{Ai}\left(\frac{x-y}{(3t)^{1/3}}\right) u_0(y)\, dy$$

where $\mathrm{Ai}(x) := \frac{1}{2\pi} \int e^{i\frac{\xi^3}{3} + i\xi x}\, d\xi$.

Lemma: $\|\mathrm{Ai}\|_{L^\infty} \lesssim 1$, and stronger, $\lim_{|x|\to\infty} \mathrm{Ai}(x) = 0$.

pf: we prove this by introducing χ 

, $\chi(x) + \chi(-x) \equiv 1$

and rewrite $\mathrm{Ai}(x) = \frac{1}{\pi} \mathrm{Re} \int e^{i\frac{\xi^3}{3} + i\xi x} \chi(\xi)\, d\xi$.

analyze using oscillatory integral.

Phase $\phi(\xi) = \frac{\xi^3}{3} + \xi x \Rightarrow \phi'(\xi) = \xi^2 + x \Rightarrow \phi''(\xi) = 2\xi$, $\phi'''(\xi) = 2$

van der Corput on $\phi''(\xi) = 2$ immediately gives $\left| \int e^{i\frac{\xi^3}{3} + i\xi x} \chi(\xi)\, d\xi \right| \lesssim \|\chi\|_\infty + \|\chi'\|_1 \lesssim 1$ (uniform in x)

For $x \geq 1$, we saw that can just IBP ($\phi'(\xi) = \xi^2 + x \geq x \geq 1 \quad \forall \xi \in \mathrm{supp}\,\chi$)

last time summarized: $\left| \int e^{i\frac{\xi^3}{3} + i\xi x} \chi(\xi)\, d\xi \right| \leq \left| \int e^{i\phi(\xi)} \left(\partial_\xi \frac{1}{\xi^2 + x} \cdot\right)^2 \chi(\xi)\, d\xi \right| \lesssim |x|^{-2}$

Remains to analyze $x \leq -1$.

$\exists$ a unique critical pt for ϕ on $\mathrm{supp}\,\chi$, namely $\xi_0 = |x|^{1/2}$.

Observe that $\phi''(\xi_0) = 2|x|^{1/2}$, so on small nbhd on ξ_0 we have $|\phi''(\xi)| \geq |x|^{1/2}$

$\phi''' \equiv 2$, so nbhd can be taken of size $|x|^{1/2}$.

By van der Corput, on this nbhd, get the bound ("δ" $= |x|^{1/2}$)

$$\lesssim (|x|^{1/2})^{-1/2} \left[\|\chi\|_\infty + \|\chi'\|_1 \right] \lesssim |x|^{-1/4}$$

away from this nbhd, can IBP as we always do to get decay in x.

around ξ_0

By Young, ($\infty = 1$, trivial case of Young)

$$\|e^{-t\partial_x^3} u_0\|_{L_x^\infty} \lesssim |t|^{-1/3} \|Ai\|_{L^\infty} \cdot \|u_0\|_1 \lesssim |t|^{-1/3} \|u_0\|_1 \quad \text{uniformly in } t \neq 0.$$

Interpolating w/ conservation law $\|e^{-t\partial_x^3} u_0\|_2 = \|u_0\|_2 \quad \forall t \in \mathbb{R}$

we get <u>generalized dispersive estimates</u> $\forall 2 \le p < \infty$, we evaluate Airy evolution in L^p

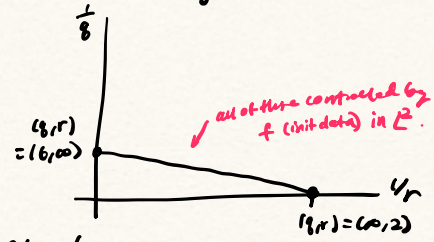
$$\|e^{-t\partial_x^3} u_0\|_{L_x^p} \lesssim |t|^{-\frac{1}{3}(1-\frac{2}{p})} \|u_0\|_{L_x^{p'}} \quad \text{uniformly in } t \neq 0.$$

$\|f\|_p \le \|f\|_\infty^{1-\frac{2}{p}} \|f\|_2^{\frac{2}{p}}$

Using this, we get the following Strichartz estimates for $2 \le q, r \le \infty$ satisfying scaling relation

$$\frac{3}{q} + \frac{1}{r} = \frac{1}{2}.$$

$$\|e^{-t\partial_x^3} f\|_{L_t^q L_x^r} \lesssim \|f\|_{L_x^2}$$



and <u>retarded</u> Strichartz estimate

(control backwards in time)

$$\left\| \int_0^t e^{-(t-\tau)\partial_x^3} F(\tau)\, d\tau \right\|_{L_t^q L_x^r} \lesssim \|F\|_{L_t^{q'} L_x^{r'}}$$

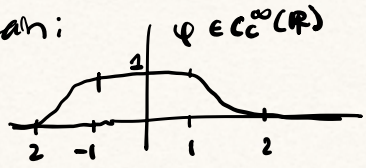
Unfortunately, all of these estimates can't help w/ KdV b/c derivative in nonlinearity and we would need H^1 control to get L^∞ control, ↻ 1:15 hr min

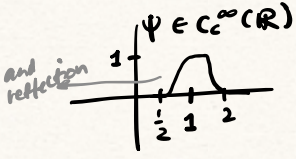
dispersive (high freqs run away very fast, ξ^2, faster than even Schrödinger running away at ξ)

→ smoothing, gain of derivatives.

To make this rigorous, need to use language of Littlewood Paley theory.

Let us set up notation again:





$$\psi(x) := \varphi(x) - \varphi(2x)$$

$N \in 2^{\mathbb{Z}} := \{2^n : n \in \mathbb{Z}\}$, let $\psi_N(x) := \psi(\frac{x}{N})$

$N = 1024$, super broad, "centered at 1024

$N = \frac{1}{1024}$, super tight! "centered at $\frac{1}{1024}$

we have partition of unity property

$$\widehat{P_{Nf}}(\xi) = \hat{f}(\xi)\,\psi(\tfrac{\xi}{N}) \quad \Leftrightarrow \quad P_N f = f * N\check{\psi}(N\cdot)$$

$P_{\leq N} f$

$P_{>N} f$

$P_{N \leq \cdot \leq M}$

Lemma: we have

(1) $$\left| \int e^{it\xi^3 + ix\xi}\, \psi(\xi)\, d\xi \right| \lesssim \begin{cases} \langle t \rangle^{-1/2} & \forall t, x \\ |x|^{-2} & |x| \geq 14\,|t| \end{cases}$$

(2) $$\left| \int e^{it\xi^3 + i\xi x}\, \psi(\tfrac{\xi}{N})\, d\xi \right| \lesssim \begin{cases} (N|t|)^{-1/2} & \forall t, x \\ N^{-1}(|x| + N^2|t|)^{-2} & |x| \not\sim N^2|t| \end{cases} \quad \leftarrow (\star)$$

(cf. Bernstein-Sobolev: $\forall 1 \leq p \leq \infty$, $\forall s \in \mathbb{R}$,

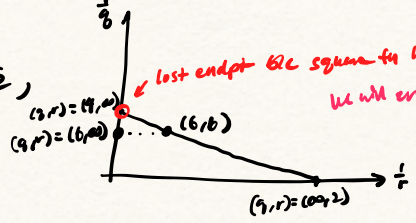
$$\| |\nabla|^s P_N f \|_p \sim N^s \|P_N f\|_p$$

)

Using $(\star)$ + square fn estimate, we get the following Strichartz estimates

$\forall 2 \leq q \leq \infty$, $\forall 2 \leq r < \infty$, $\frac{2}{q} + \frac{1}{r} = \frac{1}{2}$,

↑ control of extra gain of $\frac{1}{q}$ derivative



lost endpt b/c square fn has lost endpt.
we will eventually need to conquer the endpt, using complex interpolation method of Stein.

$$\| e^{-t\partial_x^3} |\nabla|^{\frac{1}{q}} f \|_{L_t^q L_x^r} \lesssim \|f\|_2$$

and retarded Strichartz est: $\| \int_0^t e^{-(t-\tau)\partial_x^3} |\nabla|^{\frac{2}{q}} F(\tau)\, d\tau \|_{L_t^q L_x^r} \lesssim \|F\|_{L_t^{q'} L_x^{r'}}$ ↙ gain $\frac{2}{q}$ derivs!

Interp! $\| |\nabla|^{\frac{1}{6}} e^{-t\partial_x^3} f \|_{L^6_{t,x}} \lesssim \|f\|_{L^2_x}$

cf. previously, $\| e^{-t\partial_x^3} f \|_{L_t^6 L_x^\infty} \lesssim \|f\|_{L^2_x}$

(both these satisfy same scaling... ↩ 1:30 hr min nonlogue)

Pf: $|t|^{-1/3}$ by van der Corput, get $\langle t\rangle^{-1/3}$ by ∞-meq.

∞-meq: $\left|\int e^{i\xi^3 t + i\xi x}\varphi(\xi)\,d\xi\right| \le \|\varphi\|_1 \lesssim 1.$

van der Corput: phase $\phi(\xi) = \xi^3 t + \xi x$, $\phi'(x) = 3\xi^2 t + x$, $\phi''(x) = 6\xi t$, $\phi'''(x) = 6t$

$\hookrightarrow \left|\int e^{i\xi^3 t + i\xi x}\varphi(\xi)\,d\xi\right| \lesssim |t|^{-1/3}\left[\|\varphi\|_\infty + \|\varphi'\|_1\right] \lesssim |t|^{-1/3}$

$\phi'(\xi) = 3\xi^2 t + x = 0 \Rightarrow |x| = 3\xi^2|t| \le 12|t|.$

So, $\forall c\ |\xi| > 12$, $|x| > 14|t|$ can integrate by parts

$$\left|\int e^{i\phi(\xi)}\varphi(\xi)\,d\xi\right| = \left|\int e^{i\phi(\xi)}\left(\partial_\xi \frac{1}{3\xi^2+x}\cdot\right)^2 \varphi(\xi)\,d\xi\right| \lesssim |x|^{-2}$$

same as before

Now, we approach (2):

how to get $|t|^{-1/2}$? (instead of $|t|^{1/4}$ previously?) van der Corput on 2 derivatives.

on $\operatorname{supp}\psi_N$, $|\phi''(\xi)| = 6|\xi||t| \sim N|t|$. (factor of 2)

van der Corput yields bound

$$(N|t|)^{-1/2}\left[\|\psi(\tfrac{\cdot}{N})\|_\infty + \tfrac{1}{N}\|\psi'(\tfrac{\cdot}{N})\|_1\right] \lesssim (N|t|)^{-1/2}$$

and finally, note $\phi'(\xi) = 3\xi^2 t + x = 0 \iff |x| = 3\xi^2|t| \sim 3N^2 t$ on $\operatorname{supp}\psi_N$.

So, for $x \not\sim N^2|t|$ (not comparable to (factor of 10 say)), IBP:

$$\left|\int e^{i\phi(\xi)}\psi\left(\tfrac{\xi}{N}\right)d\xi\right| = \left|\int e^{i\phi(\xi)}\left(\partial_\xi \frac{1}{3\xi^2+x}\cdot\right)^2 \psi\left(\tfrac{\xi}{N}\right)d\xi\right|$$

$$\lesssim N^{-2}\left\|\frac{1}{(3\xi^2+x)^2}\right\|_{L^1(|\xi|\sim N)} + N^{-1}\left\|\frac{\xi t}{(3\xi^2+x)^3}\right\|_{L^1(|\xi|\sim N)} + \left\|\frac{\xi^2 t^2}{(3\xi^2+x)^4}\right\|_{L^1(|\xi|\sim N)}$$

(both ∂ hit ψ; one ∂ hit ψ, one ∂_ξ hit $\frac{1}{3\xi^2+x}$)

$$\sim N^{-1}\left\|\frac{\xi t\cdot N^{-1}}{(3\xi^2+x)^3}\right\|_{L^1(|\xi|\sim N)} + \left\|\frac{\xi^2\cdot\xi^2 t^2\cdot N^{-2}}{(3\xi^2+x)^4}\right\|_{L^1(|\xi|\sim N)}$$

so all terms look like 1st term.

$$\lesssim N^{-1}\left(N^2|t| + |x|\right)^{-2}$$

Cor: $\left\| \| e^{i\xi^3 t + ix\xi} \varphi(\xi)\, d\xi \|_{L^\infty(|t|\le T)} \right\|_{L^1_x} \lesssim \langle T \rangle$

$\left\| \| e^{i\xi^3 t + ix\xi} \psi\left(\frac{\xi}{N}\right) d\xi \|_{L^\infty(|t|\le T)} \right\|_{L^1_x} \lesssim N^{3/2} \langle T \rangle^{1/2}$

unoptimal type inf., correct price, we lose derivatives "

4/5/24

Last time we proved

Lemma
$$\left|\int_{\mathbb{R}} e^{it\xi^3+ix\xi}\varphi(\xi)\,d\xi\right| \lesssim \begin{cases} \langle t\rangle^{-1/3}, & \forall t,x \\ |x|^{-2}, & |x|\geq 4|t| \end{cases}$$

$$\left|\int_{\mathbb{R}} e^{it\xi^3+ix\xi}\psi_N(\xi)\,d\xi\right| \lesssim \begin{cases} (N|t|)^{-1/2}, & \forall t,x \\ N^{-1}(|x|+N^2|t|)^{-2}, & |x|\neq N^2|t| \end{cases}$$ uniformly for $N\in 2^{\mathbb{Z}}$

This provides the key ingredient in our forthcoming proof of the maximal function estimate.

Corollary
$$\Big\| \Big| \int_{\mathbb{R}} e^{it\xi^3+ix\xi}\varphi(\xi)\,d\xi \Big|_{L^\infty(|t|\leq T)} \Big\|_{L^1_x} \lesssim \langle T\rangle \qquad (1)$$

$$\Big\| \Big| \int_{\mathbb{R}} e^{it\xi^3+ix\xi}\psi_N(\xi)\,d\xi \Big|_{L^\infty(|t|\leq T)} \Big\|_{L^1_x} \lesssim N^{3/2}\langle T\rangle^{1/2} \qquad (2)$$ only care for uniformly for $1\leq N\in 2^{\mathbb{Z}}$

uniformly for $T>0$ and $1\leq N\in 2^{\mathbb{Z}}$.

Proof: Consider (1). For $|x|\geq 4\langle T\rangle$, we may use the previous lemma to estimate
$$\Big\| \Big| \int_{\mathbb{R}} e^{it\xi^3+ix\xi}\varphi(\xi)\,d\xi \Big|_{L^\infty(|t|\leq T)} \Big\|_{L^1(|x|\geq 4\langle T\rangle)} \lesssim \big\| |x|^{-2} \big\|_{L^1(|x|\geq 4\langle T\rangle)} \lesssim \langle T\rangle^{-1} \lesssim \langle T\rangle.$$

For $|x|\leq 4\langle T\rangle$, we use the previous lemma to estimate
$$\Big\| \Big| \int_{\mathbb{R}} e^{it\xi^3+ix\xi}\varphi(\xi)\,d\xi \Big|_{L^\infty(|t|\leq T)} \Big\|_{L^1(|x|\leq 4\langle T\rangle)} \lesssim \Big\| \langle t\rangle^{-1/3} \Big\|_{L^\infty(|t|\leq T)} \Big\|_{L^1(|x|\leq 4\langle T\rangle)} \lesssim \langle T\rangle.$$
bring abs val inside, integrate φ in L^1, get constant of $\|\varphi\|_{L^1}$

We turn to (2). For $|x|\geq 4N^2\langle T\rangle$, we use the previous lemma to get
$$\Big\| \Big| \int_{\mathbb{R}} e^{it\xi^3+ix\xi}\psi_N(\xi)\,d\xi \Big|_{L^\infty(|t|\leq T)} \Big\|_{L^1(|x|\geq 4N^2\langle T\rangle)} \lesssim \big\| N^{-1}|x|^{-2} \big\|_{L^1(|x|\geq 4N^2\langle T\rangle)}$$
$$\lesssim N^{-3}\langle T\rangle^{-1} \lesssim N^{3/2}\langle T\rangle^{1/2}$$

$\left|\int e^{it\xi^3+ix\xi}\psi_N(\xi)\,d\xi\right| \lesssim N^{-1}|x|^{-2}$ uniformly in t.

For $|x|\leq N^{-1}$ we simply estimate $\left|\int e^{it\xi^3+ix\xi}\psi_N(\xi)\,d\xi\right| \leq \|\psi_N\|_{L^1}$, uniformly in t. (bring abs val inside)

(integrate in $L^1(|x|\leq N^{-1})$)
$$\Big\| \Big| \int_{\mathbb{R}} e^{it\xi^3+ix\xi}\psi_N(\xi)\,d\xi \Big|_{L^\infty(|t|\leq T)} \Big\|_{L^1(|x|\leq N^{-1})} \lesssim N^{-1}\|\psi_N\|_{L^1} \lesssim 1 \lesssim N^{3/2}\langle T\rangle^{1/2}.$$
$\leq \| \|\psi_N\|_{L^1} \|_{L^1(|x|\leq N^{-1})}$
$\lesssim N\cdot N^{-1} \lesssim 1$

Finally, for $N^{-1}\leq |x|\leq 4N^2\langle T\rangle$ we distinguish two cases:

(x fixed) Case 2: If $|x|\sim N^2|t|$ (instead t is s.t. $|x|\sim N^2|t|$), then using the previous lemma we get

$$\left|\int e^{it\xi^3+ix\xi}\psi_N(\xi)\,d\xi\right| \lesssim (N|t|)^{-\frac12} \lesssim \left(\frac{|x|}{N}\right)^{-\frac12} = N^{\frac12}|x|^{-\frac12}$$

(for fixed x, look at region of t s.t. $|x|\sim N^2|t|$)

Case 1: If instead $|x|\gtrsim N^2|t|$, then the previous lemma yields

$$\left|\int e^{it\xi^3+ix\xi}\psi_N(\xi)\,d\xi\right| \lesssim N^{-1}|x|^{-2}$$

As $N^{-1}|x|^{-2} \le N^{\frac12}|x|^{-\frac12} \iff N^{-1}\le|x|$ — so Case 2 is worse (smaller) bound, can use it in ↓

we find

$$\left\|\int e^{it\xi^3+ix\xi}\psi_N(\xi)\,d\xi\right\|_{L^\infty(|t|\le T)} \lesssim N^{\frac12}|x|^{-\frac12} \quad \text{whenever} \quad N^{-1}\le|x|\le 14N^2\langle T\rangle.$$

$$\lesssim \left\| N^{\frac12}|x|^{-\frac12}\right\|_{L^1(|x|\le 14N^2\langle T\rangle)} \lesssim N^{\frac12}\cdot(N^2\langle T\rangle)^{\frac12} \lesssim N^{\frac32}\langle T\rangle^{\frac12}.$$

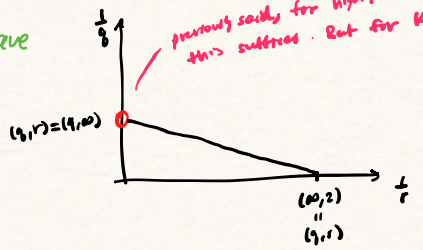
Thus

$$\left\| \left\|\int e^{it\xi^3+ix\xi}\psi_N(\xi)\,d\xi\right\|_{L^\infty(|t|\le T)}\right\|_{L^1(N^{-1}\le|x|\le 14N^2\langle T\rangle)}$$

This completes the proof of (2). ☐

Using the Lemma, TT*, Hardy-Littlewood-Sobolev, and the Littlewood-Paley square function estimate (to reassemble frequency pieces), we can derive the following Strichartz estimates:

For $2\le q\le\infty$, $2\le r<\infty$ (↑ b/c sq fn estimate fails @ ∞), $\frac{2}{q}+\frac{1}{r}=\frac12$ we have



previously sadly for high power nonlinearity KdV eq variants, this suffices. But for KdV, Need this endpt.

$$\begin{cases} \left\|e^{-t\partial_x^3}|D|^{\frac1q}f\right\|_{L_t^qL_x^r(\mathbb{R}\times\mathbb{R})} \lesssim \|f\|_2 \\ \left\|\int_0^t e^{-(t-\tau)\partial_x^3}|D|^{\frac2q}F(\tau)\,d\tau\right\|_{L_t^qL_x^r(\mathbb{R}\times\mathbb{R})} \lesssim \|F\|_{L_t^{q'}L_x^{r'}(\mathbb{R}\times\mathbb{R})} \end{cases}$$

retarded estimates

"b/c T gains $\frac1q$ der. T^* also gain $\frac1q$ derivatives"

To also obtain the endpoint estimate $\left\|e^{-t\partial_x^3}|D|^{\frac14}f\right\|_{L_t^4L_x^\infty} \lesssim \|f\|_2$ we will rely on the following

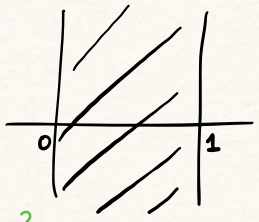
Proposition (Dispersive decay) Fix $\alpha\in[0,\frac12]$ and $2\le p\le\infty$. Then

$$\left\||D|^{\alpha(1-\frac2p)}e^{-t\partial_x^3}f\right\|_{L_x^p} \lesssim |t|^{-\frac{(\alpha+1)}{3}(1-\frac2p)}\|f\|_{L_x^{p'}}$$

uniformly for $t\neq0$.

↳ to check, $\alpha=0$, $p=\infty$, get $|t|^{-1/3}$ exactly match (van der Corput)
$\alpha=\frac12$, $p=\infty$ $|t|^{-1/2}$ (which is already proved for freq $\sim N$. So this "sums freqs" so understand f not just ψ_N.)

Proof: The claim follows from complex interpolation for the family of operators $T_z(f) = |t|^{z\frac{\alpha+1}{3}}|D|^{\alpha z}e^{-t\partial_x^3}f$ — of course analyze by looking on freq side



want bds on bdy of strip

defined for z in the strip $\{z \in \mathbb{C} : 0 \le \mathrm{Re}\, z \le 1\}$.

Note that

$$\| T_{i\sigma}(f) \|_{L^2_x} = \| |t|^{i\sigma\frac{\alpha+1}{3}} |D|^{i\sigma\alpha} e^{-t\partial_x^3} f \|_{L^2_x} = \|f\|_{L^2_x} \qquad (3)$$

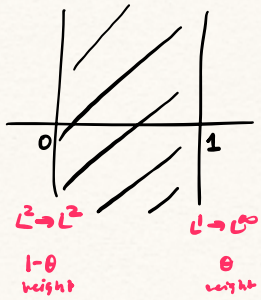
($|t|^{i\sigma\frac{\alpha+1}{3}}$: unimodular; $|D|^{i\sigma\alpha}$: goes away by Plancherel; $e^{-t\partial_x^3}$: goes away b/c $e^{i\xi^3}$ unimodular; $\mathrm{Re}=0$)

uniformly for $t \neq 0$ and $\sigma \in \mathbb{R}$. $\forall t\neq 0, \forall \sigma\in\mathbb{R}$

To complete the proof, it suffices to show

$$\| T_{1+i\sigma}(f) \|_{L^\infty_x} \lesssim \langle \sigma \rangle^2 \|f\|_{L^1_x} \qquad (4)$$

uniformly for $t \neq 0$ and $\sigma \in \mathbb{R}$. $\forall t\neq 0, \sigma\in\mathbb{R}$



$\theta = \frac{1}{p}$

Interpolating between (3) and (4) we get that

$$\| T_{1-\frac{2}{p}}(f) \|_{L^p_x} \lesssim \|f\|_{L^{p'}_x} \iff \| |D|^{\alpha(1-\frac{2}{p})} e^{-t\partial_x^3} f \|_{L^p_x} \lesssim |t|^{-\frac{\alpha+1}{3}(1-\frac{2}{p})} \|f\|_{L^{p'}_x}$$

(it was originally over on LHS, can move it over.)

unit in $t\neq 0$ b/c interpolate bdd bounds that are unit in t.

It remains to prove (4). We have

$$[T_{1+i\sigma}(f)](x) = |t|^{(1+i\sigma)\frac{\alpha+1}{3}} \frac{1}{\sqrt{2\pi}} \int e^{it\xi^3+ix\xi} |\xi|^{(1+i\sigma)\alpha} \hat{f}(\xi)\, d\xi = \frac{1}{2\pi} |t|^{\frac{\alpha+1}{3}(1+i\sigma)} \iint e^{it\xi^3+i(x-y)\xi} |\xi|^{\alpha(1+i\sigma)} f(y)\, dy\, d\xi$$

($= [|t|^{\frac{\alpha+1}{3}(1+i\sigma)} |D|^{\alpha(1+i\sigma)} e^{-t\partial_x^3} f](x)$ → inverse FT of FT)

do what we did in textbook, change var so factor out front goes away, leaving just an integral.

$$\overset{t\xi^3=\frac{\eta^3}{3}}{=} |t|^{(1+i\sigma)\frac{\alpha+1}{3}} \frac{1}{2\pi} \iint e^{i\frac{\eta^3}{3}+i\frac{x-y}{(3t)^{1/3}}\eta} \underbrace{|3t|}_{\text{indeed leave only } |t|^{\text{imag}}, \text{ as planned}} |\eta|^{(1+i\sigma)\alpha} (3t)^{-\frac{1}{3}} f(y)\, d\eta\, dy$$

$$\Rightarrow \| T_{1+i\sigma}(f) \|_{L^\infty_x} \lesssim \|f\|_{L^1_x} \Big\| \int e^{i\frac{\xi^3}{3}+ix\xi} |\xi|^{(1+i\sigma)\alpha} d\xi \Big\|_{L^\infty_x}$$

(b/c above is somehow fixed bdy fn.; rename η back to ξ.)

how to analyze? break into frequency local pieces: φ, and ψ_N w/ $N \ge 1$ (recall φ, ψ_N form partition of unity)

$$\lesssim \|f\|_{L^1_x} \Big\{ \Big\| \int e^{i\frac{\xi^3}{3}+ix\xi} |\xi|^{(1+i\sigma)\alpha} \varphi(\xi)\, d\xi \Big\|_{L^\infty_x} + \Big\| \sum_{N\ge 1} \Big| \int e^{i\frac{\xi^3}{3}+ix\xi} |\xi|^{(1+i\sigma)\alpha} \psi_N(\xi)\, d\xi \Big| \Big\|_{L^\infty_x} \Big\} \qquad (5)$$

DO NOT Δ-ineq b/c Stein Tomas, endpt ct. can't use Δ-ineq then

We have $\Big\| \int e^{i\frac{\xi^3}{3}+ix\xi} |\xi|^{(1+i\sigma)\alpha} \varphi(\xi)\, d\xi \Big\|_{L^\infty_x} \lesssim \| |\xi|^{i\sigma\alpha} \varphi(\xi) \|_{L^1} \lesssim \|\varphi\|_{L^1} \lesssim 1. \qquad (6)$

Consider $\Big| \int e^{i\frac{\xi^3}{3}+ix\xi} |\xi|^{(1+i\sigma)\alpha} \psi_N(\xi)\, d\xi \Big|$. proceed exactly using previous osc. integral technique.

The phase $\phi(\xi) = \frac{\xi^3}{3} + x\xi$ satisfies $\phi'(\xi) = \xi^2 + x$, $\phi''(\xi) = 2\xi$, $\phi'''(\xi) = 2$.

always like N on support of ψ_N

On the support of ψ_N, $|\phi''(\xi)| \sim N$. By Van der Corput we get the bound

$$N^{-\frac{1}{2}} \left[\underbrace{\| |\xi|^{(1+i\sigma)\alpha} \psi_N \|_\infty}_{N^\alpha} + \| \left(|\xi|^{(1+i\sigma)\alpha} \psi_N \right)' \|_1 \right] \lesssim N^{-\frac{1}{2}+\alpha} \cdot \langle \sigma \rangle$$

$\delta \sim N$,

scaling will be the same

Not good enough to sum in N.

derivative hit ψ_N, get $\frac{1}{N} N^\alpha \cdot N$, or derivative hit $|\xi|^{\alpha(1+i\sigma)} \rightsquigarrow \langle \sigma \rangle N^{\alpha-1} N$

Note that $\phi'(\xi) = \xi^2 + x = 0 \Rightarrow |x| \sim N^2$ for $\xi \in \operatorname{supp} \psi_N$.

find stationary pts.

"van der corput is good, but not as good as IBP!"

Thus if $|x| \not\sim N^2$ we can integrate by parts to get

$$\left| \int e^{i\frac{\xi^3}{3} + ix\xi} |\xi|^{(1+i\sigma)\alpha} \psi_N(\xi)\, d\xi \right| = \left| \int e^{i\phi(\xi)} \cdot \left(\partial_\xi \frac{1}{\xi^2 + x} \cdot \right)^2 \left[|\xi|^{(1+i\sigma)\alpha} \psi_N(\xi) \right] d\xi \right|$$

$$\lesssim \langle \sigma \rangle^2 \cdot N^{\alpha-2} \cdot \frac{1}{(N^2 + |x|)^2} \cdot \underbrace{N^{\alpha+1-2}}_{\text{coming from integral}} \lesssim \langle \sigma \rangle^2 N^{\alpha-5}$$

2 derivatives hit; ξ is like N

worst possible term, highest growth in imag part. 2 derivatives attack $|\xi|^{\alpha(1+i\sigma)}$

$$\sum_{N \geq 1} \left| \int e^{i\frac{\xi^3}{3} + ix\xi} |\xi|^{\alpha(1+i\sigma)} \psi_N(\xi)\, d\xi \right| \lesssim \sum_{N \not\sim |x|^{1/2}} \langle \sigma \rangle^2 N^{\alpha-5} + \sum_{N \sim |x|^{1/2}} N^{\alpha-\frac{1}{2}} \lesssim \langle \sigma \rangle^2$$

$N \not\sim |x|^{1/2}$: can use good IBP bd.

$N \sim |x|^{1/2}$: can only use worse vanderCorput bd.

finitely many such terms!

Still works when $\alpha = \frac{1}{2}$!

Combining this with (5,6) completes the proof of (4). ∎

9/8/24

We are now ready to prove

Theorem (Strichartz estimates) Fix $\alpha \in [0, \frac{1}{2}]$ and $2 < q < \infty$, $2 < r \leq \infty$ with $\frac{1}{q} = \frac{\alpha+1}{3}\left(\frac{1}{2} - \frac{1}{r}\right)$. Then

Recall prop

Proposition (Dispersive decay) Fix $\alpha \in [0, \frac{1}{2}]$ and $2 \leq p \leq \infty$. Then

$$\| |\nabla|^{\alpha(1-\frac{2}{p})} e^{-t\partial_x^3} f \|_{L_x^p} \lesssim |t|^{-\frac{(\alpha+1)}{3}(1-\frac{2}{p})} \| f \|_{L_x^{p'}}$$

uniformly for $t \neq 0$.

(1) $$\| |\nabla|^{\alpha(\frac{1}{2}-\frac{1}{r})} e^{-t\partial_x^3} f \|_{L_t^q L_x^r (\mathbb{R}\times\mathbb{R})} \lesssim \| f \|_{L_x^2}$$

twice as many derivatives

retarded estimate

(2) $$\left\| \int_0^t e^{-(t-\tau)\partial_x^3} |\nabla|^{2\alpha(\frac{1}{2}-\frac{1}{r})} F(\tau)\, d\tau \right\|_{L_t^q L_x^r (\mathbb{R}\times\mathbb{R})} \lesssim \| F \|_{L_t^{q'} L_x^{r'}}$$

"contribution of inhomogeneity"

Use TT^* to prove.

Proof: If $T(f) := |\nabla|^{\alpha(\frac{1}{2}-\frac{1}{r})} e^{-t\partial_x^3} f$, then its adjoint T^* is determined by

fn of space $\longmapsto$ fn of space and time

want to compute what T^* is:

$$\langle f, T^*(F) \rangle_x = \langle T(f), F \rangle_{t,x} := \int \langle |\nabla|^{\alpha(\frac{1}{2}-\frac{1}{r})} e^{-t\partial_x^3} f, F(t) \rangle_x\, dt = \int \langle f, e^{t\partial_x^3} |\nabla|^{\alpha(\frac{1}{2}-\frac{1}{r})} F(t)\, dt \rangle_x\, dt$$

self adjoint operator

$$= \langle f, \int |\nabla|^{\alpha(\frac{1}{2}-\frac{1}{r})} e^{t\partial_x^3} F(t)\, dt \rangle_x$$

can move $\int$ dt inside b/c t does not depend on t.

$$\text{i.e. } T^*(F) = \int e^{\tau\partial_x^3} |\nabla|^{\alpha(\frac{1}{2}-\frac{1}{r})} F(\tau)\, d\tau$$

Thus $T: L_x^2 \to L_t^q L_x^r \iff T^*: L_t^{q'} L_x^{r'} \to L_x^2 \iff TT^*: L_t^{q'} L_x^{r'} \to L_t^q L_x^r$

$$\| TT^*(F) \|_{L_t^q L_x^r} = \left\| \int e^{-(t-\tau)\partial_x^3} |\nabla|^{2\alpha(\frac{1}{2}-\frac{1}{r}) = \alpha(1-\frac{2}{r})} F(\tau)\, d\tau \right\|_{L_t^q L_x^r}$$

Want bd of this in terms of $\|f\|_{L_t^{q'} L_x^{r'}}$

for part (2)

TT^* each contribute $\alpha(\frac{1}{2}-\frac{1}{r})$ derivatives, hence twice as many

← first L_x^r, then L_t^q

"like composing fns"

$$\lesssim \left\| \int \| e^{-(t-\tau)\partial_x^3} |\nabla|^{\alpha((-\frac{2}{r})} F(\tau) \|_{L_x^r}\, d\tau \right\|_{L_t^q}$$

exactly **Prop: Dispersive Decay.**

$$\lesssim \Big\| \int_{\mathbb{R}} |t-\tau|^{-\frac{\alpha+1}{3}\left(1-\frac{2}{r}\right)} \|F(\tau)\|_{L_x^{r'}} \, d\tau \Big\|_{L_t^2}$$

convolution! But only in weak L^∞ space, so use **HLS!**

$$\lesssim \|F\|_{L_t^{q'} L_x^{r'}} \quad \text{by Hardy-Littlewood-Sobolev}$$

provided $\begin{cases} 0 < \frac{\alpha+1}{3}\left(1-\frac{2}{r}\right) < 1 \iff 2 < r \le \infty & \checkmark \\ 1 + \frac{1}{q} = \frac{1}{q'} + \frac{\alpha+1}{3}\left(1-\frac{2}{r}\right) \iff \frac{2}{q} = \frac{\alpha+1}{3}\left(1-\frac{2}{r}\right) & \checkmark \end{cases}$

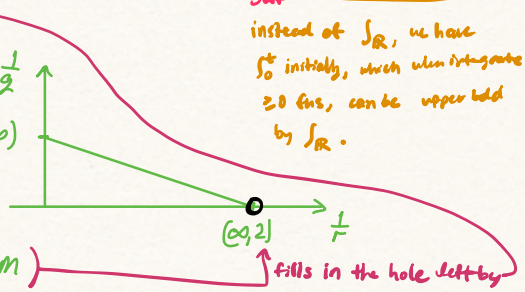
(the negative exponent "isn't too singular"; $1-\frac{1}{q'} = \frac{1}{q}$; the weak L^∞ space the $|t|^{-\cdots}$ belongs to)

$$\iff \frac{1}{q} = \frac{\alpha+1}{3}\left(\frac{1}{2}-\frac{1}{r}\right) \quad \checkmark$$

This gives (1), and (2) follows from above pf, but instead of $\int_{\mathbb{R}}$, we have $\int_0^t$ initially, which when integrate ≥ 0 fns, can be upper bdd by $\int_{\mathbb{R}}$.

Special cases:

1) $\alpha = 0 \implies \left(\frac{3}{q} + \frac{1}{r} = \frac{1}{2}, \quad 2 < r \le \infty\right)$ ($\frac{1}{q} = \frac{1}{3}(\frac{1}{2}-\frac{1}{r})$)

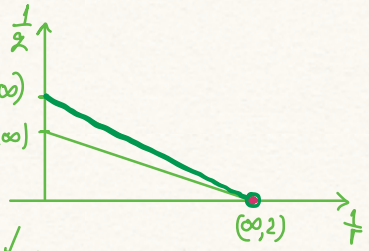


Then $\|e^{-t\partial_x^3} f\|_{L_t^q L_x^r} \lesssim \|f\|_{L_x^2}$ (*)

Note $(q,r) = (\infty, 2)$ is just the conservation of momentum — fills in the hole left by

2) $\alpha = \frac{1}{2} \implies \left(\frac{2}{q} + \frac{1}{r} = \frac{1}{2}, \quad 2 < r \le \infty\right)$ ($\frac{1}{q} = \frac{1}{2}(\frac{1}{2}-\frac{1}{r})$)

$\|e^{-t\partial_x^3} |\nabla|^{\frac{1}{2}(\frac{1}{2}-\frac{1}{r})} f\|_{L_t^q L_x^r}$



Then $\|e^{-t\partial_x^3} |\nabla|^{\frac{1}{2}} f\|_{L_t^q L_x^r} \lesssim \|f\|_{L_x^2}$ (**)

"gain of $\frac{1}{2}$ derivatives"

The case $(q,r) = (\infty, 2)$ is again the conservation of momentum

$\alpha = 0$ scaling relation in (*)

For $2 < r < \infty$, (**) implies (*) by Sobolev embedding: For $\frac{3}{q} + \frac{1}{r} = \frac{1}{2}$,

want $\frac{1}{2}$ derivative on RHS so can apply (**)

$$\|e^{-t\partial_x^3} f\|_{L_t^q L_x^r} \lesssim \|e^{-t\partial_x^3} |\nabla|^{\frac{1}{2}} f\|_{L_t^q L_x^{\tilde{r}}}$$

where $\frac{1}{\tilde{r}} = \frac{1}{r} + \frac{1}{q}$ (we are adding = 1.)

$\frac{1}{r} = \frac{1}{\tilde{r}} - \frac{1}{q} \implies \frac{2}{q} + \frac{1}{\tilde{r}} = \frac{2}{q} + \frac{1}{r} + \frac{1}{q} = \frac{1}{2}$, which is exactly the scaling relation for $\alpha = \frac{1}{2}$ in (**).

$= \frac{3}{q} + \frac{1}{r}$

12:25pm

Shelve these for now, will use them later, in conjunction of 2 more bounds:

Prop (Kato smoothing): instantaneously smoothing. $\| e^{-t\partial_x^3} f \|_{L_t^4}$ ← average in time, can actually control one whole derivative

$\| \partial_x e^{-t\partial_x^3} f \|_{L_x^\infty L_t^2} \lesssim \|f\|_{L_x^2}$ uniformly in space!

"average in time, much better regularity!"

Pf: $[\partial_x e^{-t\partial_x^3} f](x) = \frac{i}{\sqrt{2\pi}} \int e^{ix\xi + it\xi^3} \xi \hat{f}(\xi)\, d\xi$.

imagine is τ (dual of t after FT in time), set $e^{it\tau}(\ldots)$, Plancherel! ($e^{it\tau}$: FT in τ.)

$$= \frac{i}{\sqrt{2\pi}} \int e^{it\tau + ix\tau^{1/3}} \tau^{1/3} \hat{f}(\tau^{1/3}) \tfrac{1}{3}\tau^{-2/3} d\tau.$$

($= $ under $\xi^3 = \tau$, $\xi = \tau^{1/3}$)

By Plancherel, $\| (\partial_x e^{-t\partial_x^3} f)(x) \|_{L^2_t}^2 = (\tfrac{1}{3})^2 \| e^{ix\tau^{1/3}} \tau^{-\frac{1}{3}} \hat{f}(\tau^{1/3}) \|_{L^2_\tau}^2$

$$= \frac{1}{9} \int \tau^{-2/3} |\hat{f}(\tau^{1/3})|^2 d\tau$$

$$= \frac{1}{9} \int \xi^{-2} |\hat{f}(\xi)|^2 3\xi^2 d\xi = \frac{1}{3}\|f\|_{L^2}^2$$

($= $ under $\tau^{1/3} = \xi \Leftrightarrow \tau = \xi^3$; Plancherel again.)

"balanced Kato smoothing above" in sense that above had L^∞_x. Sometimes will have to use integrability in space

Prop (Maximal fn estimate): For $s > \frac{3}{4}$ — "allow decay $> \frac{3}{4}$ derivatives, one can control" (it one allows)

$$\| e^{-t\partial_x^3} f \|_{L^2_x L^\infty_t (\mathbb{R}\times[-T,T])} \lesssim \langle T \rangle^{1/2} \|f\|_{H^s}$$

worst possible behavior in time, then integrate in space

doing time first, and then space — radical idea at the time

Pf: here, always over slab $\mathbb{R} \times [-T, T]$.

Break into freq local pieces

$$\| e^{-t\partial_x^3} f \|_{L^2_x L^\infty_t} \le \| e^{-t\partial_x^3} P_{\le 1} f \|_{L^2_x L^\infty_t} + \sum_{N \in 2^{\mathbb{Z}}} \| e^{-t\partial_x^3} P_N \tilde{P}_N f \|_{L^2_x L^\infty_t}$$

flattened Littlewood-Paley projection corresponding to Fourier multiplier $\tilde{\psi}_N := \psi_{\frac{N}{2}} + \psi_N + \psi_{2N}$ which is $\equiv 1$ on supp ψ_N in particular

$\Rightarrow P_N \tilde{P}_N = P_N$

We'll use TT^*. If $T(f) := e^{-t\partial_x^3} P_{\le 1} f$, similar to above (moving all the self-adjoint ops to other side of inner prod)

$$T^*(F) = \int e^{\tau\partial_x^3} P_{\le 1} \cdot F(\tau)\, d\tau.$$

and similarly, $T(f) = e^{-t\partial_x^3} P_N f \Rightarrow T^*(F) = \int e^{\tau\partial_x^3} P_N F(\tau)\, d\tau$

$T: L^2 \to L^2_x L^\infty_t \iff T^*: L^2_x L^1_t \to L^2 \iff TT^*: L^2_x L^1_t \to L^2_x L^\infty_t$.

First, low freq $P_{\le 1}$:

$$\Big\| \int e^{-(t-\tau)\partial_x^3} P^2_{\le 1} F(\tau)\, d\tau \Big\|_{L^2_x L^\infty_t} \le \Big\| \int_{-T}^{T} \big| e^{-(t-\tau)\partial_x^3} P^2_{\le 1} F(\tau) \big|\, d\tau \Big\|_{L^2_x L^\infty_t}$$

(L^∞-abs val, Δ-freq bring abs val inside integral)

on Fourier side, it's multiplying $\widehat{F(\tau)}$ (or FT wrt x) by nice things;
on physical side, convolution in x
$= F(\tau) *_x [e^{i(t-\tau)\xi^3} \varphi(\xi)^2]^\vee$

put $\int_{-T}^{T} d\tau$ all on, remaining go in L^∞

$$\lesssim \Big\| \|F\|_{L^1_\tau} *_x \Big\| \int e^{ix\xi + i(t-\tau)\xi^3} \varphi(\xi)^2 d\xi \Big\|_{L^\infty_{t,\tau}} \Big\|_{L^2_x}$$

($*_x$: convolution in x)

Young ineq

$$\lesssim \|F\|_{L^2_x L^1_t} \cdot \Big\| \int e^{ix\xi + i(t-\tau)\xi^3} \varphi(\xi)^2 d\xi \Big\|_{L^1_x L^\infty_{t,\tau}}$$

corollary from lectures ago

$$\lesssim \|F\|_{L^2_x L^1_t} \langle T \rangle$$

So (sgn bdd TT^*), $\|e^{-t\partial_x^3} P_{\le 1} f\|_{L^2_x L^\infty_t}$ (sloppy) $\lesssim \langle T \rangle^{1/2} \|f\|_2$ — dependent on T

Similarly, again using Cor!

$$\Big\| \int e^{-(t-\tau)\partial_x^3} P^2_N F(\tau)\, d\tau \Big\|_{L^2_x L^\infty_t} \lesssim N^{\frac{3}{2}} \langle T \rangle^{1/2} \|F\|_{L^2_x L^1_t}$$

$$\Rightarrow \|e^{-t\partial_x^3} P_N \widetilde{P_N} f\|_{L^2_x L^\infty_t} \text{ (sloppy)} \lesssim N^{3/4} \langle T \rangle^{1/4} \|\widetilde{P_N} f\|_2$$

dependent on T

Finally,

$$\|e^{-t\partial_x^3} f\|_{L^2_x L^\infty_t} \le \|e^{-t\partial_x^3} P_{\le 1} f\|_{L^2_x L^\infty_t} + \sum_{N \in 2^{\mathbb{Z}}} \|e^{-t\partial_x^3} P_N \widetilde{P_N} f\|_{L^2_x L^\infty_t}$$

$$\lesssim \langle T \rangle^{\frac{1}{2}} \|f\|_2 + \sum_{N \ge 1} N^{3/4} \langle T \rangle^{1/4} \|\widetilde{P_N} f\|_2$$

(obvious; Bernstein on $\widetilde{P_N} f$ localized)

$$\lesssim \langle T \rangle^{1/2} \Big[\|f\|_{H^s} + \sum_{N \ge 1} N^{\frac{3}{4}-s} \|\widetilde{P_N} f\|_{H^s} \Big] \lesssim \langle T \rangle^{1/2} \|f\|_{H^s}$$

$\|\widetilde{P_N} f\|_{H^s} \le \|f\|_{H^s}$, $\sum_{N \ge 1} N^{\frac{3}{4}-s}$ summable.

4/10/24

KdV equation, finally

Consider initial value problem $\begin{cases} u_t + u_{xxx} = 6uu_x \\ u(0) = u_0 \end{cases}$ (1) — linear propagative terms

real valued

$u: \mathbb{R}_t \times \mathbb{R}_x \to \mathbb{R}$

History: appears first time in footnote of textbook in 1877 Boussinesq

but more prominently, 1895 Korteweg de Vries

controversy: water wave steepen and break

John Scott Russell 1930 solitary waves + waves passing thru each other w/o changing shape ; speed ~ solitons particle like

derive using physically motivated/compelling argument eg, admitting solitons

$Q_c(t,x) = -\frac{c}{2}\,\mathrm{sech}^2\left(\frac{\sqrt{c}}{2}(x-ct)\right)$, $c>0$ — speed proportional to width

1st eq (nonlinear PDE) shown to be completely integrable.. 3:35 monologue

Standard procedure: solve ODE for instance

variation of parameters, integral formulation, contraction mapping.

↓

we seek solutions of the form $u(t) = e^{-t\partial_x^3} v(t)$

$$\Downarrow$$

$$u_t + u_{xxx} = 6uu_x$$

$$-\partial_x^3 e^{-t\partial_x^3} v(t) + e^{-t\partial_x^3}\partial_t v + \partial_x^3 e^{-t\partial_x^3} v(t) = 6uu_x$$

$$\Rightarrow \quad \partial_t v = e^{t\partial_x^3}(6uu_x)$$

$$\overset{\text{FTC}}{\Rightarrow} \quad v(t) = \underbrace{v(0)}_{u(0)} + \int_0^t e^{\tau\partial_x^3}(6uu')(\tau)\,d\tau \qquad (u' = \text{space derivative})$$

$$\Rightarrow \boxed{u(t) = e^{-t\partial_x^3}u(0) + \int_0^t e^{-(t-\tau)\partial_x^3}(6uu')(\tau)\,d\tau}$$

Duhamel formula assoc'd IVP (1)

Solutions were only first constructed in 1965

Hard b/c u' in RHS, "to control LHS in H^s, need H^{s+1} control for RHS!" neverending loop !!

Question: assume u is global smooth solution. Can we control its H^s norm (for simplicity, $s \in \mathbb{N}^+$)?

w/o any extra structure, no.

recall we did this w/ Airyeq on day 1.

We compute $\partial_t \|u(t)\|_2^2 = 2\langle u, u_t\rangle = 2\langle u, -\partial_x^3 u + 6uu'\rangle$

$= 2\langle u, -\partial_x^3 u\rangle + 12\langle u, uu'\rangle$

$-\partial_x^3$ anti self adjoint, telling us this term disappears

$12\langle u, uu'\rangle = 12\int u^2 u'$ also a derivative! Special structure!

$= 4\int \partial_x(u^3) = 0$

Rephrasing, we basically observed that, $u_t = \partial_x(-\partial_x^2 u + 3u^2)$

$\Downarrow$ space integral of $u_t = 0 \Rightarrow \partial_t$ of $\int_{\mathbb{R}_x} u = 0$, "total mass of wave" doesn't change.

$\Rightarrow$ conservation law $\|u(t)\|_2 = \|u(0)\|_2 \quad \forall t \in \mathbb{R}$

As $\|u(t)\|_{H^S} \leq \|u(t)\|_2 + \|\partial_x^S u(t)\|_2$

$\partial_t \|\partial_x^S u(t)\|_2^2 = 2\langle \partial_x^S u, \partial_x^S u_t\rangle = 2\langle \partial_x^S u, \partial_x^S(-\partial_x^3)u\rangle + 12\langle \partial_x^S u, \partial_x^S(uu')\rangle$

$\partial_x^S(-\partial_x^3)$ antiself adjoint again

$\langle \partial_x^S u, \partial_x^S(uu')\rangle = \langle \partial_x^S u, (\partial_x^S u)\cdot u'\rangle + \langle \partial^S u, u\cdot(\partial^{S+1}u)\rangle + S\langle \partial^S u, u'\cdot \partial_x^S u\rangle$

∂_x^S gets all ∂_x^S; u' gets all ∂_x^S; u gets 1 ∂_x, u' gets ∂_x^{S-1}; S many choices to put ∂_x^1 on u.

"in bkt terms"

$+ \sum_{\substack{k+j=S-1 \\ 1\leq k\leq S-2}} c_k \langle \partial_x^S u, \partial_x^{k+1}u\cdot \partial_x^{S+1}u\rangle$

$= (1+S)\langle \partial_x^S u, (\partial_x^S u)\cdot u'\rangle + \int u\cdot\frac{1}{2}\partial[(\partial^S u)^2] + \sum_{\substack{k+j=S-1 \\ 1\leq k\leq S-2}} c_k \langle \partial_x^S u, \partial_x^{k+1}u\cdot\partial^{j+1}u\rangle$

$= -\frac{1}{2}\int u'\cdot(\partial^S u)^2$ IBP

same! type of term

So plugging in above,

$\partial_t\|\partial_x^S u(t)\|_2^2 \lesssim_S \|\partial_x^S u(t)\|_2^2 \cdot \|u'(t)\|_\infty + \sum_{\substack{k+j=S-1 \\ 1\leq k\leq S-2}} \|\partial_x^S u\|\cdot\|\partial_x^{k+1}u\|_{\frac{2(S-1)}{k}}\|\partial_x^{j+1}u\|_{\frac{2(S-1)}{j}}$

"Gagliardo Nirenberg type"

We'll use the ineq: (1 derivative; "maximal" # of derivatives)

$(*)\quad \|\partial^{k+1}f\|_{\frac{2(S-1)}{k}} \lesssim \|f'\|_\infty^{1-\frac{k}{S-1}}\|\partial^S f\|_2^{\frac{k}{S-1}} \quad \forall\, 1\leq k\leq S-2$

to get $\partial_t\|\partial_x^S u(t)\|_2^2 \lesssim_S \|\partial_x^S u(t)\|_2^2\cdot\|u'(t)\|_\infty + \sum_{\substack{k+j=S-1 \\ 1\leq k\leq S-2}} \|\partial_x^S u\|_2\cdot\|u'\|_\infty^{1-\frac{k}{S-1}}\|\partial_x^u u\|_2^{\frac{k}{j-1}}\cdot\|u'\|_\infty^{1-\frac{j}{S-1}}\|\partial_x^u u\|_2^{\frac{j}{j-1}}$

$k+j=S-1$

$\lesssim \|\partial_x^S u(t)\|_2^2\cdot\|u'(t)\|_\infty$

By Gronwall, + conservation of mass: we get

$$\|u(t)\|_{H^s}^2 \le \|u(0)\|_{H^s}^2 \, e^{c\int_0^t \|u'(\tau)\|_\infty \, d\tau}$$

Note (recalling Sobolev embedding)

$$\|f\|_\infty \lesssim \|f\|_{H^{\frac{1}{2}+\varepsilon}} \quad \forall \varepsilon > 0$$

$1+\frac{1}{2}+\varepsilon$ rounded up to nearest integer is 2. (so nice bcos derivs)

So $\|u(t)\|_{H^s}^2 \le \|u(0)\|_{H^s}^2 \, e^{c\int_0^t \|u(\tau)\|_{H^2} \, d\tau}$

→ To finish, we prove (*), or equivalently,

$$\|\partial^k f\|_{\frac{2(s-1)}{k}} \lesssim \|f\|_\infty^{1-\frac{k}{s-1}} \, \|\partial^{s-1} f\|_2^{\frac{k}{s-1}} \qquad \forall\, 1 \le k \le s-2.$$

Try $k=1$ first. Break into Littlewood-Paley pieces

$$\|f'\|_{2(s-1)} \lesssim \sum_{N \in 2^{\mathbb{Z}}} \|f_N'\|_{2(s-1)} \lesssim \sum_{N \in 2^{\mathbb{Z}}} N \cdot \|f_N\|_{2(s-1)}$$

Bernstein to travel to L^2

$$\lesssim N^{\frac{1}{2}-\frac{1}{2(s-1)}} \|f_N\|_2$$

insert back in ∂_x^{s-1}

$$\lesssim N^{\frac{1}{2}-\frac{1}{2(s-1)}} N^{-(s-1)} \|\partial^{s-1} f\|_2$$

this works for high freq, but middle / low freq doesn't work?

Try Hölder: $\|f_N\|_{2(s-1)} \lesssim \|f_N\|_\infty^{1-\frac{1}{s-1}} \|f_N\|_2^{\frac{1}{s-1}}$

$$\lesssim \|f\|_\infty^{\frac{s-2}{s-1}} \, N^{-1} \, \|\partial^{s-1} f\|_2^{\frac{1}{s-1}}$$

$(s-1)\cdot\frac{1}{(s-1)}$

we can't sum this $\sum_{N \in 2^{\mathbb{Z}}}$, doesn't work.

$$\|f'\|_{2(s-1)} \lesssim \|f\|_\infty^{\frac{s-2}{s-1}} \, \|\partial^{s-1} f\|_2^{\frac{1}{s-1}}$$

L^2 exactly places can sum freq pieces back to original.

Try square function ↓

$$\|f'\|_{2(s-1)}^{2(s-1)} \lesssim \|f\|_\infty^{2(s-2)} \, \|\partial^{s-1} f\|_2^2$$

$$\|f'\|_{2(s-1)}^{2(s-1)} \sim \left\| \sqrt{\sum_N N^2 |f_N|^2} \right\|_{2(s-1)}^{2(s-1)} \sim \int \left(\sum_N N^2 |f_N|^2 \right)^{s-1}$$

$$\lesssim \sum_{N_1 \le \cdots \le N_{s-1}} N_1^2 \cdots N_{s-1}^2 \int |f_{N_1}|^2 \cdots |f_{N_{s-1}}|^2$$

Want to put small freq in L^∞!

(put all but biggest one in L^∞)

$$\lesssim \|f\|_\infty^{2(s-2)} \sum_{N_1 \geq \dots \geq N_{s-1}} N_1^2 \cdots N_{s-1}^2 \|f_{N_{s-1}}\|_2^2$$

summing geom series one by one

$$\lesssim \|f\|_\infty^{2(s-2)} \sum_{N_{s-1}} N_{s-1}^{2(s-1)} \|f_{N_{s-1}}\|_2^2$$

$$\lesssim \|f\|_\infty^{2(s-2)} \|\partial^{s-1} f\|_2^2$$

square fn pieces

How to handle higher K?

- try above approach, does it end up working
- induct? LOL.
- complex interpolation

exc: cook up the operator! Tz

$$\|\partial^k f\|_{\frac{2(s-1)}{k}} \lesssim \|\partial f\|_{2(s-1)}^{1-\theta} \|\partial^{s-1} f\|_2^{\theta} \quad (*)$$

where $k = (1-\theta) + (s-1)\theta \iff \theta = \frac{k-1}{s-2}$

$$\lesssim \left[\|f\|_\infty^{\frac{s-2}{s-1}} \|\partial^{s-1} f\|_2^{\frac{1}{s-1}}\right]^{1-\theta} \cdot \|\partial^{s-1} f\|_2^{\theta}$$

and exponents work out, as they must, by scaling.

4/12/24 Bona-Smith IVP $\begin{cases} u_t + u_{xxx} = 6uu_x \\ u(0) = u_0 \in H^s(\mathbb{R}) \end{cases}$ $2 \le s \in \mathbb{N}$ (technically, can do $\frac{3}{2} < s \in \mathbb{R}$) but $\mathbb{N}$ for simplicity.

Ill-posedness to KdV via the energy method by Bona-Smith (1975)

They did it by regularizing equation: $\begin{cases} (1 - \varepsilon \partial_x^2) u_t + u_{xxx} = 6uu_x \\ u(0) = u_0 \end{cases}$ (∂_x^2: elliptic diff operator) $\boxed{\varepsilon > 0}$

- prove on time interval $(-1, 1)$, get bound indep of $\varepsilon > 0$ of these
- prove convergence in correct topology

(fatal mistake in blue book Monica showed on day 1, in this section b/c Bona-Smith is actually quite subtle.)

In similar spirit, we'll use $\begin{cases} u_t + \varepsilon u_{xxxxx} + u_{xxx} = 6uu_x \\ u(0) = u_0 \end{cases}$ $\varepsilon > 0$ (p KdV$_\varepsilon$) (parabolic)

We suppress dependence of solution u on ε. Later, we'll write u^ε for sol. to pKdV$_\varepsilon$

We proceed by integral + contraction mapping.

Duhamel formulation of (pKdV$_\varepsilon$) is

$$u(t) = \underbrace{e^{-t(\varepsilon \partial_x^4 + \partial_x^3)} u_0}_{\text{linear flow, RHS} = 0} + \underbrace{3 \int_0^t e^{-(t-\tau)(\varepsilon \partial_x^4 + \partial_x^3)} [u^2]_x (\tau)\, d\tau}_{\text{variation of parameters tells us contribution of nonlinearity}}$$

Let $\boxed{\Phi(u)(t) := e^{-t(\varepsilon \partial_x^4 + \partial_x^3)} u_0 + 3 \int_0^t e^{-(t-\tau)(\varepsilon \partial_x^4 + \partial_x^3)} [u^2]_x (\tau)\, d\tau}$

We'll show Φ is contraction on Banach space

$$B = \{ u \in C_t H^s_x([-T,T] \times \mathbb{R}) : \|u\|_{L^\infty_t H^s_x([-T,T]\times\mathbb{R})} \le 2 \|u_0\|_{H^s} \}$$

equipped w/ the distance $d(u,v) = \|u - v\|_{L^\infty_t H^s_x([-T,T]\times\mathbb{R})}$

provided $T = T(\varepsilon, \|u_0\|_{H^s})$ is ~~still~~ small.

Let's show $\boxed{\Phi : B \to B}$.

Let $u \in B$, $|t| \le T$ (chosen later.)

Fixed t

$$\|\Phi(u)(t)\|_{H^s} \overset{\Delta\text{-ineq}}{\le} \|e^{-t(\varepsilon\partial_x^4+\partial_x^3)}u_0\|_{H^s} + 3\int_0^t \|e^{-(t-\tau)(\varepsilon\partial_x^4+\partial_x^3)}[u^2]'(\tau)\|_{H^s}\,d\tau$$

Looking
Note on Fourier side

take $t>0$, we reverse time later.

(★) $\|e^{-t(\varepsilon\partial_x^4+\partial_x^3)}f\|_{H^s} := \|e^{-t(\varepsilon\xi^4 - i\xi^3)}\hat{f}\cdot\langle\xi\rangle^s\|_{L^2_\xi} \le \|f\|_{H^s}$ (unimodular)

(★★) $\|e^{-t(\varepsilon\partial_x^4+\partial_x^3)}f'\|_{H^s} := \|\underbrace{e^{-t(\varepsilon\xi^4 - i\xi^3)}\xi}_{\text{need to estimate in } L^\infty}\ \underbrace{\hat{f}\cdot\langle\xi\rangle^s}_{\text{isolate and throw in } L^2 \text{ to get } \|f\|_{H^s}}\|_2 \le \|e^{-t\varepsilon\xi^4}\xi\|_{L^\infty}\cdot\|f\|_{H^s}$

$\| t\varepsilon\xi^4 = \eta^4$

$$\|(t\varepsilon)^{-1/4}e^{-\eta^4}\eta\|_{L^\infty_\eta}\cdot\|f\|_{H^s}$$

$$\lesssim \underbrace{\varepsilon^{-1/4}}_{\text{"huge"}}\ \underbrace{|t|^{-1/4}}_{\text{may be dangerous}}\ \|f\|_{H^s}$$

For $s>\frac{1}{2}$, H^s is algebra b/c $\|fg\|_{H^s} \le \|fg\|_2 + \||\nabla|^s(fg)\|_2$

$$\lesssim \|f\|_2\|g\|_\infty + \left(\||\nabla|^s f\|_2\|g\|_\infty + \|f\|_\infty\||\nabla|^s g\|_2\right)$$

(chain rule)

$$\lesssim \|f\|_{H^s}\|g\|_\infty + \|f\|_\infty\|g\|_{H^s} \lesssim \|f\|_{H^s}\|g\|_{H^s}$$

and $\boxed{\|g\|_\infty \lesssim \|\hat{g}\|_1 \le \|\langle\xi\rangle^s\hat{g}\|_2\,\|\langle\xi\rangle^{-s}\|_2 \lesssim \|g\|_{H^s} \quad \text{if } s>\frac{1}{2}}$

so $\|\Phi(u)(t)\|_{H^s} \overset{(★)}{\le} \|u_0\|_{H^s} + C\int_0^t \varepsilon^{-1/4}\overset{(★★)}{|t-\tau|^{-1/4}}\|u(\tau)\|^2_{H^s}\,d\tau$

use algebra prop to pull square outside!

$$\le \|u_0\|_{H^s} + \|u_0\|^2_{H^s}\cdot C\,\varepsilon^{-1/4}\,t^{3/4} \quad \forall |t|\le T$$

$$\le 2\|u_0\|_{H^s} \quad \text{provided } T \text{ is chosen so that } C\,\varepsilon^{-1/4}T^{3/4}\|u_0\|_{H^s} \le 1$$

(using property of B?)

Not done w/ $\Phi : B \to B$ yet, NEED ctty in time.

Let $0 < t' < t \leq T$.

$$\|\Phi(u)(t) - \Phi(u)(t')\|_{H^s} \leq \left\| \left[e^{-t(\varepsilon\partial_x^4 + \partial_x^3)} - e^{-t'(\varepsilon\partial_x^4 + \partial_x^3)} \right] u_0 \right\|_{H^s}$$

$$+ 3 \left\| \int_0^{t'} \left[e^{-(t-\tau)(\varepsilon\partial_x^4 + \partial_x^3)} - e^{-(t'-\tau)(\varepsilon\partial_x^4 + \partial_x^3)} \right] (u^2)'(\tau)\, d\tau \right\|_{H^s}$$

$$+ 3 \left\| \int_{t'}^{t} e^{-(t-\tau)(\varepsilon\partial_x^4 + \partial_x^3)} (u^2)'(\tau)\, d\tau \right\|_{H^s}$$

$e^{-t'((\xi^4 - i\xi^3)}$ term multiplied here, bounded by 1. recall $t' > 0$

$$\left\| \left[e^{-t(\varepsilon\partial_x^4 + \partial_x^3)} - e^{-t'(\varepsilon\partial_x^4 + \partial_x^3)} \right] u_0 \right\|_{H^s} \leq \left\| \underbrace{\left[e^{-(t-t')(\varepsilon\xi^4 - i\xi^3)} - 1 \right]}_{\text{dominated by } 2\langle\xi\rangle^3 \hat{u_0}} \langle\xi\rangle^s \hat{u_0} \right\|_{H^2}$$

$\downarrow$ DCT, $t' \nearrow t$

0

$$\left\| \int_0^{t'} \left[e^{-(t-\tau)(\varepsilon\partial_x^4 + \partial_x^3)} - e^{-(t'-\tau)(\varepsilon\partial_x^4 + \partial_x^3)} \right] (u^2)'(\tau)\, d\tau \right\|_{H^s}$$

$$\lesssim_{\Delta\text{-ineq}} \int_0^{t'} \left\| \left[e^{-(t-t')(\varepsilon\partial_x^4 + \partial_x^3)} - Id \right] e^{-(t'-\tau)(\varepsilon\partial_x^4 + \partial_x^3)} (u^2)'(\tau) \right\|_{H^s} d\tau \longrightarrow 0$$

again DCT, $t' \nearrow t$

$\left\| \checkmark \right\|_{H^s} \lesssim \varepsilon^{-1/4} |t'-\tau|^{-1/4} \|u(\tau)\|_{H^s}^2$ previous estimate

algebra property

$$\left\| \int_{t'}^{t} e^{-(t-\tau)(\varepsilon\partial_x^4 + \partial_x^3)} (u^2)'(\tau)\, d\tau \right\|_{H^s} \lesssim_{\text{Δ-ineq + previous estimate}} \int_{t'}^{t} \varepsilon^{-1/4} |t-\tau|^{-1/4} \|u(\tau)\|_{H^s}^2 d\tau$$

$$\lesssim \varepsilon^{-1/4} |t-t'|^{3/4} \|u_0\|_{H^s}^2 \longrightarrow 0 \quad (t' \to t)$$

Done w/ $\Phi : B \to B$. Now, prove Φ contraction

Let $u, v \in B$.

wTS

$$d(\Phi(u), \Phi(v)) \leq \frac{1}{2} d(u, v).$$

$$d(u,v) = \|u - v\|_{L_t^\infty H_x^s([-T,T]\times\mathbb{R})}$$

$$d(\Phi(u),\Phi(v)) \le 3 \left\| \int_0^t e^{-(t-\tau)(\varepsilon\partial_x^4+\partial_x^3)} \left[u^2-v^2\right]'(\tau)\, d\tau \right\|_{L_t^\infty H_x^s}$$

same linear term, only integral remains

algebra property

$$\lesssim_{\text{prev. estimate}} \left\| \int_0^t \varepsilon^{-1/4} |t-\tau|^{-1/4} \|(u-v)(\tau)\|_{H^s} \|(u+v)(\tau)\|_{H^s} d\tau \right\|_{C_t^\infty}$$

$$\lesssim \varepsilon^{-1/4} T^{3/4} \|u_0\|_{H^s} d(u,v) < \tfrac{1}{2} d(u,v)$$

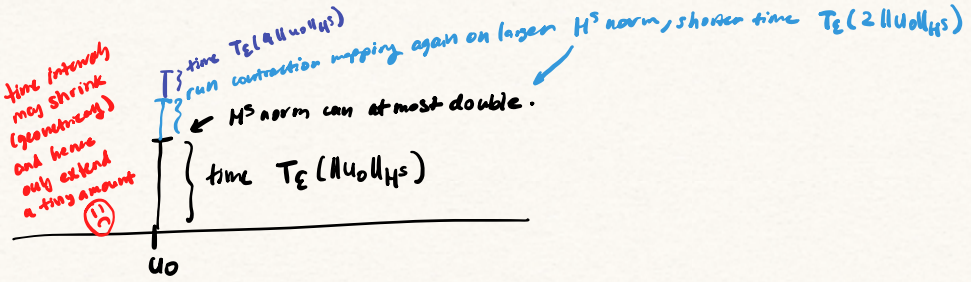
↑ for T sufficiently small.

Done w/ ~~the~~ contraction!

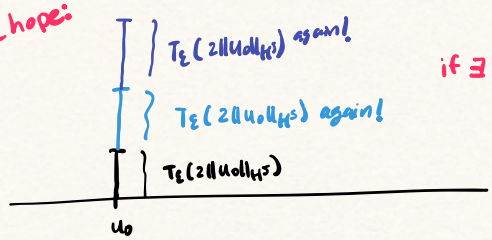
By contraction mapping thm, get unique solution in B.

Now, would like $\varepsilon \searrow 0$

Issue: ε-dependent time interval shrinking — so need to extend these sols ↑ to a time interval indep of ε.



The hope:



if $\exists$ magical T_0 s.t. H^s doesn't more than double



to do this, we seek a priori H^s bds for sols to $pkdV_\varepsilon$.

$$\partial_t \|u(t)\|_2^2 = 2\langle u, u_t\rangle = \underbrace{-2\langle u, \varepsilon\partial_x^4 u\rangle}_{\text{leave only this term.}} + 2\langle u, -\partial_x^3 u\rangle + 12\langle u, uu'\rangle$$

$-\partial_x^3$: antiself adjoint

$12\langle u, uu'\rangle = \frac{1}{3}\int (u^3)'$ "complete derivative" $= 0$

$$= \boxed{-2\varepsilon\langle \partial_x^2, \partial_x^2 u\rangle \le 0}.$$

beauty of parabolic regularization

$$\partial_t \|\partial_x^s u(t)\|_2^2 = \underbrace{-2\varepsilon\langle \partial^s u, \partial^s\partial^4 u\rangle}_{\le 0 \text{ (similar to above)}} + 2\langle \partial^s u, \partial^s(-\partial^3) u\rangle + 12\langle \partial^s u, \partial^s(uu')\rangle$$

$-\partial^3$: anti-SA

$$\lesssim \|\partial_x^s u(t)\|_2^2 \cdot \|u'(t)\|_\infty$$

square fn estimate

"Gagliardo-Nirenberg" 3-lines lemma

from last lecture

So by Gronwall,

$$\|u(t)\|_{H^s}^2 \le \|u_0\|_{H^s}^2 \, e^{C\int_0^t \|u'(\tau)\|_{L^\infty} d\tau}$$

$\frac{1}{2}$ derivative to go from L^∞ to L^2

$$\Rightarrow \|u(t)\|_{H^s} \le \|u_0\|_{H^s} \, e^{\hat{C}\int_0^t \|u(\tau)\|_{H^s} d\tau}$$

Sobolev embedding

Choosing $T_0 := T_0(\|u_0\|_{H^s})$ s.t. $\boxed{e^{\hat{C}\cdot 4 T_0 \|u_0\|_{H^s}} \le 2}$

we can extend u to $[0, T_0]$ via the usual bootstrap argument:

$$\Omega_1 = \{t \in [0, T_0] : \|u(t)\|_{H^s} \le 4\|u_0\|_{H^s}\}$$

$$\Omega_2 = \{t \in [0, T_0] : \|u(t)\|_{H^s} \le 2\|u_0\|_{H^s}\}$$

— nonempty b/c $t=0$ in both,
— closed b/c closed neq.

- so Ω_2 open!

[and then $t \in \Omega_2 \Rightarrow \exists \delta > 0$ s.t. $[t, t+\delta) \subseteq \Omega_1$

cty of solution in time

[And then prove that $\Omega_1 \subseteq \Omega_2$!

Cf Bona-Smith $(1 - \varepsilon\partial_x^2)$, Monica says she would do something like

$$u_t + u u_{xxx} = \frac{\partial_x}{1-\varepsilon\partial_x^2}(u^2)$$

4/15/24 Recall last time: For $\varepsilon > 0$, we considered

regularized KdV $\begin{cases} u_t + \varepsilon u_{xxxx} + u_{xxx} = 6uu_x \\ u(0) = u_0 \in H^s(\mathbb{R}) \end{cases}$ for $2 \le s \in \mathbb{N}$. $(pKdV_\varepsilon)$

we showed $(pKdV_\varepsilon)$ is locally well-posed in H^s: specifically, $\exists T_0 := T_0(\|u_0\|_{H^s})$

and a unique solution $u^\varepsilon \in C_t H^s_x([0,T_0]\times\mathbb{R})$ (and we used contraction mapping, ctg data $\to$ solution map)

and importantly (1) uniform in ε control

$$\|u^\varepsilon(t)\|_{H^s} \le 2\|u_0\|_{H^s} \quad \forall t \in [0,T_0]$$

we would like to send $\varepsilon \downarrow 0$ to construct solution to IVP (KdV) (w/ same u_0 as above) (initial data)

Claim: u^ε do converge as $\varepsilon \downarrow 0$ in weaker topology, $C_t L^2_x([0,T_0]\times\mathbb{R})$

Let $0 < \eta < \varepsilon$.

Grönwall to show "Cauchy"-ness

$$\partial_t \|(u^\varepsilon - u^\eta)(t)\|_2^2 = 2\langle u^\varepsilon - u^\eta, \partial_t(u^\varepsilon - u^\eta)\rangle$$

$$= 2\langle u^\varepsilon - u^\eta, -\varepsilon\partial_x^4 u^\varepsilon + \eta\partial_x^4 u^\eta\rangle + 2\langle u^\varepsilon - u^\eta, -\partial_x^3(u^\varepsilon - u^\eta)\rangle$$

"cubic terms" → 0 (∂_x^3 anti-self-adjoint)

$$+ \underbrace{6\langle u^\varepsilon - u^\eta, [(u^\varepsilon)^2 - (u^\eta)^2]'\rangle}_{\text{contributions of "nonlinearity"}}$$

$$= \underbrace{-2(\varepsilon-\eta)\langle u^\varepsilon - u^\eta, \partial_x^4 u^\varepsilon\rangle}_{\ge 0 \text{ by putting } \partial_x^2 \text{ on left half}} - 2\eta\langle u^\varepsilon - u^\eta, \partial_x^4 u^\varepsilon - u^\eta)\rangle$$

$$+ 6\langle u^\varepsilon - u^\eta, (u^\varepsilon - u^\eta)(u^\varepsilon + u^\eta)^2\rangle$$

$$+ 6\langle u^\varepsilon - u^\eta, (u^\varepsilon - u^\eta)^2(u^\varepsilon + u^\eta)\rangle$$

$= \frac{1}{2}\partial_x(u^\varepsilon - u^\eta)^2$, so upon integrate by parts, get term like but w/ different const.

$$\le 2(\varepsilon-\eta)\|u^\varepsilon - u^\eta\|_{H^2}\|u^\varepsilon\|_{H^2} + 0 + 3\|u^\varepsilon - u^\eta\|_2^2\left(\|(u^\varepsilon)'\|_\infty + \|(u^\eta)'\|_\infty\right)$$

put in $L^\infty H^2$, bdd by init data

$$\lesssim (\varepsilon-\eta)\|u_0\|_{H^2}^2 + \|u^\varepsilon - u^\eta\|_2^2\,\|u_0\|_{H^2}$$

By Grönwall

$$\|(u^\varepsilon - u^\eta)(t)\|_2^2 \lesssim (\varepsilon-\eta) + \|u_0\|_{H^2}^2 e^{C T_0 \|u_0\|_{H^2}} \lesssim_{T_0, \|u_0\|_{H^2}} (\varepsilon-\eta)$$

$\varepsilon, \eta \to 0$ ↓ 0

uniformly for $t \in [0,T_0]$ → convergence in $L^\infty_t L^2_x$

Let $u := \lim_{\varepsilon \searrow 0} u^\varepsilon$ in $C_t L^2_x([0,T_0]\times\mathbb{R})$

By (1), Fatou gives — unif in ε H^s bd

$$\|u(t)\|_{H^s} \le 2\|u_0\|_{H^s} \qquad \forall t \in [0,T_0]$$

So, $u \in (C_t L^2_x \cap L^\infty_t H^s_x)([0,T_0]\times\mathbb{R})$

By interpolation, in ptclr $u \in C_t H^r_x([0,T_0]\times\mathbb{R})$ $\forall\, 0 \le r < s$:

$$\|u(t) - u(t')\|_{H^r} \le \|u(t)-u(t')\|_2^{1-\frac{r}{s}} \|u(t)-u(t')\|_{H^s}^{r/s}$$

→ go to Fourier side, Plancherel,

$$\le \|u(t)-u(t')\|_2^{1-\frac{r}{s}} \left(2\|u\|_{L^\infty_t H^s_x}\right)^{r/s} \xrightarrow[|t-t'|\to 0]{} 0$$

we do Not yet know if u in $C_t H^s_x$.

Question: is u a solution to (KdV)?

(Duhamel for $\partial_t u^\varepsilon = -\varepsilon\partial_x^4 u^\varepsilon - \partial_x^3 u^\varepsilon + 3\partial_x(u^\varepsilon)^2$)

Recall $u^\varepsilon(t) = e^{-t(\varepsilon\partial_x^4 + \partial_x^3)} u_0 + 3\int_0^t e^{-(t-\tau)(\varepsilon\partial_x^4+\partial_x^3)} (u^2)'(\tau)\,d\tau$.

Know: $u^\varepsilon(t)$ converges in $C_t H^r_x$ $\forall 0 \le r < s$ as $\varepsilon \searrow 0$ to $u(t)$

$\varepsilon \searrow 0$ ↓ does it converge, in which regularity ??

converges in $C_t H^s$

$e^{-t\partial_x^3} u_0 + \dots$ — (Duhamel form for KdV)

Recall: $\|(e^{-\varepsilon t(\partial_x^4+\partial_x^3)} - e^{-t\partial_x^3})u_0\| \le \|(e^{-\varepsilon t\xi^4} - 1)\hat{u_0}\langle\xi\rangle^s\|_2$

↓ DCT, $\varepsilon \searrow 0$

0 — uniformly in $t \in [0,T_0]$

and similar arguments (lose a derivative though)

get convergence to $3\int_0^t e^{-(t-\tau)\partial_x^3}\partial_x(u^2)(\tau)\,d\tau$ in $C_t H_x^{s-1}$

above Duhamel DCT stuff, the integrals and series should be interpreted in C_t.

old info we already know

So u satisfies Duhamel formula in $(C_t L_x^2 \cap L_t^\infty H_x^3)([0,T_0]\times\mathbb{R})$

perhaps it should say $u \in (C_t L_x^2 \cap L_t^\infty H_x^3)([0,T_0]\times\mathbb{R})$ satisfies the Duhamel formula

---

Uniqueness of sols in $C_t L_x^2 \cap L_t^\infty H_x^3$ to KdV:

Suffices to prove uniqueness in weaker topologies, b/c once agree, intervals equal

Let $u, v \in C_t L_x^2 \cap L_t^\infty H_x^3([0,T_0]\times\mathbb{R})$

$$\partial_t \|(u-v)(t)\|_2^2 = 2\langle u-v, -\partial_x^3(u-v)\rangle + 3\langle u-v, (u^2-v^2)'\rangle$$

$\langle u-v, -\partial_x^3(u-v)\rangle = 0$

similar to how we dealt w/ it before

$$= 0 + \frac{3}{2}\langle u-v, (u-v)(u+v)'\rangle$$

$$\lesssim \|u-v\|_2^2 \left(\|u'\|_2 + \|v'\|_2\right)$$

So by Grönwall, $\|(u-v)(t)\|_2^2 \le \|(u-v)(0)\|_2^2 e^{C T_0 (\|u\|_{L_t^\infty H_x^3} + \|v\|_{L_t^\infty H_x^3})} = 0$

Issue: • above only know u, v satisfy Duhamel integral formulas, not derivative formulas

"the above derivations were fake"

(not "fatal error")

Fix:

Claim: $t \longmapsto e^{t\partial_x^3}u(t) = u_0 + 3\int_0^t e^{\tau\partial_x^3}(u^2)'(\tau)\,d\tau$ is $C_t' L_x^2$

and its derivative is exactly what we thought it was $3e^{t\partial_x^3}(u^2)'(t)$.

Assuming this claim,

stop a $e^{t\partial_x^3}$ time so now has "real" derivative,

turn remove the $e^{t\partial_x^3}$ cancel on Fourier side

then once we exploited it for its real derivative.

$$\partial_t \|u(t)\|_2^2 = \partial_t \|e^{t\partial_x^3}u(t)\|_2^2 = 2\langle e^{t\partial_x^3}u(t), 3e^{t\partial_x^3}(u^2)'(t)\rangle$$

$$= 4\int (u^3(t))'\,dx = 0$$

(This was how we proved conservation of mass for NLS!!!!)

Pf of Claim:

$$\left\| \frac{1}{t-t'}\int_{t'}^t e^{\tau\partial_x^3}(u^2)'(\tau)\,d\tau - e^{t\partial_x^3}(u^2)'(t)\right\|_2$$

$$\le \frac{1}{|t-t'|}\int_{t'}^t \|e^{\tau\partial_x^3}(u^2)'(\tau) - e^{t\partial_x^3}(u^2)'(t)\|_2\,d\tau.$$

need t, t' from integral bdd, so pw integrand in L_τ^∞.

$$\le \|(e^{\tau\partial_x^3} - e^{t\partial_x^3})(u^2)'(t)\|_{L_\tau^\infty L_x^2([t,t'])}$$

$$+ \| e^{t\partial_x^3}(u^2(\tau)-u^2(t))' \|_{L_t^\infty L_x^2[t',t]}$$

$$\downarrow \; |t-t'| \to 0$$

$$0$$

Left to show: $u \in C_t H_x^s([0,T_0]\times\mathbb{R})$, and ctd dependence of data $\mapsto$ solution map in the H^s topology.

Fix init data $u_0 \in H^s$. For dyadic freqs $N \in 2^{\mathbb{N}}$

let $u_0^N = P_{\le N} u_0$

"smooth" init data by taking low freqs.

"previous issue was init data wasn't regular enough, e.g. if $u_0 \in H^{100}$, above arguments are all good"

Note: $\|u_0^N\|_{H^s} \lesssim \|u_0\|_{H^s}$ unif in N.

$$\|u_0^N - u_0\|_{H^s} = \| P_{>N} u_0 \|_{H^s} \xrightarrow[N\to\infty]{\text{DCT}} 0$$

Let $u^N \in (C_t L_x^2 \cap L_t^\infty H_x^s)[0,T_0]\times\mathbb{R}$ be the (unique) solution to (KdV) w/ data $u^N(0) := u_0^N$.

Note: T_0 uniform in N (b/c T_0 only depend on $\|u_0\|_{H^s}$, and $\|u_0^N\|_{H^s} \lesssim \|u_0\|_{H^s}$ unif in N)

$$\|u^N(t)\|_{H^s} \le 2\|u_0^N\|_{H^s} \lesssim \|u_0\|_{H^s} \quad (1)$$

Grönwall again

$$\|u^N(t)\|_{H^{s+1}}^2 \le \|u_0^N\|_{H^{s+1}}^2 e^{CT_0\|u_0\|_{H^s}} \lesssim \|u_0\|_{H^2,\, T_0}\left(N\|u_0\|_{H^s}\right)^2$$

So for fixed N, $u^N \in C_t L_x^2 \cap L_t^\infty H_x^{s+1}([0,T_0]\times\mathbb{R}) \Rightarrow u^N \in C_b H^s([0,T_0]\times\mathbb{R})$ (interpolation from before)

If we can show $u^N \xrightarrow[N\to\infty]{L_t^\infty H_x^s} u$, then unif limit of cts fns is cts $\Rightarrow u \in C_t H_x^s$.

First step: prove convergence in weaker topology, e.g. L^2_x, so we identify a limit fn at least.

must use "Fix" above, then same as above

$$\partial_t \|(u^N - u)(t)\|_2^2 = 3\langle u^N - u, [(u^N - u)(u^N + u)]' \rangle$$

$$\lesssim \|u^N - u\|_2^2 \left[\|u^N\|_{H^2} + \|u\|_{H^2}\right] \lesssim_{\|u_0\|_{H^3}, T_0} \|u^N - u\|_2^2$$

Gronwall exponential absorbed here

So by Grönwall, $\|u^N(t) - u(t)\|_2 \lesssim_{\|u_0\|_{H^3}, T_0} \|u_0^N - u_0\|_2 \lesssim \|P_{>N} u_0\|_2 \lesssim N^{-3} \|u_0\|_{H^3}$

$\downarrow$ $N \to \infty$

0 uniform in $t \in [0, T_0)$

So, $u^N \xrightarrow[N \to \infty]{C_t L^2_x} u$.

The only thing left to show: $\{u^N\}_N$ Cauchy in $L^\infty_t H^3_x$

4/17/24 Last time, concluded:

Given $s \in \mathbb{N}$, $s \geq 2$ (regularity), $u_0 \in H^s(\mathbb{R})$ (init data), $\exists!$ sol. $u \in (C_t L^2_x \cap L^\infty_t H^3_x)([0,T_0] \times \mathbb{R})$

to (KdV) $\begin{cases} u_t + u_{xxx} = 6uu_x \\ u(0) = u_0 \in H^s(\mathbb{R}) \end{cases}$ where $T_0 = T_0(\|u_0\|_{H^s})$.

Moreover, $\|u(t)\|_{H^s} \leq 2\|u(0)\|_{H^s}$ $\forall t \in [0,T_0]$ "doesn't more than double"

Remains to show $u \in C_t H^s_x([0,T_0] \times \mathbb{R})$ and, the map $u_0 \mapsto u$ is cts. $H^s \to C_t H^s$

For $N \in 2^{\mathbb{N}}$, we defined $u_0^N := P_{\leq N} u_0$

Then $u_0^N \xrightarrow[n\to\infty]{H^s} u_0$ by DCT, and moreover, $\sup_N \|u_0^N\|_{H^s} \leq \|u_0\|_{H^s}$ (easily seen on Fourier side)

Thus, bc unif in N bd, can choose uniform time interval on which solutions w/ init data u_0 $\exists$:

Let $u^N \in (C_t L^2_x \cap L^\infty_t H^3_x)([0,T_0] \times \mathbb{R})$ be sols to KdV w/ $u^N(0) = u_0^N$ obtained by our construction

($[0,T_0]$ indep. of N)

$\|u^N(t)\|_{H^s} \leq 2\|u_0^N\|_{H^s} \leq 2\|u_0\|_{H^s}$ $\forall N \in 2^{\mathbb{N}}$, $\forall t \in [0,T_0]$.

By Grönwall arg, $\|u^N(t)\|_{H^{s+1}} \leq \|u_0^N\|_{H^{s+1}} e^{CT_0\|u_0\|_{H^2}} \lesssim N$ (from now on, all implicit constants may depend on $\|u_0\|_{H^s}$ which is fixed.)

"more regular": $u^N \in C_t L^2_x \cap L^\infty_t H^{s+1}_x \Rightarrow u^N \in C_t H^s_x$ (interpolation from beginning of last lec)

Thus, if we show that $u^N \xrightarrow[N\to\infty]{L^\infty_t H^s_x} u$, then $u \in C_t H^s_x$ (unif limit of cts fns is cts)

Recall: $\|u^N(t) - u(t)\|_2 \lesssim \|u_0^N - u_0\|_2 \lesssim \|P_{>N} u_0\|_2 \lesssim N^{-s}\|u_0\|_{H^s} \xrightarrow[n\to\infty]{} 0$ $\forall t \in [0,T_0]$

so we have convergence in weaker topology $u^N \xrightarrow[N\to\infty]{C_t L^2_x} u$. (1)

Similarly, $\|u^N(t) - u^M(t)\|_2 \lesssim \|P_{M<\cdot\leq N} u_0\|_2$ $\forall t \in [0,T_0]$

It's enough to show that $\{u^N\}_N$ is Cauchy in $L^\infty_t H^s_x$ (will converge in that topology bc something, that something must agree w/ (1) i.e. converge to u)

For simplicity, will show $\{u^N\}_N$ is Cauchy in $L^\infty_t H^2_x$. (s=2)

For $M<N$,

$$\partial_t \| \partial_x^2 (u^N-u^M)(t)\|_{L^2}^2 = 2\langle (u^N-u^M)'', -\partial_x^3 (u^N-u^M)''\rangle + 6\langle (u^N-u^M)'', [(u^N-u^M)(u^N+u^M)]'''\rangle$$

(∂x = spatial derivative; antiself adjoint → 0; using that u^N, u^M are both solutions to KdV; $uu_x = \frac{(u^2)'}{2}$; $(u^2)''' = (u^N)^2 - (u^M)^2$ difference of squares)

$$= 6\langle (u^N-u^M)'', (u^N-u^M)''' (u^N+u^M)\rangle = \text{derivative}\left(((u^N-u^M)'')^2\right)\cdots$$

(2 derivatives grouped w 3 derivative)

IBP ... ?

$$= \langle ((u^N-u^M)'')^2, (u^N+u^M)'\rangle$$

(L^2, L^∞ controlled by H^1)

product rule

$$+18\langle (u^N-u^M)'', (u^N-u^M)''(u^N+u^M)'\rangle$$

$$+18\langle (u^N-u^M)'', (u^N-u^M)'(u^N-u^M+2u^M)''\rangle$$

(write sum as difference plus lower-reg parts; $\langle (u^N-u^M)'', (u^N-u^M)'(u^N-u^M)'\rangle$ (L^2, L^2, L^∞ controlled by H^1); 3 derivatives fall on u^N-u^M, deal w/ like)

$$+6\langle (u^N-u^M)'', (u^N-u^M)(u^N-u^M+2u^M)'''\rangle$$

$$\lesssim \|(u^N-u^M)''\|_{L^2}^2 \left[\|u^N\|_{H^2} + \|u^M\|_{H^2}\right]$$

$$+ \|(u^N-u^M)''\|_{L^2}\, \|(u^M)''\|_{L^2}\, \|(u^N-u^M)'\|_{L^\infty}$$ ← Hölder

$$+ \|(u^N-u^M)''\|_{L^2}\, \|(u^M)'''\|_{L^2}\, \|u^N-u^M\|_{L^\infty}$$ ← Hölder

We use the following Gagliardo-Nirenberg type ineqs:

$$\|f\|_\infty \lesssim \|f\|_{L^2}^{\frac34}\, \|f''\|_{L^2}^{\frac14}$$ ← figure out its L^2_x by scaling argument: $f(\lambda x)$, LHS unchanged; $f(\lambda x)'' = \lambda^2 f''(\lambda x)$; $\int(\lambda^2 f''(\lambda x))^2 = \int \lambda^3 f''(u)^2\,du$ ($u=\lambda x$) $\rightsquigarrow \lambda^{3/2}$; $\int f(\lambda x)^2 \overset{u=\lambda x}{=} \int \lambda^{-1} f(u)^2 \rightsquigarrow \lambda^{-1/2}$ ← must be weighted 3:1 so $\frac34, \frac14$.

$$\|f'\|_\infty \lesssim \|f\|_{L^2}^{\frac14}\, \|f''\|_{L^2}^{\frac34}$$

$\frac{2}{p} - \frac{1}{2}a = 1$, $a+b=1$

in L^p, λ always start w $-1/p$ + 1 for every derivative

Pf: $f = \sum_N f_N$ (sum converges since $f \in L^2$), take L^∞, take L^1, take sups.

$$\|f\|_\infty \le \sum \|f_N\|_\infty \lesssim \sum \min\left\{N^{\frac12}\|f\|_2,\ N^{\frac12-2}\|f''\|_2\right\}$$ optimize and sum.

Bernstein: $P_{\le N} f \in L^q$, $q \le p$, $\|P_{\le N} f\|_p \lesssim N^{d(\frac1q - \frac1p)}\|P_{\le N} f\|_q$

$$\|f'\|_\infty \lesssim \sum N\|f_N\|_\infty \lesssim \sum \min\left\{N^{\frac32}\|f\|_2,\ N^{-\frac12}\|f''\|_2\right\}$$ · — —

$\|(u^N-u^M)''\|_{L^2} \le M\|u_0\|_{H^2}$

So,

$$\partial_t \|(u^N-u^M)''\|_{L^2}^2 \lesssim \|(u^N-u^M)''\|_{L^2} + \|(u^N-u^M)''\|_{L^2}^{1+\frac34}\, \|u^N-u^M\|_{L^2}^{\frac14} + M\,\|(u^N-u^M)''\|_{L^2}^{1+\frac14}\, \|u^N-u^M\|_{L^2}^{3/4}$$

other factors $\lesssim \|u^N(0)\|_{H^2}$ uniform in N $\lesssim \|u(0)\|_{H^2}$

"A"

$\lesssim \|P_{M<\cdot\le N} u_0\|_{L^2}^{1/4} \lesssim M^{-1/2}\|u_0\|_{H^2}$

"B"

$\lesssim \|P_{M<\cdot\le N} u_0\|_{L^2}^{3/4} \lesssim M^{-3/2}$

Young ineq, A raised to $\frac87$, so $\frac74\cdot\frac87 = 2$; B raised to dual of $\frac87$, means $\frac81$.

again Young ineq, "A" raise to $\frac85$, "B" raise to $\frac83$.

Get $\partial_t \|(u^N-u^M)''\|_{L^2}^2 \lesssim \|(u^N-u^M)''\|_{L^2}^2 + M^{-\frac43}$

$\Rightarrow$ Gronwall

$$\|(u^N - u^M)''\|_{L^2_x}^2 \lesssim \|(u_0^N - u_0^M)''\|_{L^2}^2 + C T_0 M^{-\frac{4}{3}}$$

$$\lesssim \|P_{>M} u_0\|_{H^2}^2 + C T_0 M^{-\frac{4}{3}}$$

$\downarrow$ DCT, $M \to \infty$

0

unif in $t \in [0, T_0]$

only "win" by DCT, not quantitatively; i.e. rate of this convergence depend on u_0, not on anything we can control.

Finally, let's show cty of data $\mapsto$ sol map in H^s.

Let $u_{0,n} \xrightarrow[n\to\infty]{H^s} u_0$.

Let $u_n \in C_t H^s_x([0,T_0] \times \mathbb{R})$ be unique sols to (KdV) w/ init data $u_n(0) = u_{0,n}$

and $T_0 = T_0(2\|u_0\|_{H^s})$

eventually $\|u_{0,n}\|_{H^s} \le 2\|u_0\|_{H^s}$ for n large enough,

so time interval uniform in n.

$$\|u_n(t) - u(t)\|_{H^s} \le \underbrace{\|u_n^N(t) - u_n(t)\|_{H^s}}_{(I)} + \underbrace{\|u^N(t) - u(t)\|_{H^s}}_{(II)} + \underbrace{\|u_n^N(t) - u^N(t)\|_{H^s}}_{(III)}$$

(u_n^N: regularizations from before, sol to KdV w/ init data $u_{0,n}^N$)

$$(I) := \|u_n^N(t) - u_n(t)\|_{H^s} = \lim_{M\to\infty} \|u_n^N(t) - u_n^M(t)\|_{H^s} \lesssim \lim_{M\to\infty} \|u_n^N(0) - u^M(0)\|_{H^s} + N^{-c(s)}$$

(b/c $u_n(t) = H^s\text{-}\lim_{M\to\infty} u_n^M(t)$; $c(s)$: some negative exponent depending on s; smaller of 2 terms)

$$\lesssim \|P_{>N} u_{0,n}\|_{H^s} + N^{-c(s)}$$

$$\lesssim \|P_{>N} u_0\|_{H^s} + \|u_{0,n} - u_0\|_{H^s} + N^{-c(s)}$$

$\downarrow$ DCT $N \to \infty$

0

Similarly, (even easier!):

$$(II) := \| u^N(t) - u(t) \|_{H^s} = \lim_{M \to \infty} \| u^N(t) - u^M(t) \|_{H^s}$$

$\therefore$ above steps

$$\lesssim \| P_{>N} u_0 \|_{H^s} + N^{-(CS)}$$

$\downarrow$ DCT, $N \to \infty$

0

$$(III) := \| u_n^N(t) - u^N(t) \|_{H^s}$$

$$\frac{d}{dt} \| \partial_x^2 (u_n^N(t) - u^N(t)) \|_{H^s}^2 \lesssim \| \partial^s (u_n^N - u^N) \|_{L^2}^2 + \| \partial^s (u_n^N - u^N) \|_{L^2} \left[\| u_n^N \|_{H^{s+1}} + \| u^N \|_{H^{s+1}} \right] \| u_n^N - u^N \|_{\infty}$$

worst possible term

same long argument as above

$\lesssim N$ up to const of init data in H^s

$\lesssim \| u_n^N - u^N \|_{L^2}^{\frac{2s-1}{2s}} \cdot \| \partial^s (\cdots) \|_{L^2}^{\frac{1}{2s}}$

$\lesssim N \cdot \| P_{>N} (u_{0,n} - u_0) \|_{L^2}^{\frac{2s-1}{2}}$

$\lesssim N \cdot N^{-\frac{2s-1}{2}}$

$\| \partial^s (u_n^N - u^N) \|_{L^2}^{\frac{2s+1}{2s}}$

raise to $\frac{4s}{2s+1}$ — Young ineq. — raise to $\frac{4s}{2s-1}$

$$\lesssim N^{\frac{(3-2s)2s}{2s-1}}$$ ($s=2$, agree w/ our $N^{-\frac{4}{3}}$ above!)

$$\lesssim \| \partial^s (u_n^N - u^N) \|_{L^2}^2 + N^{\frac{(3-2s)2s}{2s-1}}$$

$\lesssim$ Gronwall $\Rightarrow$

$$(III) := \| u_n^N(t) - u^N(t) \|_{H^s} \lesssim \underbrace{\| P_{\leq N} (u_{0,n} - u_0) \|_{H^s}}_{\lesssim \| u_{0,n} - u_0 \|_H} + C T_0 N^{\frac{(3-2s)2s}{2s-1}} \xrightarrow[\text{then } n \to \infty]{N \to \infty} 0.$$

LWP for KdV: The Oscillatory Integral Method

Consider $\begin{cases} u_t + u_{xxx} = 6uu_x \\ u(0) = u_0 \in H^s(\mathbb{R}), \end{cases}$ $\frac{3}{4} < s \le 1$ (suffices to treat fractional part b/c it >1, use ordinary product rule, reduce to studying fractional)

Let $\Phi(u)(t) := e^{-t\partial_x^3} u_0 + 3\int_0^t e^{-(t-\tau)\partial_x^3} (u^2)'(\tau)\, d\tau$.

will show Φ is contraction on Banach space

$$B := \left\{ u \in C_t H^s_x([-T_0, T_0] \times \mathbb{R}) \ : \ \|u\|_{X([-T_0,T_0])} \le C_0 \|u_0\|_{H^s} \right\}$$

universal / absolute constant

equipped w/ metric $d(u,v) := \|u - v\|_{X([-T_0,T_0])}$

provided $T_0 = T_0(\|u_0\|_{H^s}) \le 1$

i.e. take min of what contraction mapping require, & 1).

C_0 is absolute const,

and

$$\|u\|_{X([-T_0,T_0])} := \|u\|_{L^\infty_t H^s_x} + \||\nabla| u\|_{L^4_t L^\infty_x} + \|u\|_{L^2_x L^\infty_t} + \|\langle \nabla \rangle^s \partial_x u\|_{L^\infty_x L^2_t}$$

"the very least needed" — recall Strichartz has $\frac{1}{4}$ derivative, but here we have $s > \frac{3}{4}$. — "Kato smoothing norm"

"Strichartz norm"

here and below, all spacetime norms are on $[-T_0, T_0] \times \mathbb{R}$.

Show $\Phi : B \to B$ Fix $u \in B$.

started linear part. $\|e^{-t\partial_x^3} u\|_{L^\infty_t H^s_x} = \|u_0\|_{H^s_x}$. spatial interest first, at before "conservation law" $e^{-t\partial_x^3}$ disappears

Strichartz norm $\||\nabla| e^{-t\partial_x^3} u_0\|_{L^4_t L^\infty_x} = \||\nabla|^{\frac{1}{4}} e^{-t\partial_x^3} |\nabla|^{\frac{3}{4}} u_0\|_{L^4_t L^\infty_x}$

Fourier multipliers commutative

$\lesssim_{\text{Strichartz}} \||\nabla|^{\frac{3}{4}} u_0\|_2 \lesssim \|u_0\|_{H^s_x}$ if $s \ge \frac{3}{4}$

$\|e^{-t\partial_x^3} u_0\|_{L^2_x L^\infty_t} \lesssim_{\text{maximal fn inequality}} \langle T_0 \rangle^{1/2} \|u_0\|_{H^s_x} \le \|u_0\|_{H^s_x}$ for $s > \frac{3}{4}$.

assumed $T_0 \le 1$

"Kato smoothing" $\| \langle \nabla \rangle^s \partial_x e^{-t\partial_x^3} u_0 \|_{H^s} \lesssim_{\text{Kato smoothing}} \| \langle \nabla \rangle^s u_0 \|_2 \lesssim \|u_0\|_{H^s}$.

So outcome is free evolution $\| e^{-t\partial_x^3} u_0 \|_{X([-T_0,T_0])} \leq C \|u_0\|_{H^s_x}$.

Turn to nonlinear term.

$$\| \int_0^t e^{-(t-\tau)\partial_x^3} (u^2)'(\tau)\, d\tau \|_{L_t^\infty H_x^s} \lesssim_{\substack{\Delta\text{-ineq} \\ \text{to move} \\ H^s \text{ norm} \\ \text{inside}}} \| \int_0^t \| e^{-(t-\tau)\partial_x^3} (u^2)'(\tau) \|_{H_x^s} d\tau \|_{L_t^\infty}$$

(in H^s, unimodular constant)

$$\lesssim \int_{-T_0}^{T_0} \| (u^2)'(\tau) \|_{H_x^s} d\tau$$

$$\lesssim_{(2,2)\text{Hölder in}\tau} T_0^{1/2} \| (u^2)' \|_{L_t^2 H_x^s}$$

$$\| \int_0^t e^{-(t-\tau)\partial_x^3} |\nabla| (u^2)'(\tau)\, d\tau \|_{L_t^4 L_x^\infty}$$

using stuchwartz, $\lesssim \| |\nabla|^{\frac{1}{4}} (uu') \|_{L_t^{4/3} L_x^1}$

Unfortunately, not make us happy b/c Hölder would have to put u or u' in L^2, and $\frac{1}{2}+1$ derivatives in L^2 is too many for us to handle.

Instead,

By duality,

$$\| \int_0^t e^{-(t-\tau)\partial_x} |\nabla| (u^2)'(\tau) d\tau \|_{L_t^4 L_x^\infty} = \sup_{\|G\|_{L_t^{4/3} L_x^1} \leq 1} \left\langle \int_0^t e^{-(t-\tau)\partial_x^3} |\nabla| (u^2)'(\tau)\, d\tau,\ G(t) \right\rangle_{t,x}$$

$$= \sup_G \iint_{0<\tau<t} \langle e^{\tau\partial_x^3} |\nabla|^{\frac{3}{4}} (u^2)'(\tau),\ e^{t\partial_x^3} |\nabla|^{\frac{1}{4}} G(t) \rangle_x \, dt\, d\tau$$

$$= \sup_G \left\langle e^{\tau\partial_x^2} |\nabla|^{\frac{3}{4}} (u^2)'(\tau),\ \int e^{t\partial_x^3} |\nabla|^{\frac{1}{4}} \chi_{[\tau,\infty)}(t)\, G(t)\, dt \right\rangle_{\tau,x}$$

L^2 ; $(2,2)$ Hölder ; L^2

$|\nabla|^{3/4}$ controlled by H^s, $s \geq \frac{3}{4}$. and in H^s $e^{\tau\partial_x^3}$ goes away

adjoint $\| |\nabla|^{\frac{1}{4}} e^{-t\partial_x^3} f \|_{L_t^4 L_x^\infty} \lesssim \|f\|_2$

$\Leftrightarrow$

$\| \int e^{t\partial_x^3} |\nabla|^{\frac{1}{4}} F(t)\, dt \|_2 \lesssim \|F\|_{L_t^{4/3} L_x^1}$

$$\leq \sup_\tau \left\| \|(u^2)'(\tau)\|_{H^3_x} \cdot \left\| \chi_{[\tau,\infty)} G \right\|_{L^{3/4}_t L^1_x} \right\|_{L^1_\tau}$$

$$\lesssim \|(u^2)'\|_{L^1_t H^3_x}$$

with $\|G\|_{L^{4/3}_t L^1_x} \leq 1$

$$\lesssim T_0^{1/2} \|(u^2)'\|_{L^2_t H^3_x}$$

CBS

$$\left\| \int_0^t e^{-(t-\tau)\partial_x^3} (u^2)'(\tau)\, d\tau \right\|_{L^2_x L^\infty_t} \leq \left\| \int_0^t \left| e^{-(t-\tau)\partial_x^3} (u^2)'(\tau) \right| d\tau \right\|_{L^2_x L^\infty_t}$$

$$\leq \left\| \int_{-T_0}^{T_0} \left| e^{-(t-\tau)\partial_x^3} (u^2)'(\tau) \right| d\tau \right\|_{L^2_x L^\infty_t}$$

$$\leq \int_{-T_0}^{T_0} \underbrace{\left\| e^{-(t-\tau)\partial_x^3} (u^2)'(\tau) \right\|_{L^2_x L^\infty_t}}_{\text{maximal fn inequ to free evolution}} d\tau \lesssim \left\| e^{\tau\partial_x^3} (u^2)'(\tau) \right\|_{H^3_x} \underbrace{\langle T_0 \rangle^{1/2}}_{\leq 1}$$

(again invisible to H^3_x)

$$\lesssim T_0^{1/2} \|(u^2)'\|_{L^2_t H^3_x}.$$

(again CBS in time)

Last term.

Similarly to

$$\left\| \int_0^t e^{-(t-\tau)\partial_x^3} \partial_x \langle\nabla\rangle^3 (u^2)'(\tau)\, d\tau \right\|_{L^\infty_x L^2_t} \leq \int_{-T_0}^{T_0} \left\| e^{-(t-\tau)\partial_x^3} \partial_x \langle\nabla\rangle^3 (u^2)'(\tau) \right\|_{L^\infty_x L^2_t} d\tau$$

(∂_x Riesz smoothing.)

$$\lesssim \int_{-T_0}^{T_0} \left\| e^{\tau\partial_x^3} \langle\nabla\rangle^3 (u^2)'(\tau) \right\|_2 d\tau$$

$$\lesssim T_0^{1/2} \|(u^2)'\|_{L^2_t H^3_x}$$

So we have shown

$$3 \left\| \int_0^t e^{-(t-\tau)\partial_x^3} (u^2)'(\tau)\, d\tau \right\|_{X([-T_0,T_0])} \leq C T_0^{1/2} \|uu'\|_{L^2_t H^3_x}$$

If we can show $\|uu'\|_{L^2_t H^3_x} \leq C_0^2 \|u_0\|^2_{H^3_x}$,

then we accomplish $\|\Phi(u)\|_{X([-T_0,T_0])} \lesssim C\|u_0\|_{H^3} + C T_0^{1/2} \|u_0\|^2_{H^3}$

$C_0 = 2C$, T_0 absorbs everything else?

$\leq C_0 \|u_0\|_{H^3}$ provided $C_0 T_0^{1/2} \|u_0\|_{H^3} \leq 1$ and $T_0 \leq 1$.

recall $u \in B$, Strichartz norm can control 1 derivative

$$\|uu'\|_{L^2_{t,x}} \lesssim \|u'\|_{L^4_t L^\infty_x} \cdot \|u\|_{L^\infty_t L^2_x} \cdot T_0^{1/4} \text{ (Hölder in the remaining time)} \lesssim \|u\|^2_{X([-T_0,T_0])} \lesssim \|u_0\|^2_{H^3}$$

$T_0 \leq 1$?

must exhaust all of space available since only control u in L^2

$\|$ s derivative of $uu'\|_{L^2_{t,x}}$

If $s=1$,

$$\|(uu')'\|_{L^2_{t,x}} \le \|uu''\|_{L^2_{t,x}} + \|u'\cdot u'\|_{L^2_{t,x}}$$

on below

$$\lesssim \|u''\|_{L^\infty_x L^2_t}\,\|u\|_{L^2_x L^\infty_t} + \|u'\|_{L^q_t L^\infty_x}\,\|u'\|_{L^\infty_t L^2_x}\,T_0^{1/q}$$

"maximal fn estimate in space"

Sobolev

(?)

$$\lesssim \|u\|^2_{X([-T_0,T_0])}$$

For $\frac{3}{4}<s<1$, we'll show

"pretend that s derivative either hit f, or g"

bound as if derivatives fell on strangers.

$$\||\nabla|^s(fg) - f|\nabla|^s g\|_2 \lesssim \||\nabla|^s f\|_2\,\|g\|_\infty$$

"peel away most problematic term"

"Monica believes the paper in which this is isolated, actually did it wrong"

Choosing $f:=u$, $g:=u'$, get

"take nasty term away" ... "put it back"

kato smoothing kills 1 than derivative

$$\||\nabla|^s(uu')\|_{L^2_{t,x}} \le \underbrace{\||\nabla|^s(uu') - u|\nabla|^s u'\|_{L^2_{t,x}}}_{\lesssim \|\,|\nabla|^s u\|_{L^\infty_t L^2_x}\,\|u'\|_{L^q_t L^\infty_x}\,T_0^{1/q}} + \underbrace{\|u\,|\nabla|^s u'\|_{L^2_{t,x}}}_{\lesssim \||\nabla|^s u'\|_{L^\infty_t L^2_x}\cdot\|u\|_{L^2_t L^\infty_x}}$$

again "maximal in space"

$$\lesssim \|u\|^2_{X([-T_0,T_0])}$$

"all things balanced"

To show $\Phi(u)\in B$, it remains to show ctg in time.

For $-T_0 < t' < t < T_0$,

$$\|\Phi(u)(t) - \Phi(u)(t')\|_{H^s_x} \le \underbrace{\|\,[e^{-(t-t')\partial_x^3} - 1]\,u_0\|_{H^s_x}}_{\text{Fourier side}} \xrightarrow[|t-t'|\to 0]{\text{DCT}} 0$$

linear terms

$$+ 3\,\Big\|\int_0^t [e^{-(t-t')\partial_x^3} - 1]\,e^{\tau\partial_x^3}(u^2)'(\tau)\,d\tau\Big\|_{H^s_x} \lesssim T_0^{1/2}\,\|(u^2)'\|_{L^2_t H^s_x} \xrightarrow[|t-t'|\to 0]{\text{DCT}} 0.$$

move abs val inside, move to $\int_{-T_0}^{T_0}$

$$+ 3\,\Big\|\int_{t'}^t e^{-(t-\tau)\partial_x^3}(u^2)'(\tau)\,d\tau\Big\|_{H^s_x} \lesssim |t-t'|\,\|(u^2)'\|_{L^2_t H^s_x} \xrightarrow[|t-t'|\to 0]{} 0.$$

use smooth cuts of intervals, i.e. DCT on $1_{[t',t]}$

4/22/24

WTS $\Phi(u)(t) = e^{t\partial_x^3} u_0 + 3\int_0^t e^{-(t-\tau)\partial_x^3} (u^2)'(\tau)\, d\tau$

is contraction on Banach space

$$B = \{ u \in C_t H^3_x : \|u\|_{X([-T_0,T_0])} \le C_0 \|u_0\|_{H^s} \}$$

where

$$\|u\|_{X([-T_0,T_0])} = \|u\|_{L^\infty H^s} + \||\nabla| u\|_{L^4_t L^\infty_x} + \|u\|_{L^3_x L^\infty_t} + \||\nabla|^s \partial_x u\|_{L^\infty_x L^2_t}$$

all spacetime norms on $[-T_0,T_0]\times\mathbb{R}$

and $\frac{3}{4} < s \le 1$, $T_0 = T_0(\|u_0\|_{H^s}) \le 1$.

Long time reshaved

$$\|\Phi(u)\|_{X([-T_0,T_0])} \lesssim \|u_0\|_{H^s} + \underbrace{T_0^{1/2}}_{\text{source of smallness}} \|uu'\|_{L^2_t H^s_x}$$

will show "can pretend all derivatives either hit f or g"

$|\nabla|^s(fg)$

$$\||\nabla|^s(fg) - f|\nabla|^s g\|_2 \lesssim \||\nabla|^s f\|_2 \|g\|_\infty.$$

using this, $\|uu'\|_{L^2_t H^s_x} \lesssim \|uu'\|_{L^2_{t,x}} + \underbrace{\||\nabla|^s(uu') - u|\nabla|^s u'\|_{L^2_{t,x}}}_{\text{worst parts of derivatives}} + \underbrace{\|u|\nabla|^s u'\|_{L^2_{t,x}}}_{\text{only tool to eat derivatives is Kato smoothing but loses in t switches time $\leftrightarrow$ space}}$

$$\lesssim \|u'\|_{L^4_t L^\infty_x} \|u\|_{L^\infty_t L^2_x} \cdot T_0^{1/4} + \underbrace{\||\nabla|^s u\|_{L^2_{t,x}}}_{\text{pretend } |\nabla|^s \text{ hit } u \text{ instead}} \cdot \|u'\|_{L^4_t L^\infty_x} \cdot T_0^{1/4}$$

$$+ \|u\|_{L^2_x L^\infty_t} \||\nabla|^s u\|_{L^\infty_x L^2_t}$$

$$\lesssim_{(\text{using } T_0 \le 1)} \|u\|^2_{X([-T_0,T_0])} \lesssim_{(u\in B)} \|u_0\|^2_{H^s}$$

So, taking $T_0 \lesssim \min\{1, \|u_0\|_{H^s}^{-2}\}$ and $C_0 \gg 1$, we get $\|\Phi(u)\|_X \le C_0 \|u_0\|_{H^s}$.

Long-time also showed cty in time, so indeed Φ maps B to B.

Φ contraction

for $u, v \in B$, may bound

$$\|\Phi(u) - \Phi(v)\|_X \lesssim \|(u^2 - v^2)'\|_{L^2_t H^s_x} \cdot \underbrace{T_0^{1/2}}_{\text{source of smallness}}$$

fractional product rule

$$\||\nabla|^s(fg)\|_2 \lesssim \||\nabla|^s f\|_2 \|g\|_\infty + \|f\|_\infty \||\nabla|^s g\|_2$$

important that $L^2_{t,x}$ same exponent so can switch order of integration (time $\leftrightarrow$ space) so fract product rule not good enough ☹.

"either no derivatives, or s derivatives" → in which case can pull out in front ∂_x fall on one or other

$$\|\nabla[(u-v)(u+v)]'\|_{L_t^2 H_x^s} \lesssim \underbrace{\|(u-v)'(u+v)\|_{L_{t,x}^2}}_{(I)} + \underbrace{\|(u-v)(u+v)'\|_{L_{t,x}^2}}_{(II)} + \underbrace{\|\nabla^s((u-v)'(u+v)) - (u+v)\nabla^s(u-v)'\|_{L_{t,x}^2}}_{(III)}$$

$$+ \underbrace{\|(u+v)\nabla^s(u-v)'\|_{L_{t,x}^2}}_{(IV)} + \underbrace{\|\nabla^s((u-v)(u+v)') - \nabla^s(u+v)'\cdot(u-v)\|_{L_{t,x}^2}}_{(V)} + \underbrace{\|(u-v)\nabla^s(u+v)'\|_{L_{t,x}^2}}_{(VI)}$$

$$(I) \lesssim \|(u-v)'\|_{L_t^4 L_x^\infty} \|u+v\|_{L_t^\infty L_x^2} \quad T_0^{1/4}$$

$$(II) \lesssim \|(u+v)'\|_{L_t^4 L_x^\infty} \|u-v\|_{L_t^\infty L_x^2} \quad T_0^{1/4}$$

$$(III) \lesssim \|\nabla^s(u+v)\|_{L_t^\infty L_x^2} \cdot \|(u-v)'\|_{L_t^4 L_x^\infty} \quad T_0^{1/4}$$

Hölder-then L^2 into L^4 and L^4

$$(IV) \lesssim \|\nabla^s(u-v)'\|_{L_x^\infty L_t^2} \|u+v\|_{L_x^2 L_t^\infty}$$

switch order! to Kato smooth

$$(V) \lesssim \|\nabla^s(u-v)\|_{L_t^\infty L_x^2} \|(u+v)'\|_{L_t^4 L_x^\infty} \quad T_0^{1/4}$$

$$(VI) \lesssim \|\nabla^s(u+v)'\|_{L_x^\infty L_t^2} \|(u-v)\|_{L_x^2 L_t^\infty}$$

all of these $\lesssim \|u-v\|_{X([-T_1,T_0])} \left[\|u\|_X + \|v\|_X\right]$

$\lesssim \|u_0\|_{H^s} \|u-v\|_X$ $\quad (u,v \in B)$

Choosing T_0 small enough that $\|u_0\|_{H^s} T_0^{1/2} < \frac{1}{2}$

$T_0 \lesssim \min\{1, \|u_0\|_{H^s}^{-2}\}$ we get

$$\|\Phi(u) - \Phi(v)\|_X \leq \tfrac{1}{2}\|u-v\|_X.$$

Lemma: for $0<s<1$,

$$\|\nabla^s(fb) - f\nabla^s b\|_2 \lesssim \|\nabla^s f\|_2 \|b\|_\infty.$$

Pf: begin as we began for fractional product rule in 2970A.

Decompose $fb = \sum_N f_N b_{\sim N} + \sum_N f_{\ll N} b_N + \sum_N f_{\sim N} b_N$

so we need to understand/bound

$$|\nabla|^s(fb) - f|\nabla|^s b = \sum_N |\nabla|^s(f_N b_{\ll N}) - \sum_N f_N |\nabla|^s b_{\ll N} + \sum_N |\nabla|^s(f_{\ll N} b_N) - \sum_N f_{\ll N} |\nabla|^s b_N$$

$$+ \sum_N |\nabla|^s(f_{\sim N} b_N) - \sum_N f_{\sim N} |\nabla|^s b_N$$

only this term needs tools beyond 2014, invented for Carleson measures.

$$\Big\| \sum_N |\nabla|^s(f_N b_{\ll N}) \Big\|_2 \lesssim \Big\| \sqrt{\sum_N N^{2s} |\tilde{P}_N(f_N b_{\ll N})|^2} \Big\|_2$$

sq fn estimate, $N + (\ll N)$ is like tailored $\tilde{P}_N$

$|f_N| \lesssim [Mf]$ — f convolve at rapidly decayed fn, easily proven to be controlled by MO'd fn.

$$\lesssim \Big\| \sqrt{\sum_N N^{2s} |M[f_N \cdot Mb]|^2} \Big\|_2$$

$|M[f_N \cdot Mb]| = |M(N^s f_N \cdot Mb)|$ — vector valued maximal fn!

$$\lesssim \Big\| \sqrt{\sum_N N^{2s} |f_N|^2 |Mb|^2} \Big\|_2$$

$$\lesssim \underbrace{\|Mb\|_\infty}_{\lesssim \|b\|_\infty} \Big\| \underbrace{\sqrt{\sum_N N^{2s} |f_N|^2}}_{\sim \||\nabla|^s f\|_2 \text{ sq fn}} \Big\|_2 \lesssim \|b\|_\infty \||\nabla|^s f\|_2.$$

$$\Big\| \sum_N f_N |\nabla|^s b_{\ll N} \Big\|_2^2 \lesssim \Big\| \sqrt{\sum_N \tilde{P}_N[f_N |\nabla|^s b_{\ll N}]^2} \Big\|_2^2$$

sq fn estimate

$\tilde{P}_N$ bdd by M, as before

$$\lesssim \Big\| \sqrt{\sum_N |M[f_N |\nabla|^s b_{\ll N}|^2} \Big\|_2^2$$

$$\lesssim \Big\| \sqrt{\sum_N |f_N|^2 \, \||\nabla|^s b_{\ll N}\|^2} \Big\|_2^2 \sim \int \sum_N |f_N|^2 \, \||\nabla|^s b_{\ll N}\|^2 dx$$

$$\leq \sum_N \|f_N\|_2^2 \, \||\nabla|^s b_{\ll N}\|_\infty^2 \lesssim_{\text{Bernstein}} \sum_N N^{2s} \|f_N\|_2^2 \|b\|_\infty^2 \lesssim \|b\|_\infty^2 \||\nabla|^s f\|_2^2.$$

sq fn pull back together

$$\Big\| \sum_N |\nabla|^s(f_{\sim N} b_N) \Big\|_2^2 \lesssim_{\text{sq fn est.}} \Big\| \sqrt{\sum_K K^{2s} |P_K(\sum_N f_{\sim N} b_N)|^2} \Big\|^2$$

2 high freq can give low freq by destructive interference but 2 low can never get 2 high

$$\sim P_K\Big(\sum_{N \gtrsim K} f_{\sim N} b_N\Big)$$

P_K bdd by M,

$$\lesssim \Big\| \sqrt{\sum_K |M[K^s \sum_{N \gtrsim K} f_{\nu N} b_N]|^2} \Big\|_2^2$$

vector valued max inq

$$\lesssim \Big\| \sqrt{\sum_K K^{2s} \Big| \sum_{N \gtrsim K} f_{\nu N} b_N \Big|^2} \Big\|_2^2$$

$(\sum_{N_1})(\sum_{N_2})$, pay factor of 2, can assume $N_1 \geq N_2$.

$$\lesssim \int \sum_K K^{2s} \sum_{N_1 \geq N_2 \gtrsim K} |f_{\nu N_1}| \cdot |b_{N_1}| \cdot |f_{\nu N_2}| \cdot |b_{N_2}|$$

(sum in k, geom series, $s>0$, gets to biggest value N_2.)

$$\lesssim \sum_{N_1 \geq N_2} N_2^{2s} \|f_{\nu N_1}\|_2 \|f_{\nu N_2}\|_2 \|b_{N_1}\|_\infty \|b_{N_2}\|_\infty$$

L-P proj bdd in L^∞, $\lesssim \|b\|_\infty^2$

Bernstein:

$$\lesssim \|b\|_\infty^2 \sum_{N_1 \geq N_2} N_2^s \cdot \frac{1}{N_1^s} \| |\nabla|^s f_{\nu N_1}\|_2 \cdot \| |\nabla|^s f_{\nu N_2}\|_2$$

Schur test

$$\lesssim \|b\|_\infty^2 \Big(\sum_N \| |\nabla|^s f_{\nu N}\|_2^2 \Big)$$

sqfn est.

$$\lesssim \|b\|_\infty^2 \| |\nabla|^s f\|_2^2$$

For orange pieces, must exploit cancellation, i.e. can't handle each orange term separately.

$$\Big\| \sum_N |\nabla|^s (f_{\ll N} b_N) - \sum_N f_{\ll N} |\nabla|^s b_N \Big\|_2 \lesssim \Big\| \sqrt{\sum_N |\tilde{P}_N [|\nabla|^s (f_{\ll N} b_N) - f_{\ll N} |\nabla|^s b_N]|^2} \Big\|_2$$

$\tilde{P}_N$ bdd by M, M is bdd

$$\lesssim \Big\| \sqrt{\sum_N |\,|\nabla|^s (f_{\ll N} b_N) - f_{\ll N} |\nabla|^s b_N|^2} \Big\|_2$$

So want to understand:

$$|\nabla|^s (f_{\ll N} b_N) - f_{\ll N} |\nabla|^s b_N$$

$$= |\nabla|^s \tilde{P}_N (f_{\ll N} b_N) - f_{\ll N} |\nabla|^s \tilde{P}_N b_N = [|\nabla|^s \tilde{P}_N, f_{\ll N} \cdot] b_N$$

think of as a commutator; multiplication operator

$K(x,y)$ third kernel, i.e. thing that is convolve by to produce this operator

here it is just almost convolution

$$\Big\{ [|\nabla|^s \tilde{P}_N, f_{\ll N} \cdot] b_N \Big\}(x) = \int N^{s+1} \check{\chi}(N(x-y)) [f_{\ll N}(y) - f_{\ll N}(x)] b_N(y)$$

on Fourier side it multiplies by

$$|\xi|^s \tilde{\psi}\Big(\frac{\xi}{N}\Big) = N^s \underbrace{\Big(\frac{|\xi|^s}{N^s}\Big) \tilde{\psi}\Big(\frac{\xi}{N}\Big)}_{=: \chi(\frac{\xi}{N})}$$

4/24/24

Thm: for $0<s<1$, (bdd)

$$\| |\nabla|^s(fb) - f|\nabla|^s b\|_2 \lesssim \| |\nabla|^s f\|_2 \|b\|_\infty .$$

Recall: decompose:

$$|\nabla|^s(fb) - f|\nabla|^s b = \sum_N |\nabla|^s(f_N b_{\ll N}) - \sum_N f_N |\nabla|^s b_{\ll N} + \sum_N |\nabla|^s(f_{\ll N} b_N) - \sum_N f_{\ll N} |\nabla|^s b_N$$

$$+ \sum_N |\nabla|^s(f_{\sim N} b_N) - \boxed{\sum_N f_{\sim N} |\nabla|^s b_N}$$

only this term needs tools beyond 24A, invented for Carleson measures.

We showed (using sq fn estimates, Bernstein, vector valued max'l fn estimates (and 1st blue term, Schur test)) red terms, 1st blue term acceptable, $\lesssim \| |\nabla|^s f\|_2 \cdot \|b\|_\infty$.

For orange pieces, must exploit cancellations, i.e. can't handle each orange term separately.

$$\| \sum_N |\nabla|^s(f_{\ll N} b_N) - \sum_N f_{\ll N} |\nabla|^s b_N \|_2 \lesssim \| \sqrt{\sum_N |\tilde{P}_N[|\nabla|^s(f_{\ll N} b_N) - f_{\ll N}|\nabla|^s b_N]|^2} \|_2$$

($\tilde{P}_N$ bdd by M, M is bdd)

$$\lesssim \| \sqrt{\sum_N |\,|\nabla|^s(f_{\ll N} b_N) - f_{\ll N}|\nabla|^s b_N|^2} \|_2$$

$$|\nabla|^s(f_{\ll N} b_N - f_{\ll N}|\nabla|^s b_N) = |\nabla|^s \tilde{P}_N(f_{\ll N} b_N) - f_{\ll N}|\nabla|^s \tilde{P}_N b_N = [|\nabla|^s \tilde{P}_N, \underbrace{f_{\ll N}\cdot}_{\text{mult. operator}}] b_N$$

on Fourier side,

$$\widehat{|\nabla|^s \tilde{P}_N g}(\xi) = |\xi|^s \tilde{\psi}(\tfrac{\xi}{N}) \hat{g}(\xi) = N^s \underbrace{\left[(\tfrac{|\xi|}{N})^s \tilde{\psi}(\tfrac{\xi}{N})\right]}_{=:\chi(\frac{\xi}{N})} \hat{g}(\xi)$$

χ pretty much away from 0 like ψ.

$$\Rightarrow \widehat{|\nabla|^s \tilde{P}_N g} = N^{s+1} \underbrace{\check{\chi}(N\cdot)}_{\text{just another L-P projection to frequencies like } N} * g .$$

$[|\nabla|^s \tilde{P}_N, f_{\ll N}\cdot] b_N$ (convolution operator; mult. operator)

So $\left| |\nabla|^s(f_{\ll N} b_N) - f_{\ll N}|\nabla|^s b_N\right|(x) = \left| \int N^{s+1} \underbrace{\check{\chi}(N(x-y))}_{\text{fun convolve}} \underbrace{f_{\ll N}(y)}_{\text{multiply}} b_N(y)\, dy - \int N^{s+1} \underbrace{\check{\chi}(N(x-y))}_{\text{convolve}} b_N(y) \underbrace{f_{\ll N}(x)}_{\text{then multiply}} dy \right|$

$$\lesssim \int \frac{N^{s+1}}{\langle N(x-y)\rangle^{10}} |b_N(y)| \cdot |f_{\ll N}(x) - f_{\ll N}(y)|\, dy$$

$$\lesssim \|b\|_\infty \sum_{K \ll N} \int \frac{N^{s+1}}{\langle N(x-y)\rangle^{10}} |f_K(x) - f_K(y)|\, dy .$$

If $|x-y| \geq \frac{1}{K}$,

$$|f_K(x) - f_K(y)| \leq |f_K(x)| + |f_K(y)| \lesssim [Mf_K](x) + [Mf_K](y)$$

— Mf_K show up in next bound, and also M play well w/ L^2 norms!

If $|x-y| < \frac{1}{K}$, want to exploit cancellation:

$$|f_K(x) - f_K(y)| = |\widetilde{P}_K[f_K(x) - f_K(y)]| = \left| \int K \check{\Psi}(Kz) \left[f_K(x-z) - f_K(y-z) \right] dz \right|$$

want to FTC, but first change var to put difference in $\check{\Psi}$ and do FTC there

$y-z = x-t$, $z = y-x+t$

$$= \left| \int f_K(x-z)\, K \left[\check{\Psi}(Kz) - \check{\Psi}(Kz - K(x-y)) \right] dz \right|$$

↓ FTC

$$\lesssim \int |f_K(x-z)| \cdot K \cdot \underbrace{K|x-y|}_{\text{difference of arguments}} \int_0^1 \left| \check{\Psi}'(Kz - \theta K(x-y)) \right| d\theta\, dz$$

$\check{\Psi}$ decay to any order, say 10,

$$\lesssim \frac{1}{\langle Kz - \theta K(x-y) \rangle^{10}} \sim \frac{1}{\langle Kz \rangle^{10}} \quad \text{for } |x-y| < \tfrac{1}{K}$$

$$\lesssim K|x-y| \int \frac{K}{\langle Kz \rangle^{10}} |f_K(x-z)|\, dz$$

$$\lesssim K|x-y| \, [Mf_K](x).$$

Combining both cases above,

So, $|f_K(x) - f_K(y)| \lesssim K|x-y| \left\{ [Mf_K](x) + [Mf_K](y) \right\}$.

So then continuing above

$$\left| |D|^s (f_{<cN} b_N) - f_{<cN} |D|^s b_N \right|(x) \lesssim \|b\|_\infty \sum_{K<cN} \int \frac{N^{s+1}}{\langle N(x-y) \rangle^{10}} |f_K(x) - f_K(y)|\, dy.$$

$$\lesssim \|b\|_\infty \sum_{K<cN} \int \underbrace{\frac{N^{s+1} K|x-y|}{\langle N(x-y) \rangle^{10}}}_{N^{s-1} \cdot K \cdot \frac{N|x-y|}{\langle N(x-y) \rangle^{10}} \lesssim \frac{N}{\langle N(x-y) \rangle^9}} \left\{ [Mf_K](x) + [Mf_K](y) \right\} dy$$

N — comes normalization factor for next step.

$$\lesssim \|b\|_\infty \sum_{K<cN} N^{s-1} \cdot K \underbrace{\left\{ [Mf_K](x) + [M[Mf_K]](x) \right\}}_{=: c_K}$$

So we are now at.

$$\left\| \sum_N |D|^s (f_{<cN} b_N) - f_{<cN} |D|^s b_N \right\|_2 \lesssim \left\| \left\| \sum_{K<cN} N^{s-1} K c_K \right\|_{\ell^2_N} \right\|_{L^2_x} \cdot \|b\|_\infty.$$

$\sum_N \left| \sum_{K \ll N} N^{s-1} K C_K \right|^2 \lesssim \sum_N \sum_{K \le L \ll N} N^{2(s-1)} K^1 L^1 C_K C_L$ ($\sum_K \sum_L$ usual, pay factor of 2 to assume $K \le L$)

K, L entangled w/ C_K, C_L, and can sum in N b/c $s<1$!

$\lesssim \sum_{K \le L} L^{2s-1} K C_K C_L$

$\lesssim \sum_{K \le L} \underbrace{\left(\frac{K}{L}\right)^{1-s}}_{\text{good kernel: } \left(\frac{\text{low freq}}{\text{high freq}}\right)^{\text{positive}}} \underbrace{K^s C_K \; L^s C_L}_{\text{correct normalization for maximal fns ??}}$

$\{$ Schur test

$\lesssim \sum_K K^{2s} C_K^2 .$

So

$$\left\| \left\| \sum_{K \ll N} N^{s-1} K C_K \right\|_{\ell^2_N} \right\|_{L^2_x} \lesssim \left\| \left\| K^s C_K \right\|_{\ell^2_K} \right\|_{L^2_x}$$

Now remember what C_K is:

$$\lesssim \left\| \left\| M[K^s f_K] \right\|_{\ell^2_K} \right\|_{L^2_x} + \left\| \left\| M[M[K^s f_K]] \right\|_{\ell^2_K} \right\|_{L^2_x}$$

(M bdd; one more application of vector-val max'l fn yields something bdd by; one more application of vector-val)

$$\lesssim \left\| \sqrt{\sum_K K^{2s} |f_K|^2} \right\|_{L^2_x} \sim \| |\nabla|^s f \|_2$$

One last term: 2nd blue term.

$$\left\| \sum_N f_{\sim N} |\nabla|^s b_N \right\|_2 \lesssim \| |\nabla|^s f \|_2 \cdot \|b\|_{BC^s}$$

Proceed by duality. For $g \in L^2$

$$\left\langle \sum_N f_{\sim N} |\nabla|^s b_N , g \right\rangle = \sum_N \left\langle \underbrace{P_{\lesssim N}}_{\text{self adj operator}} (f_{\sim N} |\nabla|^s b_N), g \right\rangle$$

$$\lesssim \sum_N \left\langle N^s f_{\sim N}, \frac{(|\nabla|^s b_N)}{N^s} \cdot g_{\lesssim N} \right\rangle$$

CBS in N, CBS in x:

$$\lesssim \underbrace{\left\| \sqrt{\sum_N N^{2s} |f_{\sim N}|^2} \right\|_2}_{\sim \| |\nabla|^s f \|_2} \cdot \left\| \sqrt{\sum_N \left| \frac{|\nabla|^s}{N^s} P_{\sim N} b \right|^2 |g_{\lesssim N}|^2} \right\|_2$$

essentially just again $P_{\sim N} b$ proj to freq like N, so will just rotate as b_N.

also ignore squiggle $\lesssim$.

WTS $\int \sum_N |b_N|^2 |g_{\le N}|^2 \, dx \lesssim \|g\|_2^2 \|b\|_\infty^2$ (1)

low freq annoying b/c sum them over and over again, so need "b" terms to have good tail decay.

"weights"

so b_N must have special property so that (1) possible but not contradict

cf. $\| \sqrt{\sum_N |g_{\le N}|^2} \|_2 \not\lesssim \|g\|_2$

The special property satisfied by b_N that makes (1) possible is:

$$\sup_{x_0 \in \mathbb{R}} \sup_N N \int_{|x-x_0| \le \frac{1}{N}} \sum_{k \ge N} |b_k|^2(x) \, dx \lesssim \|b\|_\infty^2 \quad (2)$$

avg.

$\frac{1}{N}$ about the size of an interval at one ?

Interlude. on Carleson measure.

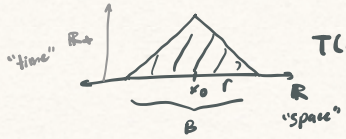
Given $d\mu$ a Borel msr on $\mathbb{R} \times \mathbb{R}_+$ (could be $\mathbb{R}^d \times \mathbb{R}_+$) we define

$$C(d\mu)(x_0) := \sup_{B \ni x_0} \frac{1}{|B|} \int_{T(B)} |d\mu|$$

↑ Carleson

where: sup taken over balls B containing x_0

and $T(B)$ = tent above B when if $B = B(x_0, r)$ then

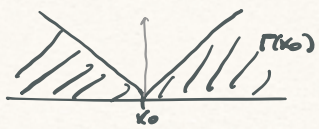


$T(B) = \{(x,t) : |x - x_0| \le r - t\}$.

we say $d\mu$ is Carleson msr if $\sup_{x_0 \in \mathbb{R}} C(d\mu)(x_0) < \infty$. (3)

Given $F : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{C}$, its nontangential maximal fn is

$$F^*(x_0) = \sup_{(x,t) \in \Gamma(x_0)} |F(x,t)| \quad \text{where } \Gamma(x_0) = \{(x,t) : |x - x_0| \le t\}$$



the relation is:

Thm: If $d\mu$ is Carleson msr and $F^* \in L^1(\mathbb{R})$, then

$$\left| \int_{\mathbb{R} \times \mathbb{R}_+} F(x,t) \, d\mu(x,t) \right| \lesssim \|F^*\|_{L^1(\mathbb{R})}$$

We think of $\{b_N\}$ as defining a msr $\int h(x,N) \, d\mu := \int \sum_N |b_N|^2 h(x,N) \, dx$ and $\frac{1}{N}$ as the "vertical" variable

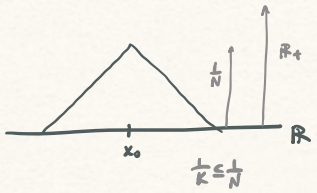
so that $\int |g_{\le N}|^2 d\mu = \int \sum_N |g_{\le N}|^2 |b_N|^2 dx$

the object we want to study.

Then (3) is similar to (2).

$$\sup_{x_0}\sup_{N}\frac{1}{|B(x_0,\frac{1}{N})|}\int_{T(B(x_0,\frac{1}{N}))}d\mu<\infty \quad\Longleftrightarrow\quad \sup_{x_0}\sup_{N} N\int_{|x-x_0|\le N}\sum_{k\ge N}|b_k|^2(x)\,dx<\infty.$$

$$\sup_{x_0}\sup_{N}\ \Updownarrow\ \sup_{B(x_0,\frac{1}{N})\ni x_0}\ \Updownarrow\ \sup_{B\ni x_0}$$



$\frac{1}{k}\le\frac{1}{N}$

4/26/24

final lecture on

Thm: for $0<s<1$, and

$$\| |\nabla|^s(fb) - f|\nabla|^s b\|_2 \lesssim \| |\nabla|^s f\|_2 \, \|b\|_\infty .$$

One last term — 2nd blue term.

$$\Big\| \sum_N f_{\sim N} |\nabla|^s b_N \Big\|_2 \lesssim \| |\nabla|^s f\|_2 \cdot \|b\|_\infty$$

(difficulty is our demand to put b in L^∞.)

If we didn't have to put b in L^∞, could very easily estimate:

CBS $\hookrightarrow$ $$\Big|\sum_N f_{\sim N} |\nabla|^s b_N\Big| \underset{CBS}{\lesssim} \sqrt{\sum_N N^{2s} |f_{\sim N}|^2} \sqrt{\sum_N \Big|\tfrac{|\nabla|^s}{N^s} P_N b\Big|^2}$$

$$\Big\| \sum_N f_{\sim N} |\nabla|^s b_N \Big\|_2 \underset{CBS}{\lesssim} \Big\| \sqrt{\sum_N N^{2s} |f_{\sim N}|^2} \Big\|_{L^p} \Big\| \sqrt{\sum_N \Big|\tfrac{|\nabla|^s}{N^s} P_N b\Big|^2} \Big\|_{L^q}$$

(sq fn est)

$$\lesssim \| |\nabla|^s f\|_p \, \|b\|_q$$

provided $\frac{1}{2} = \frac{1}{p} + \frac{1}{q}$ and $2 < p, q \le \infty$ ☹.

Pf. Last time: Arguing by duality, we said it sufficed to show

$$\int \sum_N |b_N|^2 \, |g_{\le N}|^2 \, dx \lesssim \|g\|_2^2 \, \|b\|_\infty^2 \qquad (1)$$

Write $\int \sum_N |b_N|^2(x) \, |g_{\le N}|^2(x)\, dx =: \int |g_{\le N}|^2(x) \, d\mu(x,N)$

(a nicemeasure, b_N are nice)

Let's show b_N satisfy Carleson condition in the sense that

$$\sup_{x_0} \sup_N N \int_{|x-x_0| \le \frac{1}{N}} \sum_{K \ge N} |b_K|^2(x)\, dx \lesssim \|b\|_\infty^2 \qquad (2)$$

By symmetry (i.e. left/right shift b, doesn't change L^∞ norm), we may assume $x_0 = 0$.

We decompose $b = \underbrace{b 1_{\{|x| \le \frac{10}{N}\}}}_{b_1} + \underbrace{b 1_{\{|x| > \frac{10}{N}\}}}_{b_2}$

b_1:

$$N \int_{|x| \le \frac{1}{N}} \sum_{K \ge N} |P_K b_1|^2(x)\, dx \lesssim N \int \sum_K |P_K b_1|^2(x)\, dx \overset{\text{sq fn est}}{\lesssim} N \|b_1\|_2^2 \lesssim N \cdot \|b\|_\infty^2 \, N^{-1} = \|b\|_\infty .$$

$|P_K b_1|^2(x)$: blurring b_1 at scale $\frac{1}{K}$, still mostly supp on $|x| \le \frac{10}{N}$, so is "distraction"

b2: For $|x| \le \frac{1}{N}$, $k \ge N$,

$$|P_k b_2|(x) = \left| \int K \check{\psi}(K(x-y))\, b_2(y)\, dy \right| \lesssim \int_{|y| \ge \frac{10}{N}} \frac{K}{\langle K(x-y)\rangle^{10}} |b(y)|\, dy$$

$|y| \ge \frac{10}{N}$, $|x| \le \frac{1}{N}$ $\Rightarrow$ Japanese bracket can be replaced by abs val

$$\lesssim \|b\|_\infty \cdot \int_{|x-y| \ge \frac{9}{N}} \frac{K}{(K|x-y|)^{10}}\, dy$$

$$\frac{1}{K^9} \int_{|y| \ge \frac{9}{N}} \frac{1}{|y|^{10}}\, dy$$

$$\lesssim \|b\|_\infty \left(\frac{N}{K}\right)^9$$

So

$$N \int_{|x| \le \frac{1}{N}} \sum_{K \ge N} |P_k b_2|^2(x)\, dx \lesssim N \int_{|x| \le \frac{1}{N}} \sum_{K \ge N} \|b\|_\infty^2 \left(\frac{N}{K}\right)^{18} dx$$

$$\lesssim \|b\|_\infty.$$

sum geom series in K

We now use (2) to prove (1).

Let $\check{\varphi}$ denote kernel corresponding to L^1 proj to freq ≤ 1, $P_{\le 1}$.

Take slight majorization $\Phi(x) = \sup_{|y| \le 1} |\check{\varphi}(x-y)|$

(so $\Phi(x)$ is bigger than $\check{\varphi}$ on 1-ball around x).

Define the maximal fn "glorified proj to low freq"

$$[\mathcal{M} f](x) := \sup_N \int |f(y)|\, N\, \Phi(N(x-y))\, dy.$$

Clearly

$$|f_{\le N}| \le \mathcal{M} f \lesssim Mf$$

$\mathcal{M}$ b/c $\sup$ over N; LHS is one N.

M = H-L max^l fn

Fact

$$|f_{\leq N}|(x) \leq \int |f(y)| \cdot N\,|\check{\varphi}(N(x-y))|\,dy \leq \inf_{|\theta|\leq \frac{1}{N}} \int |f(y)|\,N\,|\Phi(N(x-\theta-y))|\,dy$$

all "smudging" of φ, since Φ majorizes 1-ball smudging of $\check{\varphi}$.

$$\leq \inf_{|\theta|\leq \frac{1}{N}} [\mathcal{M}f](x-\theta)$$

$$\int \sum_N |b_N|^2(x)\,|\mathbb{S}_{\leq N}|^2(x)\,dx = \int |g_{\leq N}|^2(x)\,d\mu(x,N) = \int_0^\infty \mu\{(x,N) : |g_{\leq N}|^2(x) > \lambda\}\,d\lambda$$

so if $|g_{\leq N}|^2(x) > \lambda \Rightarrow [\mathcal{M}g]^2(y) > \lambda \quad \forall\, |y-x| \leq \frac{1}{N}$

i.e. $y \in \overline{B(x,\frac{1}{N})}$, interval

so $\{(x,N) : |g_{\leq N}|^2(x) > \lambda\} \subseteq \{(x,N) : x \in \bigcup_j I_j \text{ s.t. } [\mathcal{M}g]^2(y) > \lambda \ \forall y \in \bigcup_j I_j \text{ and } |I_j| \geq \frac{2}{N}\}$

took maximal interval

thus $\int_0^\infty \mu\{(x,N) : |g_{\leq N}|^2(x) > \lambda\}\,d\lambda \lesssim \int_0^\infty \sum_j \left(\int_{I_j} \sum_{k \geq \frac{2}{|I_j|}} |b_k|^2(x)\,dx \right) d\lambda$

i.e. integrate indicator of $d\mu$

cut off in "vertical direction"

integrating indicator of on (x,N) space

vertical

b/c $|I_j| \leq \frac{2}{N}$.

$$\lesssim_{(2)} \int_0^\infty \sum_j \left(|I_j| \cdot \|b\|_{\ell^2}^2 \right) d\lambda$$

using I_j disjoint

$$\lesssim \|b\|_{\ell^2}^2 \cdot \int_0^\infty |\{x : [\mathcal{M}g]^2(x) > \lambda\}|\,d\lambda$$

$$\lesssim \|\mathcal{M}g\|_2^2$$

$$\lesssim \|Mg\|_2^2$$

$$\lesssim \|g\|_2^2$$

so we have completed (1)

We have thus completed the proof of well-posedness of KdV using oscillatory integral approach.

Well-posedness for KdV, the $X^{s,b}$ method

(sometimes called "Bourgain spaces")

pinnacle of marriage between harmonic analysis and PDE

Consider $\begin{cases} u_t + u_{xxx} = 6uu_x \\ u(0) = u_0 \in H^s(\mathbb{R}) \end{cases}$ for $0 \le s \le 1$.

can go all the way down to $-\frac{3}{4}$, but we'll stop at 0

Later, Monica will present her own work going down to -1 anyways.

We denote by $X^{s,b}$ the Banach space of fns on $\mathbb{R}\times\mathbb{R}$ defined by norm

$$\|u\|_{X^{s,b}} := \| \langle \xi \rangle^s \langle \tau - \xi^3 \rangle^b \hat{u}(\tau,\xi) \|_{L^2_{\tau,\xi}}$$

twisted derivative

unlike before, $\hat{u}$ denote FT wrt. BOTH time & space

$$\hat{u}(\tau,\xi) = \frac{1}{2\pi} \iint e^{-it\tau - ix\xi} u(t,x)\, dx\, dt.$$

Some words: if u solved Airy eq $u_t + u_{xxx} = 0$,

then $\hat{u}(\tau,\xi)[\tau - \xi^3] = 0$

Explanation 1.

Tells us that when (τ,ξ) don't live on surface $\tau = \xi^3$, $\hat{u}$ must be 0

i.e. $\hat{u}$ is supported on $\{(\tau,\xi) : \tau = \xi^3\}$.

So twist $\langle \tau - \xi^3 \rangle^b$ penalizes τ,ξ off the surface.

But $X^{s,b}$ still perturbative approach, treat nonlinearity as perturbation of Airy eq.

Explanation 2: "interaction variables".

The Duhamel formulation of KdV is as we know

$$u(t) = e^{-t\partial_x^3} u_0 + 3\int_0^t e^{-(t-\tau)\partial_x^3} \partial_x(u^2)(\tau)\, d\tau$$

Switch to interaction variables:

$$e^{t\partial_x^3} u(t) = u_0 + 3\int_0^t e^{\tau\partial_x^3} \partial_x (u^2)(\tau)\, d\tau$$

$f(t)$?!! "pull sol u of KdV via Airy operator back down to $t=0$."

the interaction variable

so eq becomes $f(t) = u_0 + 3\int_0^t e^{\tau\partial_x^3}\,\partial_x\left[e^{-\tau\partial_x^3} f(\tau)\right]^2 d\tau.$

↓ on Fourier side, square ⇝ convolution, $e^{-\tau\partial_x^3}$ ⇝ phase

Observe $\widehat{f}(\tau,\xi) = \widehat{e^{t\partial_x^3} u(t)}\,(\tau,\xi)$

$$= \frac{1}{\sqrt{2\pi}}\int e^{-it\tau}\,\mathcal{F}_x\left[e^{it\partial_x^3} u(t)\right](\xi)\,dt$$

$$= \frac{1}{\sqrt{2\pi}}\int e^{-it\tau - it\xi^3}\left[\mathcal{F}_x(u(t))\right](\xi)\,dt$$

$$= \hat{u}(\tau+\xi^3,\xi)$$

Hence standard def of interaction variable version $X^{s,b}$ norm.

$$\|f\|_{H_t^b H_x^s} = \|\langle\tau\rangle^b\langle\xi\rangle^s \hat{f}(\tau,\xi)\|_{L^2_{\tau,\xi}} = \|\langle\tau-\xi^3\rangle^b\langle\xi\rangle^s \hat{u}(t,\xi)\|_{L^2_{\tau,\xi}}.$$

For $T_0>0$, we define the restricted $X^{s,b}$ norm via

$$\|u\|_{X^{s,b}_{T_0}} := \inf\left\{\|v\|_{X^{s,b}} : v\equiv u \text{ on } |t|\le T_0\right\}$$

all possible extensions of u outside interval.

We'll construct solutions via contraction mapping w/

$$\Phi[u](t) = \eta(t)\,e^{-t\partial_x^3}u_0 + 3\eta(t)\int_0^t e^{-(t-\tau)\partial_x^3}\partial_x(u^2)(\tau)\,d\tau$$

where $\eta\in C_c^\infty([-2,2])$, $\eta\equiv 1$ on $[-1,1]$

we'll take $T_0 = T_0(\|u_0\|_{H^s}) \le 1$.

The Banach space for contraction will be:

$$B := \left\{u\in X^{s,b}_{T_0} : \|u\|_{X^{s,b}_{T_0}} \le C_0\|u_0\|_{H^s}\right\}.$$

(note did not write cty in time. we will prove that $B \hookrightarrow$ cts in time fns)

w/ $0\le s\le 1$ and $\frac{1}{2} < b < \frac{3}{4}$ fixed.

→ to embed into cts fns. $H_t^{\frac{1}{2}+} \hookrightarrow C_t$.

4/29/24 10:40 Either cases about $X^{s,b}$ space "taking" being away from $\tau = \xi^3$ surface.
Monica saying $X^{s,b}$ norm "engineered" to suit the problem.

$$\|u\|_{X^{s,b}} := \| \langle \xi \rangle^s \langle \tau - \xi^3 \rangle^b \hat{u}(\tau,\xi) \|_{L^2_{\tau,\xi}} = \| e^{t\partial_x^3} u(t) \|_{H^b_t H^s_x}$$

"regular sobolev norm in interaction variable"

$$\|u\|_{X^{s,b}_{T_0}} := \inf \{ \|v\|_{X^{s,b}} : v \equiv u \text{ on } [-T_0, T_0] \times \mathbb{R}^2 \}$$

We'll show that $\Phi[u](t) = \eta(t) e^{-t\partial_x^3} u_0 + 3\eta(t) \int_0^t e^{-(t-\tau)\partial_x^3} \partial_x (u^2)(\tau)\, d\tau$

where $\eta \in C_c^\infty([-2,2])$, $\eta \equiv 1$ on $[-1,1]$

we'll take $T_0 = T_0(\|u_0\|_{H^s}) \le 1$. (so $[-T_0, T_0] \subseteq [-1,1]$, where $\eta \equiv 1$)

The Banach space for contraction will be:

$$B := \{ u \in X^{s,b}_{T_0} : \|u\|_{X^{s,b}_{T_0}} \le C_0 \|u_0\|_{H^s} \}.$$

(Quote did not write cty in time. We will prove that $B \hookrightarrow$ cts in time fns)

w/ $0 \le s \le 1$ and $\frac{1}{2} < b < \frac{3}{4}$ fixed.

Lemma: For $b > \frac{1}{2}$, $X^{s,b}_{T_0} \hookrightarrow C_t H^s_x([-T_0, T_0] \times \mathbb{R})$ "whatever solution we get will be cts in time"

Pf: hard analysis: bound.
Soft analysis to prove cty.

Show: $\|u\|_{L^\infty_t H^s_x([-T_0,T_0]\times\mathbb{R})} \lesssim \|u\|_{X^{s,b}_{T_0}}$ (1)

By def of restricted $X^{s,b}_{T_0}$ norm,
Let $v \in X^{s,b}$ w/ $v \equiv u$ on $|t| \le T_0$

For $|t| \le T_0$,

$$\|u(t)\|_{H^s_x} = \|v(t)\|_{H^s_x} = \|e^{t\partial_x^3} v(t)\|_{H^s_x} = \| \langle \xi \rangle^s e^{-it\xi^3} [\mathcal{F}_x v](t,\xi) \|_{L^2_\xi}$$

(can pass to interaction variable for free)

$$= \frac{1}{\sqrt{2\pi}} \| \langle \xi \rangle^s \int e^{it\xi - it\xi^3} \hat{v}(\tau,\xi)\, d\tau \|_{L^2_\xi}$$

$$= \frac{1}{\sqrt{2\pi}} \left\| \langle \xi \rangle^s \int e^{it\xi} \hat{v}(\tau+\xi^3, \xi)\, d\tau \right\|_{L^2_\xi}$$

$\tau \to \tau + \xi^3$ to simplify phase

$$\lesssim \int \| \langle \xi \rangle^s \hat{v}(\tau+\xi^3, \xi) \|_{L^2_\xi}\, d\tau$$

$$\lesssim \| \langle \xi \rangle^s \langle \tau \rangle^b \hat{v}(\tau+\xi^3, \xi) \|_{L^2_{\tau,\xi}} \| \langle \tau \rangle^{-b} \|_{L^2_\tau}$$

also how we prove $H^{\frac{1}{2}+} \hookrightarrow C$ — bring up $X^{s,b}$ norm w/ CBS

$$\lesssim \|v\|_{X^{s,b}} \quad \text{if } b > \tfrac{1}{2}.$$

Taking infimum over such v we get $\|u(t)\|_{H^s_x} \lesssim \|u\|_{X^{s,b}_{T_0}}$ unif in $|t| \le T_0$.

If remains to prove cty in t.

For $t' < t$ in $[-T_0, T_0]$,

again v agrees w/ u on $[-T_0, T_0]$

$$\|u(t) - u(t')\|_{H^s_x} = \|v(t) - v(t')\|_{H^s_x} \le \underbrace{\| e^{t\partial_x^3} v(t) - e^{t'\partial_x^3} v(t') \|_{H^s_x}}_{\text{(I)}} + \underbrace{\| (e^{t\partial_x^3} - e^{t'\partial_x^3}) v(t') \|_{H^s_x}}_{\text{(II)}}$$

move to interaction variables

$$\| e^{t\partial_x^3} v(t) - e^{t'\partial_x^3} v(t') \|_{H^s_x} \overset{\text{on Fourier side, and inverse FT}}{=} \frac{1}{\sqrt{2\pi}} \left\| \langle \xi \rangle^s \left[\int e^{it\tau - it\xi^3} \hat{v}(\tau,\xi)\, d\tau - \int e^{it'\tau - it'\xi^3} \hat{v}(\tau,\xi)\, d\tau \right] \right\|_{L^2_\xi}$$

(I)

$$= \frac{1}{\sqrt{2\pi}} \left\| \langle \xi \rangle^s \int \left[e^{it\tau} - e^{it'\tau} \right] \hat{v}(\tau+\xi^3, \xi)\, d\tau \right\|_{L^2_\xi}$$

$\tau \to \tau + \xi^3$ var change again

Δ-ineq to bring L^2_ξ inside

$$\lesssim \underbrace{\| \langle \xi \rangle^s \langle \tau \rangle^b \left| e^{it\tau} - e^{it'\tau} \right| \hat{v}(\tau+\xi^3, \xi) \|_{L^2_{\tau,\xi}}}_{\text{bdd by } X^{s,b} \text{ norm, as DCT applies and } |e^{it\tau} - e^{it'\tau}| \text{ sinks it to 0.}} \underbrace{\| \langle \tau \rangle^{-b} \|_{L^2_\tau}}_{\lesssim 1 \text{ if } b > \frac{1}{2}}$$

again sandwich $\langle \tau \rangle^b$ and CBS

$$\underbrace{\| (e^{t\partial_x^3} - e^{t'\partial_x^3}) v(t') \|_{H^s_x}}_{\text{(II)}} \lesssim \left\| \langle \xi \rangle^s \int \left[e^{it\tau - it\xi^3} - e^{it'\tau - it'\xi^3} \right] \hat{v}(\tau,\xi)\, d\tau \right\|_{L^2_\xi}$$

on Fourier side

$$\lesssim \| \langle \xi \rangle \langle \tau \rangle^b \left| e^{-i(t-t')\xi^3} - 1 \right| \hat{v}(\tau+\xi^3, \xi) \|_{L^2_{\tau,\xi}} \cdot \underbrace{\| \langle \tau \rangle^{-b} \|_{L^2_\tau}}_{\lesssim 1 \text{ for } b > 1} \longrightarrow 0.$$

$\tau \to \tau + \xi^3$ only eliminate one term + usual CBS, etc.

again as before (DCT as $|t'-t| \to 0$)

Proving $\Phi : B \to B$ and is contraction.

The linear term in $\Phi(u)$:

$$\| \eta(t) e^{-t\partial_x^3} u_0 \|_{X^{s,b}_{T_0}} \le \| \eta(t) e^{-t\partial_x^3} u_0 \|_{X^{s,b}} = \| e^{t\partial_x^3} \eta(t) e^{-t\partial_x^3} u_0 \|_{H^b_t H^s_x}$$

b/c $\eta(t) e^{-t\partial_x^3} u_0$ is extension of to all spacetime.

$e^{t\partial_x^3}$ fn of time only so commutes

$$= \| \eta \|_{H^b_t} \| u_0 \|_{H^s_x} \lesssim \| u_0 \|_{H^s_x}.$$

The nonlinear term in $\Phi(u)$:

Lemma. For $b > \frac{1}{2}$, $\| \eta(t) \int_0^t e^{-(t-\tau)\partial_x^3} F(\tau)\, d\tau \|_{X^{s,b}_{T_0}} \lesssim \| F \|_{X^{s,b-1}_{T_0}}$

"smoothing effect: if F any $b-1$ level of regular, then integrate, gain 1 level of regularity."

Pf. w/ usual starting w/ restr. norms, consider extension $\tilde{F} \equiv F$ on $|t| \le T_0$.

again is extension of; "doubling trick" – Ettas!

$$\| \eta(t) \int_0^t e^{-(t-\tau)\partial_x^3} F(\tau)\, d\tau \|_{X^{s,b}_{T_0}} \le \| \eta(t) \int_0^t e^{-(t-\tau)\partial_x^3} \tilde{F}(\tau)\, d\tau \|_{X^{s,b}}$$

pass to inhomog. sol'n

$$= \| e^{t\partial_x^3} \eta(t) \int_0^t e^{-(t-\tau)\partial_x^3} \tilde{F}(\tau)\, d\tau \|_{H^b_t H^s_x}$$

fn of time only, so commute

$$= \| \eta(t) \int_0^t e^{\tau\partial_x^3} \tilde{F}(\tau)\, d\tau \|_{H^b_t H^s_x}$$

$$\lesssim \| \eta(t) \int_0^t \| e^{\tau\partial_x^3} \tilde{F}(\tau) \|_{H^s_x} d\tau \|_{H^b_t}$$

So if we can show $\boxed{\| \eta(t) \int_0^t f(\tau)\, d\tau \|_{H^b_t} \lesssim \| f \|_{H^{b-1}_t}}$ (2)

then would get $\| \eta(t) \int_0^t e^{-(t-\tau)\partial_x^3} F(\tau)\, d\tau \|_{X^{s,b}_{T_0}} \lesssim \| e^{\tau\partial_x^3} \tilde{F}(\tau) \|_{H^{b-1}_t H^s_x} = \| \tilde{F} \|_{X^{s,b-1}}$

So, boil down to proving (2).

$$\int_0^t f(\tau)\, d\tau = \int \chi_{[0,t]}(\tau) f(\tau)\, d\tau \sim \int \hat{\chi}_{[0,t]}(-\omega) \hat{f}(\omega)\, d\omega \sim \int \frac{e^{it\omega} - 1}{\omega} \hat{f}(\omega)\, d\omega$$

factors of 2π

$$\hat{\chi}_{[0,t]}(\omega) \sim \int e^{i\omega\tau} \chi_{[0,t]}\, d\tau \sim \frac{e^{it\omega} - 1}{\omega}$$

factors of 2π

so, $\| \eta(t) \int_0^t f(\tau)\,d\tau \|_{H^b_t} \overset{\text{on Fourier side}}{=} \| \langle\tau\rangle^b \int e^{-it\tau} \eta(t) \left(\int \frac{e^{i\omega t}-1}{\omega} \hat{f}(\omega)\,d\omega \right) dt \|_{L^2_\tau}$

$\overset{\text{Fubini}}{=} \| \langle\tau\rangle^b \int \frac{\hat{\eta}(\tau-\omega) - \hat{\eta}(\tau)}{\omega} \hat{f}(\omega)\,d\omega \|_{L^2_\tau}$

$|\omega|\le 1$: FTC, expect cancellation. $|\omega|\ge 1$: replace with $\langle\omega\rangle$.

$|\omega|\le 1$ cancellation

$$\| \langle\tau\rangle^b \int_{|\omega|\le 1} \frac{\hat{\eta}(\tau-\omega)-\hat{\eta}(\tau)}{\omega} \hat{f}(\omega)\,d\omega \|_{L^2_\tau} \lesssim \| \langle\tau\rangle^b \int_{|\omega|\le 1} \left(\int_0^1 |\hat{\eta}'(\tau-\theta\omega)|\,d\theta \right) |\hat{f}(\omega)|\,d\omega \|_{L^2_\tau}$$

$$\lesssim \frac{1}{\langle\tau-\theta\omega\rangle^{10}} \sim \frac{1}{\langle\tau\rangle^{10}}$$

$$\lesssim \| \langle\tau\rangle^{b-10} \|_{L^2_\tau} \int_{|\omega|\le 1} |\hat{f}(\omega)|\,d\omega$$

finite for "10" much larger

$$\lesssim \| \langle\omega\rangle^{b-1} \hat{f} \|_2 \, \| \langle\omega\rangle^{1-b} \|_{L^2(|\omega|\le 1)}$$

finite no matter what b is.

$$\lesssim \|f\|_{H^{b-1}}$$

$|\omega|\ge 1$ no cancellation, Δ-ing b

$$\| \langle\tau\rangle^b \int_{|\omega|\ge 1} \frac{|\hat{\eta}(\tau)| \cdot |\hat{f}(\omega)|}{\langle\omega\rangle} d\omega \|_{L^2_\tau} \lesssim \underbrace{\| \langle\tau\rangle^b \hat{\eta} \|_{L^2_\tau}}_{\lesssim 1 \text{ b/c } \hat{\eta} \text{ decay}} \| \langle\omega\rangle^{-1} \hat{f} \|_1$$

decoupled integrals

$$\overset{\text{CS}}{\lesssim} \| \langle\omega\rangle^{b-1} \hat{f} \|_2 \, \| \langle\omega\rangle^{-b} \|_2 \lesssim \|f\|_{H^{b-1}} \quad \text{since } b > \tfrac{1}{2}.$$

if only $\lesssim \langle\tau-\omega\rangle^b$, since then get clean convolution. not true, but it is true that $\lesssim \langle\tau-\omega\rangle^b \langle\omega\rangle^b$! (consider $\tau \sim \omega$, $\tau \gtrsim \omega$)

$$\| \langle\tau\rangle^b \int_{|\omega|\ge 1} \frac{|\hat{\eta}(\tau-\omega)| \cdot |\hat{f}(\omega)|}{\langle\omega\rangle} d\omega \|_{L^2_\tau} \lesssim \| \int \langle\tau-\omega\rangle^b |\hat{\eta}(\tau-\omega)| \, \langle\omega\rangle^{-1+b} |\hat{f}(\omega)|\,d\omega \|_{L^2_\tau}$$

convolution

$\langle\omega\rangle^{-1+b}$ put in L^2 means $\hat{f}$ has to go in L^1, but $\hat{\eta}$ has fast decay.

$$\lesssim \| \langle\omega\rangle^{b-1} \hat{f} \|_2 \, \underbrace{\| \langle\tau-\omega\rangle^b \hat{\eta}(t-\omega) \|_1}_{\lesssim 1}$$

$$\lesssim \|f\|_{H^{b-1}}$$

So far we have proven

$$\| \mathcal{I}(u) \|_{X^{s,b}_{T_0}} \lesssim \|u_0\|_{H^s_x} + \| \partial_x(u^2) \|_{X^{s,b-1}_{T_0}}$$

using Lemma.

$$\|u\|_{X^{s,b}_{T_0}} \lesssim T_0^{\tilde{b}-b} \|u\|_{X^{s,\tilde{b}}_{T_0}}$$

source of smallness, as long as $\tilde{b} > b$. But fortunately, we gained a whole "1"!

Thus far, we have showed that the mapping

$$\Phi(u)(t) = \eta(t)\, e^{-t\partial_x^3} u_0 + 3\eta(t) \int_0^t e^{-(t-\tau)\partial_x^3} \partial_x (u^2)(\tau)\, d\tau$$

where $\eta \in C_c^\infty((-2,2))$ with $\eta \equiv 1$ on $|t| \le 1$ satisfies

$$\begin{cases} \|\Phi(u)\|_{X^{s,b}_{T_0}} \lesssim \|u_0\|_{H^s_x} + \|\partial_x(u^2)\|_{X^{s,b-1}_{T_0}} \\ \|\Phi(u) - \Phi(v)\|_{X^{s,b}_{T_0}} \lesssim \|\partial_x [u^2 - v^2]\|_{X^{s,b-1}_{T_0}} \end{cases}$$

for all $s \ge 0$, $b > 1/2$.

It remains to estimate the nonlinear terms above. We start by exhibiting the small parameter necessary to prove Φ is a contraction on

$$B = \{ u \in X^{s,b}_{T_0} : \|u\|_{X^{s,b}_{T_0}} \le C_0 \|u_0\|_{H^s} \}$$

where C_0 is an absolute constant.

Lemma For any $-1/2 < b' < b < 1/2$ and $s \in \mathbb{R}$ we have

$$\|u\|_{X^{s,b'}_{T_0}} \lesssim T_0^{b-b'} \|u\|_{X^{s,b}_{T_0}}$$

Proof: We'll prove the claim for $0 \le b' < b < 1/2$. Then, by duality, for $-1/2 < b' < b \le 0$ we have

$$\begin{aligned} \|u\|_{X^{s,b'}_{T_0}} = \sup_{\|g\|_{X^{-s,-b'}_{T_0}} \le 1} \langle u, g \rangle &\lesssim \sup_g \|u\|_{X^{s,b}_{T_0}} \|g\|_{X^{-s,-b}_{T_0}} \\ &\lesssim \sup_g \|u\|_{X^{s,b}_{T_0}} T_0^{-b'+b} \|g\|_{X^{-s,-b'}_{T_0}} \qquad 0 \le -b < -b' < 1/2 \\ &\lesssim \|u\|_{X^{s,b}_{T_0}} \end{aligned}$$

Finally, for $-1/2 < b' < 0 < b < 1/2$ we have

$$\|u\|_{X^{s,b'}_{T_0}} \le T_0^{-b'} \|u\|_{X^{s,0}_{T_0}} \le T_0^{b-b'} \|u\|_{X^{s,b}_{T_0}}$$

So fix $0 \le b' < b < 1/2$. Let's show

$$\|u\|_{X^{s,b'}_{T_0}} \lesssim T_0^{b-b'} \|u\|_{X^{s,b}_{T_0}}$$

Let $v(t) = u(t)$ for $|t| \le T_0$. Using the fractional product rule and Sobolev embedding, we may bound.

$$\begin{aligned}
\|u\|_{X^{s,b'}_{T_0}} &\le \|\eta(t/T_0)\, v(t)\|_{X^{s,b'}} = \|\eta(\tfrac{t}{T_0})\, e^{t\partial_x^3} v(t)\|_{H^{b'}_t H^s_x} \\
&\lesssim \Big\| \|\eta(\tfrac{t}{T_0})\|_{L^{\frac{1}{b}}} \|e^{t\partial_x^3} v(t)\|_{L^{\frac{2}{1-2b}}_t} \Big\|_{H^s_x} \\
&\quad + \Big\| \| |\nabla|_t^{b'} \eta(\tfrac{t}{T_0})\|_{L^{1/b}} \|e^{t\partial_x^3} v(t)\|_{L^{\frac{2}{1-2b}}_t} \Big\|_{H^s_x} \\
&\quad + \Big\| \|\eta(\tfrac{t}{T_0})\|_{L^{\frac{1}{b-b'}}} \| |\nabla|_t^{b'} e^{t\partial_x^3} v(t)\|_{L^{\frac{2}{1-2(b-b')}}_t} \Big\|_{H^s_x} \\
&\lesssim \|e^{t\partial_x^3} v(t)\|_{H^b_t H^s_x} \cdot \{ T_0^{b} + T_0^{b-b'} \} \\
&\lesssim T_0^{b-b'} \|v\|_{X^{s,b}}
\end{aligned}$$

Taking the infimum over such v, we derive the claim ∎

<u>Lemma</u> For $b > 1/2$, $s \ge 0$, and $b' \le -1/4$ we have

$$\|\partial_x (uv)\|_{X^{s,b'}_{T_0}} \lesssim \|u\|_{X^{s,b}_{T_0}} \|v\|_{X^{s,b}_{T_0}}$$

<u>Remark</u>: For $\frac{1}{2} < b < \frac{3}{4}$ and $s \ge 0$, this gives

$$\|\partial_x(u^2)\|_{X^{s,b-1}_{T_0}} \lesssim T_0^{\frac{3}{4}-b} \|\partial_x(u^2)\|_{X^{s,-\frac{1}{4}}_{T_0}} \lesssim T_0^{\frac{3}{4}-b} \|u\|^2_{X^{s,b}_{T_0}}$$

and so $$\|\Phi(u)\|_{X^{s,b}_{T_0}} \lesssim \|u_0\|_{H^s} + T_0^{\frac{3}{4}-b} \|u\|^2_{X^{s,b}_{T_0}}$$

Thus for $T_0 = T_0(\|u_0\|_{H^s})$, $\Phi : B \to B$.

Finally, to see that Φ is a contraction, for $u, v \in B$ we estimate

$$\begin{aligned}
\|\phi(u)-\phi(v)\|_{X^{s,b}_{T_0}} &\lesssim \|\partial_x(u^2-v^2)\|_{X^{s,b-1}_{T_0}} \\
&\lesssim T_0^{\frac{3}{4}-b} \|\partial_x[(u-v)(u+v)]\|_{X^{s,-\frac{1}{4}}_{T_0}} \\
&\lesssim T_0^{\frac{3}{4}-b} \|u-v\|_{X^{s,b}_{T_0}} \left[\|u\|_{X^{s,b}_{T_0}} + \|v\|_{X^{s,b}_{T_0}}\right] \\
&\lesssim T_0^{\frac{3}{4}-b} \|u-v\|_{X^{s,b}_{T_0}} \cdot \|u_0\|_{H^s} \\
&< \tfrac{1}{2} \|u-v\|_{X^{s,b}_{T_0}} \qquad \text{provided } T_0 = T_0(\|u_0\|_{H^s}) \text{ is small enough.}
\end{aligned}$$

It remains to prove the lemma.

Proof: We'll ignore the T_0 and argue by duality. We have to show

$$\left|\iint \partial_x(uv)\cdot w \, dx\,dt\right| \lesssim \|u\|_{X^{s,b}} \|v\|_{X^{s,b}} \|w\|_{X^{s,-b'}}$$

Recall $\|u\|_{X^{s,b}} = \|\langle\tau-\xi^3\rangle^b \langle\xi\rangle^s \hat{u}(\tau,\xi)\|_{L^2_{\tau,\xi}}$

$$\begin{aligned}
\left|\iint \partial_x(uv)\cdot w \, dx\,dt\right| &= \left|\iint \xi \cdot \widehat{uv}(\tau,\xi)\, \hat{w}(-\tau,-\xi)\, d\tau\, d\xi\right| \\
&= \left|\iiiint \xi\, \hat{u}(\tau_1,\xi_1)\, \hat{v}(\tau-\tau_1,\xi-\xi_1)\, \hat{w}(-\tau,-\xi)\, d\tau_1 d\tau\, d\xi_1 d\xi\right| \\
&\lesssim \iiiint \frac{\langle\xi\rangle^s |\xi|\, \langle\tau-\xi^3\rangle^{b'}}{\langle\xi_1\rangle^s \langle\xi-\xi_1\rangle^s \langle\tau-\xi_1^3\rangle^b \langle\tau-\tau_1-(\xi-\xi_1)^3\rangle^b} f_1(\tau_1,\xi_1)\, f_2(\tau-\tau_1,\xi-\xi_1)\, f_3(\tau,\xi)\, d\tau_1 d\tau\, d\xi_1 d\xi
\end{aligned}$$

where $f_1(\tau,\xi) = \langle\tau-\xi^3\rangle^b \langle\xi\rangle^s |\hat{u}(\tau,\xi)|$

$f_2(\tau,\xi) = \langle\tau-\xi^3\rangle^b \langle\xi\rangle^s |\hat{v}(\tau,\xi)|$

$f_3(\tau,\xi) = \langle\tau-\xi^3\rangle^{-b'} \langle\xi\rangle^{-s} |\hat{w}(-\tau,-\xi)|$

By Cauchy-Schwarz,

$$\left|\iint \partial_x(uv)\cdot w\,dx\,dt\right|^2 \lesssim \iiiint |f_1(\tau_1,\xi_1)|^2 |f_2(\tau-\tau_1,\xi-\xi_1)|^2 d\tau_1 d\tau\, d\xi_1 d\xi$$

$$\cdot \iiiint \frac{\langle\xi\rangle^{2s}|\xi|^2\langle\tau-\xi^3\rangle^{2b'}}{\langle\xi_1\rangle^{2s}\langle\xi-\xi_1\rangle^{2s}\langle\tau_1-\xi_1^3\rangle^{2b}\langle\tau-\tau_1-(\xi-\xi_1)^3\rangle^{2b}} |f_3(\tau,\xi)|^2 d\tau_1 d\tau\, d\xi_1 d\xi.$$

$$\lesssim \|f_1\|^2_{L^2_{\tau,\xi}} \|f_2\|^2_{L^2_{\tau,\xi}} \cdot \iiiint \frac{|\xi|^2\langle\tau-\xi^3\rangle^{2b'}|f_3(\tau,\xi)|^2}{\langle\tau_1-\xi_1^3\rangle^{2b}\langle\tau-\tau_1-(\xi-\xi_1)^3\rangle^{2b}} d\tau_1 d\xi_1 d\tau d\xi.$$

We use the fact that

$$\int_{\mathbb{R}} \frac{1}{\langle x\rangle^{2b}\langle x-\alpha\rangle^{2b}} dx \lesssim \langle\alpha\rangle^{-2b} \qquad \text{for } b>1/2$$

Indeed $\displaystyle\int_{|x-\alpha|\le\frac{|\alpha|}{2}} \frac{1}{\langle x\rangle^{2b}\langle x-\alpha\rangle^{2b}} dx \lesssim \langle\alpha\rangle^{-2b} \int_{\mathbb{R}} \frac{1}{\langle x-\alpha\rangle^{2b}} dx \lesssim \langle\alpha\rangle^{-2b}$

and $\displaystyle\int_{|x-\alpha|>\frac{|\alpha|}{2}} \frac{1}{\langle x\rangle^{2b}\langle x-\alpha\rangle^{2b}} dx \lesssim \langle\alpha\rangle^{-2b} \int_{\mathbb{R}} \frac{1}{\langle x\rangle^{2b}} dx \lesssim \langle\alpha\rangle^{-2b}$

Thus $\displaystyle\int \frac{d\tau_1}{\langle\tau_1-\xi_1^3\rangle^{2b}\langle\tau-\tau_1-(\xi-\xi_1)^3\rangle^{2b}} = \int \frac{d\tau_1}{\langle\tau\rangle^{2b}\langle\tau_1-\tau+\xi_1^3+(\xi-\xi_1)^3\rangle^{2b}} \lesssim \langle\tau-\xi_1^3-(\xi-\xi_1)^3\rangle^{-2b}$

$$\lesssim \langle\tau-3\xi_1^2\xi+3\xi_1\xi^2-\xi^3\rangle^{-2b}$$

Now $\displaystyle\int_{\mathbb{R}} \langle\tau-3\xi_1^2\xi+3\xi_1\xi^2-\xi^3\rangle^{-2b} d\xi_1 = \int \langle\xi_1^2-\xi_1\cdot\sqrt{3}|\xi|^{3/2}+\xi^3-\tau\rangle^{-2b} |3\xi|^{-1/2} d\xi_1$

$\xi_1 \mapsto \xi_1+\frac{\sqrt{3}}{2}|\xi|^{3/2}$ $\displaystyle= \int \langle\xi_1^2+\xi^3-\tau-\frac{3|\xi|^3}{4}\rangle^{-2b} |3\xi|^{-1/2} d\xi_1$

$\xi_1^2 \mapsto \eta$ $\displaystyle\lesssim |\xi|^{-1/2} \int \langle\eta+\xi^3-\tau-\frac{3|\xi|^3}{4}\rangle^{-2b} |\eta|^{-1/2} d\eta$

$$\lesssim |\xi|^{-1/2}\cdot\langle\xi^3-\tau-\tfrac{3|\xi|^3}{4}\rangle^{-1/2}$$

$$\lesssim |\xi|^{-1/2}\langle 4\xi^3-3|\xi|^3-4\tau\rangle^{-1/2}$$

Here we use the easy bound

$$\int_{\mathbb{R}} \frac{dx}{|x|^{1/2}\langle x-\alpha\rangle^{2b}} \lesssim \langle\alpha\rangle^{-1/2} \qquad \text{for } b>1/2$$

Hint: This is immediate if $|\alpha|\le 1$. If $|\alpha|>1$ treat separately the cases $|x|\ge 2|\alpha|$, $|x|\le\frac{|\alpha|}{2}$, and $\frac{|\alpha|}{2}<|x|<2|\alpha|$.

Get $\left|\iint \partial_x(uv)\cdot w\,dx\,dt\right|^2 \lesssim \|f_1\|^2_{L^2_{\xi,\xi}} \|f_2\|^2_{L^2_{\xi,\xi}} \cdot \underbrace{\iint \frac{|\xi|^{3/2}\,|f_3(\xi,\xi)|^2}{\langle\xi-\xi^3\rangle^{-2b'}\langle 4\xi^3-3|\xi|^3-4\xi\rangle^{1/2}}\,d\xi\,d\xi}_{\lesssim \|f_3\|^2_{L^2_{\xi,\xi}}}$

provided $\left\| \frac{|\xi|^{3/2}}{\langle\xi-\xi^3\rangle^{-2b'}\langle 4\xi^3-3|\xi|^3-4\xi\rangle^{1/2}} \right\|_{L^\infty_{\xi,\xi}} = \left\| \frac{\xi}{\langle\xi-\xi\rangle^{-4b'}\langle 4\xi-3|\xi|-4\xi\rangle} \right\|_{L^\infty_{\xi,\xi}} \lesssim 1$

If $|\xi| \geq 2|\xi| \Rightarrow \frac{|\xi|}{\langle\xi-\xi\rangle^{-4b'}} \sim \frac{|\xi|}{\langle\xi\rangle^{-4b'}} \lesssim 1$ for $b' \leq -\frac{1}{4}$

$|\xi| \leq \frac{|\xi|}{2} \Rightarrow \frac{|\xi|}{\langle\xi-\xi\rangle^{-4b'}} \lesssim \frac{|\xi|}{\langle\xi\rangle^{-4b'}} \lesssim 1$ for $b' \leq -\frac{1}{4}$

$\frac{|\xi|}{2} \leq |\xi| \leq 2|\xi| \Rightarrow \frac{|\xi|}{\langle 4\xi-3|\xi|-4\xi\rangle} = \begin{cases} \frac{|\xi|}{\langle\xi-4\xi\rangle} \lesssim 1 & \text{if } \xi \geq 0 \\ \frac{|\xi|}{\langle 4\xi+7|\xi|\rangle} \lesssim 1 & \text{if } \xi < 0 \end{cases}$ ▨

→ global well posed b/c local well posed$^{\text{time}}$ & deterrance by L^2 norm of init data and L^2 norm conserved.

5/6/24 **Conservation laws** ("little cute" (historical approach, will look like "voodoo" b/c physicists derived w/o even knowing solutions ∃))

Rmk: If v is solution to **modified KdV** (mKdV)

$$v_t + v_{xxx} = 6v^2 v_x$$

@ 11:34 why ↑ nicer.

then $u := v^2 + v_x$ solves KdV!

$$u_t + u_{xxx} - 6uu_x = (2v + \partial_x)(v_t + v_{xxx} - 6v^2 v_x) = 0.$$

Amazingly, $\tilde{u} = v^2 - v_x$ **also** solves KdV!

$$\tilde{u}_t + \tilde{u}_{xxx} - 6\tilde{u}\tilde{u}_x = (2v - \partial_x)(v_t + v_{xxx} - 6v^2 v_x) = 0$$

⟶ Darboux transformation, "create solitons out of thin air!"

The mapping $v \longmapsto u := v^2 + v_x$ is called the **Miura transform**

sol to mKdV ⟶ sol to KdV

If could invert, would conclude that well-posedness theory for mKdV and KdV are the same.

But to invert Miura transform, want to find v s.t. $v^2 + v_x = u$.

Looks like Riccati eq, so inspired by theory of Riccati equations,

seek solutions of the form $v = \frac{\psi'}{\psi}$ for nowhere vanishing fn. ψ.

Then $u = v^2 + v_x = \left(\frac{\psi'}{\psi}\right)^2 + \frac{\psi''}{\psi} - \frac{(\psi')^2}{\psi^2} = \frac{\psi''}{\psi}$ i.e. $-\psi'' + u\psi = 0$.

$\iff (-\partial^2 + u)\psi = 0$

ψ is solution to Schrödinger eq,

i.e. formal eigenfn for operator w/ eigenvalue 0.

i.e. we want nowhere vanishing solution ψ to $(-\partial^2 + u)\psi = 0$.

Sadly, this doesn't exist even if u is really nice.

If u is of form  (negative Gaussian or whatever)

then $-\partial^2+u$ has negative eigenvalue. @ 11:41 cpct perturbation... essential spectrum $[0,\infty)$ so any negative... must automatically be eigenvalue.

$\langle(-\partial^2+u)f, f\rangle = \int |f'|^2+u|f|^2<0$

By Sturm-Liouville oscillation theory, ψ (which is formal eigenfunction of eigenvalue 0) must vanish at least once.

(u positive this isn't problem, but we want robust well posedness theory, not require silly restriction like init data >0 or whatever)

If instead we considered $(-\partial^2+u)\psi = -k^2\psi$. (1)

then we could find a nowhere vanishing sol. ψ provided $k = k(u) \gg 1$.
(k suff large depending on u).

e.g. if $k^2 > \|u\|_\infty$, then

$$\langle(-\partial^2+u+k^2)f, f\rangle = \int |f'|^2 + \underbrace{(u+k^2)}_{>0}|f|^2 > 0$$

so $\nearrow$ is positive definite operator.

Why does pos. def operator $\longrightarrow$ $\exists$ nowhere vanishing ψ satisfying (1)?

If $u \in C_c^\infty$, then (1) becomes $-\psi'' = -k^2\psi$ as $x \to \pm\infty$.

So $\exists$ a solution ψ s.t. $\psi(x) = e^{kx}$ as $x \to -\infty$.

Also by ODE theory, at ∞, (let's pretend we know $a \neq 0$ @ 11:50) $\psi(x) = ae^{kx} + be^{-kx}$ as $x \to +\infty$.

Note $\frac{\psi'}{\psi} = \frac{kae^{kx} - kbe^{-kx}}{ae^{kx} + be^{-kx}} \xrightarrow[x\to\infty]{} k$



In middle, such a solution can't vanish. Indeed if $\exists x_0 \in \mathbb{R}$ s.t. $\psi(x_0)=0$,

then $0 < \int_{-\infty}^{x_0} |\psi'|^2 + (u+k^2)|\psi|^2 = \int_{-\infty}^{x_0} (\psi\psi')' - \psi\psi'' + (u+k^2)|\psi|^2\,dx$
(0 b/c k large)

$$= \psi\psi'\Big|_{-\infty}^{x_0} + \int_{-\infty}^{x_0} \psi\cdot(-\partial^2+u+k^2)\psi\,dx$$

$$= 0 + 0 \quad \not\!\!\not$$

modified Miura transform

So can invert this, but so what?

Eqn (1) corresponds to seeking a solution of the form

$$w = \frac{\psi'}{\psi} - k.$$

to the eqn $u = w^2 + w_x + (\dots)$ (+2kw)

$$= \left(\frac{\psi'}{\psi}\right)^2 - 2k\frac{\psi'}{\psi} + k^2 + \frac{\psi''}{\psi} - \left(\frac{\psi'}{\psi}\right)^2 + 2k\frac{\psi'}{\psi} - 2k^2$$

choose so that these things in circle cancel.

$$= \frac{\psi''}{\psi} - k^2$$

$$\Leftrightarrow \quad -\psi'' + u\psi = -k^2\psi.$$

The mapping $w \longmapsto u = w^2 + w_x + 2kw$ is called the Gardner transform.

We saw that it is invertible if $k = k(w) \gg 1$.

What eq does w satisfy? Well of course, the Gardner eq!

$$w_t + w_{xxx} = \underbrace{6w^2 w_x}_{\text{"mKdV" term}} + \underbrace{12k\, w w_x}_{\text{"KdV term"}}$$

$$\text{i.e.} \quad \partial_t w + \partial_x(w_{xx} - 2w^3 - 6kw^2) = 0 \qquad (2)$$

Why Gardner interested in this? He wanted conservation laws for KdV.

$u = w^2 + w_x + 2kw.$

expand w as power series in $\frac{1}{k}$:

plug it in to itself!

$$w = \frac{1}{2k}(u - w^2 - w')$$

$$= \frac{1}{2k}u - \left(\frac{1}{2k}\right)^3 (u - w^2 - w')^2 - \left(\frac{1}{2k}\right)^2 (u - w^2 - w')' + \frac{1}{8k^3}(u - w^2 - w')''$$

$$= \frac{1}{2k}u - \frac{1}{4k^2}u' - \frac{1}{8k^3}(u^2 - u'') + O\left(\frac{1}{k^4}\right)$$

Plug this expansion into eqn (2):

we get infinitely many conservation laws! of the type: $\partial_t \underbrace{\rho}_{\text{density}} + \partial_x \underbrace{j}_{\text{"currents"}} = 0$

the first one is:

only keep the terms that give $\frac{1}{k}$

$$\frac{1}{2k}\partial_t u + \partial_x\left(\frac{1}{2k}u'' - 6k\frac{1}{(2k)^2}u^2\right) = 0$$

$$\Rightarrow \partial_t u + \partial_x(u'' - 3u^2) = 0, \quad \text{the KdV eqn!}$$

Integrating these microscopic conservation laws, like ($\int u(t,x)\,dx$ holds physical momentum) $\Rightarrow \int u(t,x)\,dx = \int u(0,x)\,dx$

macroscopic conservation laws

But microscopic conservation laws more powerful than macroscopic, b/c can integrate micro laws w/ all sorts of fns!

↳ cf. previous 2SLR how we derived virial & Hamiltonian.

The next two are:

$$\partial_t(u^2) + \partial_x(-4u^3 + 2uu_{xx} - u_x^2) = 0$$

$$\partial_t\left(u^3 + \tfrac{1}{2}u_x^2\right) + \partial_x\left(-\tfrac{9}{2}u^4 + 3u^2u_{xx} - 6uu_x^2 + u_xu_{xxx} - \tfrac{1}{2}u_{xx}^2\right) = 0$$

(linearly independent, b/c each next density or current involves higher and higher derivatives)

known to KdV

$$\Rightarrow \int u(t,x)^2\,dx = \int u_0(x)^2\,dx$$

Whitham 1965

$$\int u^3(t,x) + \tfrac{1}{2}(u'(t,x))^2\,dx = \int u_0^3 + \tfrac{1}{2}(u')^2\,dx$$

"energy", or Hamiltonian. KdV is Hamiltonian flow!

2 more conservation laws were discovered by Kruskal–Zabusky 1967

4 . — — — — in quick succession.

Miura spent a week discovering the tenth.

Computer program for the 11th.

The above $\frac{1}{k}$ expansion is method of Gardner that produces all these, and infinitely many more.

↻ 12:15 on different "definitions" of complete integrability

Rmk: KdV is Galilean invt (mKdV is NOT)

meaning if u is sol to KdV, then $\tilde{u}(t,x) = u(t, x+6ht) + h$

is also sol to KdV.

check: $\tilde{u}_t + \tilde{u}_{xxx} - 6\tilde{u}\tilde{u}_x = u_t + 6hu_x - 6(u+h)u_x = u_t + u_{xxx} - 6uu_x = 0.$

Taking $h = k^2$, $u = 0$, we get a solution to KdV $\tilde{u} = k^2$.

Let $\psi = \cosh(kx) \Rightarrow v = \frac{\psi'}{\psi} = k\tanh(kx)$.

then $v^2 + v_x = \frac{\psi''}{\psi} = k^2 = u$

Compute $v^2 - v_x = 2\left(\frac{\psi'}{\psi}\right)^2 - \frac{\psi''}{\psi} = 2k^2\tanh(kx)^2 - k^2$

$= 2k^2(\tanh^2(kx) - 1) + k^2$

$= k^2 - \frac{2k^2}{\cosh^2(kx)}$ (to cancel)

Undo the Galilean transform, $h = -k^2$,

we get a solution to KdV $\tilde{u}(t,x) = \frac{-2k^2}{\cosh(k(x - 12k^2t))}$

Start w/ 0 solution → boosted water level $+k^2$, $\cosh(kx)$ is crazy that when makes $v+v_x$ is constant.

↓

Darboux transform $v+v_x$ to $v-v_x$

← miraculously end w/ soliton solution!

5/8/24

Poisson structure and Hamiltonian flows

Def: Poisson str is a mapping $\{\cdot,\cdot\}: A \times A \to A$ (A an algebra) that satisfies the algebraic properties

Example: A = set of smooth functions on manifold

1) antisymmetry $\{F,G\} = -\{G,F\}$.

2) Linearity in both entries $\{aF+bG, H\} = a\{F,H\} + b\{G,H\}$.
(2nd entry follows by (1)).

3) Jacobi identity $\{\{F,G\},H\} + \{\{H,F\},G\} + \{\{G,H\},F\} = 0$.
"cycle last position to 1st position"

4) Leibniz rule $\{FG,H\} = \{F,H\}G + F\{G,H\}$
"differentiate FG in direction H"

Examples: 1) $A = M_{n\times n}$ matrices

$\{F,G\} := [F,G]$ commutator $FG-GF$.

2) $A = C^\infty(\underbrace{\mathbb{R}^n_x \times \mathbb{R}^n_p}_{\mathbb{R}^{2n}})$ closer to finite dim DOE

$\mathbb{R}^{2n}$ split into 2 groups, position & momentum

$$\{F,G\} := \sum_{j=1}^{n} \frac{\partial F}{\partial x_j}\frac{\partial G}{\partial p_j} - \frac{\partial F}{\partial p_j}\frac{\partial G}{\partial x_j}$$

Note that $(x_j, p_j)_{1\le j\le n}$ define canonical coordinates, i.e. (def of "canonical coord")

$\{x_j, x_k\} = 0$ (b/c $\frac{\partial x}{\partial p} = 0$), and similarly $\{p_j, p_k\} = 0$

But $\{x_j, p_k\} = \delta_{jk}$.

$u \in \mathcal{H} \longrightarrow [v \mapsto \lim_{\varepsilon\to 0} \frac{F(u+\varepsilon v)-F(u)}{\varepsilon}] \in \mathcal{H}^*$ ↔ representative given by Riesz repr., $\frac{\delta F}{\delta u} \in \mathcal{H}$

$\mathcal{H}^* \ni$ I denote as $\frac{\delta F}{\delta u}$

i.e. $\frac{\delta F}{\delta u} = \langle \frac{\delta F}{\delta u}, \cdot \rangle_{\mathcal{H}}$, $\frac{\delta F}{\delta u} \in \mathcal{H}$, $\cdot : \mathcal{H} \to \mathbb{R}$

so the map $[u \mapsto \frac{\delta M}{\delta u}]$ $\mathcal{H} \to \mathcal{H}$

3) A is set of smooth real valued fns on complex Hilbert space $\mathcal{H}$

and $\{F,G\} := \mathrm{Im}\langle \frac{\delta F}{\delta u}, \frac{\delta G}{\delta u}\rangle_{\mathcal{H}}$

($\frac{\delta F}{\delta u}$: conjugate linear in 1st entry; $\frac{\delta G}{\delta u}$: linear in 2nd entry)

$\frac{\delta F}{\delta u}$ functional derivative $\mathcal{H} \to \mathcal{H}$ (so better notation $\frac{\delta F}{\delta \bar{u}}$?)

meaning

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} F(u+\varepsilon v) =: \mathrm{Re}\langle \frac{\delta F}{\delta u}, v\rangle$$

"directional derivative in direction v"

"actually easy though notationally dense"

Example: $\mathcal{H} = L^2(\mathbb{R})$, $F[u] := \frac{1}{2}\int |u(x)|^2 dx$.

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} F(u+\varepsilon v) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \frac{1}{2}\int |u(x)|^2 + 2\varepsilon\,\mathrm{Re}(\bar{u}v)(x) + \varepsilon^2|v(x)|^2\,dx$$

$$= \mathrm{Re}\int (\bar{u}v)(x)\,dx \Rightarrow \boxed{\frac{\delta F}{\delta u} := u.}$$

inspired by (3) but generalizing (2) to infinite dim

3') complex Hilbert space $\mathcal{H} := \ell^2(\mathbb{Z}, \mathbb{C})$

$\mathcal{A} = C^\infty(\mathcal{H}, \mathbb{R})$ (smooth functions on Banach space $F: E \to \mathbb{R}$: $\lim_{t\to 0} \frac{F(v+tw)-F(v)}{t}$)

For $u \in \mathcal{H}$, write $u =: (u_j)_{j\in\mathbb{Z}} =: (x_j + ip_j)_{j\in\mathbb{Z}}$. Then for $F: \mathcal{H} \to \mathbb{R}$ (smooth),

By chain rule $\frac{\delta F}{\delta u}: \mathcal{H} \to \mathcal{H}$ has jth entry $\in \mathbb{C} = \mathcal{H} = \ell^2(\mathbb{Z})$, $\left(\frac{\delta F}{\delta u}\right)_j := \frac{\partial F}{\partial x_j} + i\frac{\partial F}{\partial p_j} : \mathcal{H} \to \mathbb{Z}$

So $\{F, G\} := \operatorname{Im} \left\langle \frac{\delta F}{\delta u}, \frac{\delta G}{\delta u} \right\rangle_{\ell^2(\mathbb{Z},\mathbb{C})}$ (conjugate-lin!)

$= \sum_{j\in\mathbb{Z}} \frac{\partial F}{\partial x_j}\frac{\partial G}{\partial p_j} - \frac{\partial F}{\partial p_j}\frac{\partial G}{\partial x_j}$

} Poisson bracket responsible for Schrödinger eqs.

$u: \mathbb{R}^d \to \mathbb{C}$

$\{\operatorname{Re} u(x), \operatorname{Im} u(x)\}_{x\in\mathbb{R}^d}$

canonical coords indexed over uncountable

Phase space is $(x_j, p_j)_{j\in\mathbb{Z}} = (\operatorname{Re} u_j, \operatorname{Im} u_j)_{j\in\mathbb{Z}}$

↑ canonical coordinates

---

4) Poisson bracket for kdV.

$\mathcal{H} = L^2(\mathbb{R}, \mathbb{R})$ $\quad \mathcal{A} = C^\infty(L^2(\mathbb{R},\mathbb{R}), \mathbb{R}) \Rightarrow F, G$

$\{F, G\}_{(u)\in\mathcal{H}} := \int \frac{\delta F}{\delta u}(x) \left(\frac{\delta G}{\delta u}\right)'(x)\, dx$ ← pretend can take derivative of $\frac{\delta G}{\delta u} \in L^2(\mathbb{R},\mathbb{R})$. (Since $\frac{\delta G}{\delta u}$ is probably pretty nice, hard to prove it $\frac{\delta G}{\delta u}$ is smooth or etc.)

also pretend the result is smooth $\mathcal{H} \to \mathbb{R}$.

where again functional derivative $\frac{\delta F}{\delta u} := \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} F(u+\varepsilon v) = \int \frac{\delta F}{\delta u}(x)\cdot v(x)\, dx$ (Riesz representation?)

$\left[v \to \lim_{\varepsilon\to 0} \frac{F(u+\varepsilon v)-F(u)}{\varepsilon}\right]$

To find canonical coords, we compute

$\{F, G\} = \int \frac{\delta F}{\delta u}(x) \left(\frac{\delta G}{\delta u}\right)'(x)\, dx = \int \widehat{\frac{\delta F}{\delta x}}(\xi) \cdot \widehat{\left(\frac{\delta G}{\delta u}\right)'}(\xi)\, d\xi$ (Plancherel to Fourier side)

$= \int \widehat{\frac{\delta F}{\delta x}}(\xi) \cdot (-i\xi)\widehat{\left(\frac{\delta G}{\delta u}\right)}(\xi)\, d\xi$

$= \int_0^\infty \widehat{\frac{\delta F}{\delta u}}(\xi) \cdot (-i\xi)\widehat{\frac{\delta G}{\delta u}}(\xi)\, d\xi + \int_0^\infty \widehat{\frac{\delta F}{\delta u}}(-\xi)\cdot(i\xi)\overline{\widehat{\frac{\delta G}{\delta u}}(-\xi)}\, d\xi$ (variable change $\xi \to -\xi$)

$= \int_0^\infty \widehat{\frac{\delta F}{\delta u}}(\xi)\cdot(i\xi)\overline{\widehat{\frac{\delta G}{\delta u}}(\xi)}\, d\xi$

$= 2\operatorname{Re}\int_0^\infty \widehat{\frac{\delta F}{\delta u}}(\xi)\cdot(-i\xi)\overline{\widehat{\frac{\delta G}{\delta u}}}(\xi)\, d\xi$

$= 2\operatorname{Im}\int_0^\infty \xi\, \widehat{\frac{\delta F}{\delta u}}(\xi)\cdot\overline{\widehat{\frac{\delta G}{\delta u}}}(\xi)\, d\xi$

$$= 2 \int_0^\infty \xi \cdot \left[-\mathrm{Re}\, \widehat{\frac{\delta F}{\delta u}}(\xi) \cdot \mathrm{Im}\, \widehat{\frac{\delta G}{\delta u}}(\xi)\, d\xi + \mathrm{Im}\, \widehat{\frac{\delta F}{\delta u}}(\xi) \cdot \mathrm{Re}\, \widehat{\frac{\delta G}{\delta u}}(\xi)\, d\xi \right] d\xi$$

canonical coord given by pieces of Fourier transform.

So the canonical coordinates are

$$C_\xi(u) = \frac{1}{\sqrt{2\pi}} \int \cos(x\xi)\, u(x)\, dx \qquad \xi > 0.$$

$$S_\xi(u) = \frac{1}{\sqrt{2\pi}} \int \frac{\sin(x\xi)}{\xi}\, u(x)\, dx$$

need to divide by 1 ξ, decide to put it w/ the sin.

So e.g. Poisson bracket of position (indexed by $\xi > 0$) & momentum (indexed by $\eta > 0$)

differentiate wrt x

$$\{C_\xi(u), S_\eta(u)\} = \int \frac{\delta C_\xi}{\delta u} \left(\frac{\delta S_\eta}{\delta u}\right)' = \frac{1}{2\pi} \int \cos(x\xi) \left(\frac{\sin(x\eta)}{\eta}\right)' dx$$

$$= \frac{1}{2\pi} \int \cos(x\xi) \cos(x\eta)\, dx$$

$$= \delta_0(\xi - \eta)$$

interpreted in sense of distributions

write as $e^{ix\xi}$ $e^{ix\eta}$, integrate against Schwarz, get Fourier transform

Exc: $\{C_\xi(u), C_\eta(u)\} = 0 \quad \forall \xi, \eta > 0.$

same as S.

12:35pm.

---

e.g. one for Schrödinger, or one for KdV.

Def: given a Poisson bracket, and real-valued Hamiltonian H (abstract element of algebra)

we define a vector field X_H w/ $X_H(f) := \{f, H\}$.

$: \mathcal{H} \to \mathcal{H}$

Lemma: $\{F, G\} = -[X_F, X_G]$ (commutator)

i.e. 2 Hamiltonians that Poisson-commute $\Rightarrow$ vector fields commute!

pf: symbol pushing.

$$-[X_F, X_G](f) = -X_F X_G(f) + X_G X_F(f)$$

$$= -\{X_G(f), F\} + \{X_F(f), G\}$$

$$= -\{\{f, G\}, F\} + \{\{f, F\}, G\}$$

$= -\{\{F, f\}, G\}$ (antisym)

$= + \{\{G, F\}, f\}$ (Jacobi id)

$= \{f, \{F, G\}\}$ (antisym)

cycle!

G-cycle to beginning

by definition, $X_{\{F,G\}}(f)$.

Def: given Poisson bracket and real-valued Hamiltonian H. (abstract element of algebra)

the associated Hamiltonian flow (flow generated by H)

$$\frac{d}{dt} u = X_H(u) := \{u, H\}.$$

Example: $A = C^\infty(\mathbb{R}^n_x \times \mathbb{R}^n_p)$, recall $\{F,G\} := \sum_{j=1}^n \frac{\partial F}{\partial x_j}\frac{\partial G}{\partial p_j} - \frac{\partial F}{\partial p_j}\frac{\partial G}{\partial x_j}$

Let $H(\vec{x},\vec{p}) := \frac{1}{2m}|\vec{p}|^2 + V(\vec{x})$

Kinetic energy

Then evolution under Hamiltonian?

only term that contribute

$$\begin{cases} \frac{d}{dt} x_j := \{x_j, H\} = \frac{\partial H}{\partial p_j} = \frac{1}{m} p_j \\ \frac{d}{dt} p_j := \{p_j, H\} = -\frac{\partial H}{\partial x_j} = -\frac{\partial}{\partial x_j} V(\vec{x}). \end{cases}$$

(Newton's equations).

is the flow associated to H

Example KdV: use Poisson bracket from Ex 4 — 4) Poisson bracket for KdV.

$\mathcal{H} = L^2(\mathbb{R}, \mathbb{R})$ $\quad F, G : \mathcal{H} \to \mathbb{R}$ smooth for $\mathcal{A}$ algebra

$$\{F,G\} := \int \frac{\delta F}{\delta u}(x) \left(\frac{\delta G}{\delta u}\right)'(x)\, dx.$$

$\{F,G\}$: smooth to $\mathcal{H} \to \mathbb{R}$, $\mathcal{A}$

$\left(\frac{\delta G}{\delta u}\right)'$: this is derivative

Claim: Hamiltonian flow associated to H is

$$\frac{d}{dt} u = \left(\frac{\delta H}{\delta u}\right)'$$

Indeed, if $G(u) := u(x)$ (functional; "point where we can evaluate L^2 fun. Hilbert space. e.g.") then $\frac{\delta G}{\delta u} := \delta_x$

$$\Rightarrow \frac{d}{dt} u(t,x) := \{u(x), H\} = \int \frac{\delta u}{\delta u}_m \left(\frac{\delta H}{\delta u}\right)'_m dx = \left(\frac{\delta H}{\delta u}\right)'(x).$$

The Hamiltonian flow corresponding to $P(u) = \frac{1}{2}\int |u(x)|^2 dx$

is $\frac{d}{dt} u = \left(\frac{\delta P}{\delta u}\right)' = u' \Rightarrow u(t,x) = f(t+x)$

This is why we call P the momentum (it generates translations)!

"mass" in Schrödinger but "momentum" here!

The Hamiltonian flow corresponding to $H(u) = \int \frac{1}{2}|u'(x)|^2 + u^3(x)\, dx$

is $\frac{d}{dt} u = \left(\frac{\delta H}{\delta u}\right)' = \partial_x(-u'' + 3u^2)$

$\frac{1}{2}(u'+\varepsilon v')^2 = \frac{1}{2}u'^2 + \varepsilon u'v' + \frac{\varepsilon^2}{2}v'^2$; $(u+\varepsilon v)^3 = u^3 + \varepsilon(\cdots)v + \frac{\varepsilon^2}{2}(\cdots)v^2$

$$= -u_{xxx} + 6uu_x$$

KdV!

(KdV realized as Hamiltonian flow wrt. Poisson str.

(the same Hamiltonian will give rise to NLS! wrt. Poisson str in Ex 3

8/10/24 **Recall** the Poisson bracket: $\{F,G\} := \int \frac{\delta F}{\delta u}(x) \left(\frac{\delta G}{\delta u}\right)'(x)\,dx$. (always space derivative)

Hamiltonian flow associated to Hamiltonian H is

$$\frac{d}{dt}u = \left(\frac{\delta H}{\delta u}\right)'$$

The Hamiltonian flow corresponding to $P(u) = \frac{1}{2}\int |u(x)|^2\,dx$

is $\frac{d}{dt}u = \left(\frac{\delta P}{\delta u}\right)' = u' \Rightarrow u(t,x) = f(t+x)$

This is why we call P the momentum (it generates translations)!

The Hamiltonian flow corresponding to $H(u) = \int \frac{1}{2}|u'(x)|^2 + u^3(x)\,dx$ (b/c generates KdV, call it the Hamiltonian for KdV.)

is $\frac{d}{dt}u = \left(\frac{\delta H}{\delta u}\right)' = \partial_x(-u'' + 3u^2)$ ($\frac{1}{2}(u'+\varepsilon v')^2$ has $\int u'v'$, $\int 2Pv'$, $\int (u')^3 v$)

$= -u_{xxx} + 6uu_x$ KdV! (KdV realized as Hamiltonian flow wrt Poisson str)

---

do a computation:

$$\{P,H\} := \int \frac{\delta P}{\delta u}(x)\cdot\left(\frac{\delta H}{\delta u}\right)'(x)\,dx$$

$$= \int u\cdot(-u''' + 6u')\,dx \overset{\text{plug in results above}}{=} \int u'u'' + 6u^2u'\,dx \overset{\text{using IBP}}{=} \int \partial_x\left[\tfrac{1}{2}(u'(x))^2 + 2(u(x))^3\right]dx$$

$$= 0$$

for nice enough fns u, P & H commute!

This has physical interpretation, but first a

Remark: Let H be Hamiltonian and u flow by this Hamiltonian $\frac{d}{dt}u = \left(\frac{\delta H}{\delta u}\right)'$ (*) (Hamiltonian flow corresponding to H.)

Given another Hamiltonian F (meaning element of algebra $\mathcal{A}$), how does $F(u(t))$ evolve in time?

$$\frac{d}{dt}F(u(t)) \overset{\text{chain rule}}{=} \left\langle \frac{\delta F}{\delta u}(u(t)), \frac{d}{dt}u \right\rangle = \left\langle \frac{\delta F}{\delta u}(u(t)), \left(\frac{\delta H}{\delta u}\right)'(u(t)) \right\rangle = \{F,H\}(u(t))$$

(Poisson bracket evaluated at?)

$$= (X_H F)(u(t))$$

vector field at the pt $u(t)$ on manifold.

Similarly, w/ ϕ_t denoting flow corresponding to (*),

i.e. $u_0 \longmapsto u(t)$ where $u(t)$ solution to $\begin{cases} \frac{d}{dt}u = \left(\frac{\delta H}{\delta u}\right)' \\ u(0) = u_0 \end{cases}$

we can also compute (u initial data u_0)

$$\frac{d}{dt}\widetilde{F\circ\phi_t} = \lim_{s\to 0}\frac{F\circ\phi_t\circ\phi_s - F\circ\phi_t}{s} = \frac{d}{ds}\Big|_{s=0}(\underbrace{F\circ\phi_t}_{\text{"F"}})\circ\phi_s = \{F\circ\phi_t, H\}$$

$\widetilde{F\circ\phi_t} = F(u(t))$

def of derivative

Thus we have written in 2 ways:

$$\underbrace{\{F,H\}\circ\phi_t}_{\text{Hamiltonian flow } (\star)} = \frac{d}{dt}F\circ\phi_t = \{F\circ\phi_t, H\}.$$

In plain, solutions to Hamiltonian flow ($\star$) conserve the Hamiltonian b/c

$$\frac{d}{dt}H(u(t)) = \{H,H\}(u(t)) = 0$$

anti-commute

bringing this back to thing above $\{P,H\}=0$,

we have also found that solutions to KdV also conserve the momentum!

(again, $\frac{d}{dt}P(u(t)) = \{P,H\}(u(t)) = 0$)

or equivalently!, the class of solutions to KdV is left-invariant under spatial translations.

(as above)

The following theorem explains this best, and in general (not just for Poisson bracket for KdV above)

Thm: Poisson commuting Hamiltonians yield commuting flows.

Pf: start w/ 2 Hamiltonians F, G that Poisson commute $\{F,G\}=0$.

Let X_F, X_G be vector fields corresponding to F, G.

Then, recalling final last time, $-[X_F, X_G] = X_{\{F,G\}} = 0$.

We want to prove: Let ϕ_t be flow corresponding to F,

ψ_s " " " " G.

WTS $\phi_t\circ\psi_s = \psi_s\circ\phi_t$ $\forall s,t$ (i.e. $\forall$ pts on manifold u_0, $\phi_t\psi_s u_0 = \psi_s\phi_t u_0$)

e.g. ϕ = KdV flow, ψ = translations, then it is the statement

We do one sample computation: (2nd order Taylor series)

Let f be a smooth functional on the manifold

$$\frac{\partial^2}{\partial s\partial t}\Big|_{(s,t)=(0,0)} f(\phi_t\psi_s u) = \frac{\partial}{\partial s}\Big|_{s=0}\underbrace{(X_F f)}_{\text{functional on manifold, the new type}}(\psi_s u) = (X_G(X_F f))(u)$$

taking derivative wrt. t using above fact

taking derivative wrt. s.

OTOH, $\frac{\partial^2}{\partial s\partial t}\Big|_{(s,t)=(0,0)} f(\psi_s\phi_t u) = \frac{\partial}{\partial t}\Big|_{t=0} (X_G f)(\phi_t u) = (X_F(X_G f))(u)$

(first deriv. w.r.t. s; again)

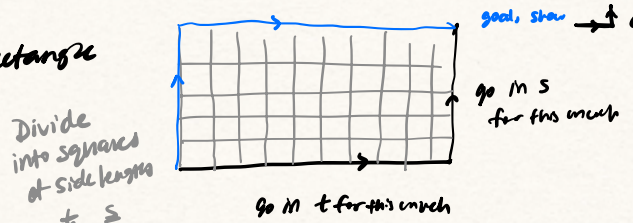
Thus, $\frac{\partial^2}{\partial s\partial t}\Big|_{(s,t)=(0,0)} f(\phi_t\psi_s u) - f(\psi_s\phi_t u) = ([X_G, X_F]f)(u) = 0.$

By Taylor Prod, for general smooth fn,

$$f(\phi_t\psi_s u) - f(\psi_s\phi_t u) = \underbrace{0}_{} + \underbrace{0}_{\text{1st derivatives}} + \underbrace{0}_{\text{2nd derivative}} + O(s^3+t^3)$$

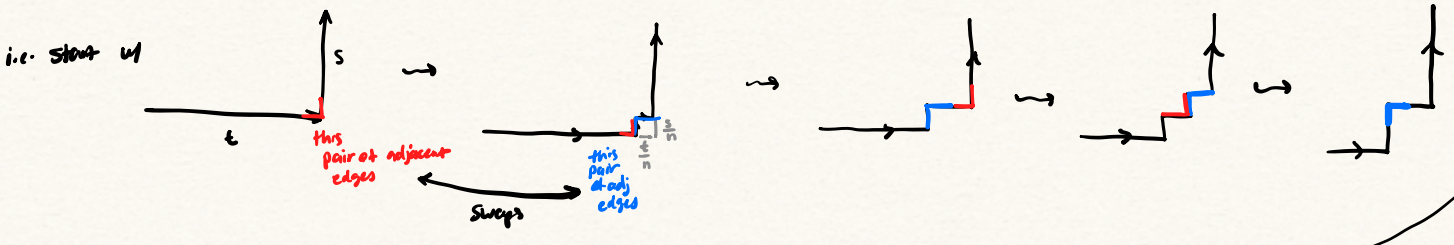
for s,t ∈ nbhd of $(0,0)$

Divide rectangle



Divide into squares of side length $\frac{t}{n} \times \frac{s}{n}$ (so n^2 many squares)

we express $\phi_t\psi_s - \psi_s\phi_t$ as n^2 many "telescopic" differences where each difference swaps one pair of adjacent edges of a small square by the other pair:



Note that each difference is of the order of $O(\frac{1}{n^3})$

But we only have n^2 many pieces (i.e. terms in sum!)

Letting $n\to\infty$, we get that $f(\phi_t\psi_s) = f(\psi_s\phi_t)$ ∀ smooth f.

$\Rightarrow \phi_t\psi_s = \psi_s\phi_t$.

more naive idea/pairs:
$\frac{d}{dt}\phi_t\psi_s(u) = X_F(\phi_t\psi_s u)$
$\frac{d}{dt}u = X_F(u) = \{u, F\}$

Prop: Hamiltonian flows preserve Poisson structure.

i.e. Let H be Hamiltonian, let ϕ_t denote its flow.
Say have 2 other Hamiltonians F, G
WTS $\{F\circ\phi_t, G\circ\phi_t\} = \{F,G\}\circ\phi_t$

PF: Recall already proved this if $G = H$: $\frac{d}{dt} F \circ \phi_t = \{F \circ \phi_t, H\}$ and we know $H = H \circ \phi_t$ b/c H preserved by flow.

$= \{F, H\} \circ \phi_t$

well, let's just calculate $\frac{d}{dt}\left[\{F\circ\phi_t, G\circ\phi_t\} - \{F,G\}\circ\phi_t\right]$

$= \{\{F\circ\phi_t, H\}, G\circ\phi_t\} + \{\{F\circ\phi_t, \{G\circ\phi_t, H\}\}\} - \{\{F,G\}\circ\phi_t, H\}.$

(Leibniz rule)

$\{G\circ\phi_t, H\} = -\{H, G\circ\phi_t\}$ (antisym)

$-\{\{G\circ\phi_t, H\}, F\circ\phi_t\}$

$= \{\{F\circ\phi_t, G\circ\phi_t\}, H\} + \{\{H, F\circ\phi_t\}, G\circ\phi_t\}.$ (Jacobi cycle.)

(antisym)

$= \{\{F\circ\phi_t, G\circ\phi_t\} - \{F,G\}\circ\phi_t, H\}$

$= X_H(\{F\circ\phi_t, G\circ\phi_t\} - \{F,G\}\circ\phi_t)$

So $\{F\circ\phi_t, G\circ\phi_t\} - \{F,G\}\circ\phi_t$ solves init value problem $\begin{cases} \frac{d}{dt} f = X_H(f) \\ f(0) = 0 \end{cases} \Rightarrow f(t) = e^{tX_H} f(0) = 0$

X_F vector field nice as can be

(i.e. uniqueness of sol of ODE, 0 is solution, hence is only solution)

We are now ready for **Lax Pair Formulation of KdV**

Main objects of study for remainder of the quarter:

Lax operator $L(t,x)f = (-\partial_x^2 + q(t,x))f$ (operator acting on f) self adjoint ($q: \mathbb{R}\times\mathbb{R} \to \mathbb{R}$)

"Peter" operator $P(t)f = (-4\partial_x^3 + 3(q\cdot\partial_x + \partial_x q\cdot))f$ antiself adjoint.

and Lax pair formulation to KdV is

q solves KdV $\iff \frac{d}{dt} L(t) = [P(t), L(t)]$.

5/13/24

Lax pair formulation of KdV

Introduced last time,

$$L(t) = -\partial_x^2 + q(t,x)$$

(only dependence on time)

$$P(t) = -4\partial_x^3 + 3(q\partial + \partial q) = -4\partial_x^3 + 6q\partial + 3q'.$$

(q multiply by q operator; ∂ w/o subscript is ∂_x)

as we are only interested in real-valued q, L is self adjoint, P is antiself adjoint

we have $\frac{d}{dt}L(t) = [P(t), L(t)] \iff q$ solves KdV.

pf: $\frac{d}{dt}L(t) - [P(t), L(t)] = q_t - [-4\partial^3 + 6q\partial + 3q', -\partial^2 + q]$

(exc. in expanding commutator)

$$= q_t - \{[-4\partial^3, -\partial^2+q] + 6[q^2, -\partial^2+q] + 3[q', -\partial^2+q]\}$$

(b/c ∂^3 and ∂^2 commute; Leibniz rule $[AB,C] = [A,C]B + A[B,C]$; b/c mult by q and mult by q' commute)

$$= q_t - \{-4(q''' + 3q''\partial + 3q'\partial^2 + 0) + 6[q, -\partial^2+q]\partial + 6q[\partial, -\partial^2+q] + 3(q''' + 2q''\partial + 0)\}$$

(all ∂^3 hit f, commutator doesn't see it; mult by q and mult by q commute; ∂ and ∂^2 commute)

$$= q_t - \{-4q''' - 12q''\partial - 12q'\partial^2 + 6(q'' + 2q'\partial)\partial + 6qq' + 3q''' + 6q''\partial\}$$

$$= q_t - \{-q''' + 6qq'\}$$

$$= q_t + q''' - 6qq'.$$

---

restrict for now to finite dim setting to explain why Lax pair formulation useful.

Rmk: If $L(t), P(t)$ were finite dimensional matrices s.t. $P(t)$ is antiselfadjoint and $\frac{d}{dt}L(t) = [P(t), L(t)]$, then the solutions to the initial value problem $\begin{cases} \frac{d}{dt}U(t) = P(t)U(t) \\ U(0) = Id \end{cases}$ would be unitary.

Indeed: $\frac{d}{dt}(U^*(t)U(t)) = [P(t)U(t)]^* U(t) + U^*(t)P(t)U(t) = U^*(t)\underbrace{[P^*(t) + P(t)]}_{=0 \text{ by antiselt adj}}U(t) = 0$

$\Rightarrow U^*(t)U(t) = U^*(0)U(0) = Id \Rightarrow U$ is an isometry

$\Rightarrow U$ is unitary

Antiselfadjoint matrices: Lie algebra for unitary matrices...?

In infinite dim, the computation above still $\rightsquigarrow$ U is isometry, though can not immediately conclude unitarity.

"explained by Fredholm theory".

Unitarity of $U(t,s)$ follows from the well-posedness theory of $\frac{d}{dt}\psi = P(t)\psi$

Then $U(t,s)$ takes $\psi(s)$ and outputs $\psi(t)$ "fundamental solution for this problem"

Good uniqueness of

gives $U(t,s)\,U(s,t) = Id$

can read out adjoint of $U(t,s)$.

time derivative hits U^* time der. hits L time der. hits U

Going back to finite dim. $\frac{d}{dt}(U^* L\, U)(t) = U^* P^* L U + U^*[P,L]U + U^* L P U$

$= U^*[-PL + PL - LP + LP]U = 0.$

So $U^*(t)\,L(t)\,U(t) = U^*(0)\,L(0)\,U(0) = L(0)$.

$\Rightarrow$ $\det((U^*LU)(t) - \lambda) = \det(L(0) - \lambda)$

characteristic poly

$\Rightarrow$ $\det(L(t) - \lambda) = \det(L(0) - \lambda)$ $\Rightarrow$ eigenvalues of $L(0)$ preserved as time t

U unitary

Spectrum of L preserved by flow.

14:43 for some back and forth about these topics.

Back in infinite dim,

Would like to build the "characteristic polynomial" of $L(t) = -\partial^2 + q(t)$

"Fredholm $\det(-\partial^2 + q + k^2)$"

Sadly this doesn't make sense even if $q \equiv 0$.

E.g. on 1 dim torus $\mathbb{T}$, can compute eigenvalues of $-\partial^2 + k^2$ explicitly: $\{n^2 + k^2\}_{n \in \mathbb{Z}}$

b/c $(-\partial^2 + k^2)\, e^{inx} = (n^2 + k^2)\, e^{inx}$

So $\det(-\partial^2 + k^2) = \prod_{n \in \mathbb{Z}} (n^2 + k^2) = \infty$.

However, if we <u>normalize</u> correctly, do get something that converges.

By comparison $\prod_{n \in \mathbb{Z}\setminus\{0\}} \frac{n^2+k^2}{n^2} = \frac{1}{\pi k^2}\sinh(\pi k^2)$

Inspired by this, ppl lead to <u>perturbation determinants</u>

written poetically as: $\det\left(\frac{-\partial^2+q^2+k^2}{-\partial^2+k^2}\right)$

Here $\frac{1}{-\partial^2+k^2}$ stands for the resolvent $R_0(k) = (-\partial^2+k^2)^{-1}$.

↑ correspond to $q=0$

On Fourier side, $\widehat{R_0(k)f}(\xi) = \frac{1}{\xi^2+k^2}\hat{f}(\xi)$ → inverse FT.

$$[R_0(k)f](x) = \frac{1}{2k}\int e^{-k|x-y|} f(y)\,dy$$

Now ambiguity of if $R_0(k)$ acting on left or right?

i.e. Can interpret $\det\left(\frac{-\partial^2+q+k^2}{-\partial^2+k^2}\right)$ as $\det(1+R_0 q)$ or as $\det(1+qR_0)$.

Note $qR_0(k)$, $R_0(k)q$ is bdd map $L^2 \to L^2$ if $q \in L^2_{loc}$.

adjoints b/c q real-valued

14:56 long story on history of KdV problem pushing down to H^{-1}. (H^{-1} first done on Torus, using Birkhoff coordinates)

Ultimately want potential q in H^{-1}. The above interpretations of $\det(\ldots)$ do not work.

Instead of R_0 on left <u>or</u> right, we do R_0 on <u>left and</u> right (by splitting it in half):

→ interpret as $\det\left(1+\sqrt{R_0(k)}\,q\,\sqrt{R_0(k)}\right)$

We'll show $\sqrt{R_0}\,q\,\sqrt{R_0}$ is <u>Hilbert Schmidt</u> iff $q \in H^{-1}$.

Def: Let A is an operator of L^2 w/ integral kernel $K(x,y)$

$$(Af)(x) = \int K(x,y) f(y)\, dy$$

We say that A is **Hilbert-Schmidt** if $\iint |K(x,y)|^2 dx\, dy < \infty$ $\quad = \|K\|_{L^2_{x,y}} =: \|A\|_{HS}$

(Lin finite-dim, $K(x,y)$ is matrix, A is mult by matrix, and $\|\cdot\|_{HS}$ is sum of squares of all entries)

Note $\|A\|_{op} = \sup_{\|f\|=\|g\|=1} |\langle Af, g\rangle| \geq \sup_{f,g} \left| \iint K(x,y) f(y)\, dy\, \bar{g}(x)\, dx \right|$

$$\leq \sup_{\substack{g \\ \text{CBS in} \\ \text{variable } x}} \underbrace{\|g\|_2}_{=1} \left\| \int K(x,y) f(y)\, dy \right\|_{L^2_x}$$

$$\leq \sup_{\substack{f \\ \text{CBS} \\ \text{in var } y}} \underbrace{\|f\|_2}_{=1} \|K(x,y)\|_{L^2_{x,y}} =: \|A\|_{HS}.$$

Rmk: A is Hilbert Schmidt $\Rightarrow$ A is cpct operator, i.e. $A(B_{0,1}(L^2))$ is precompact in L^2.

How to show set in L^2 is precompact?

Recall in 247, Riesz thm (for all L^p!) says this equiv to $A(B_{0,1})$ being bdd, tight, equicts

Bdd: Let $\|f\|_2 < 1$. Then $\|Af\|_2 \leq \|A\|_{op} \|f\|_2 \leq \|A\|_{HS}$.

Tight: (very little L^2 norm in spatial infinity) unif in f.

$$\|Af\|_{L^2(|x|>R)} \leq \|K\|_{L^2_y L^2_x (|x|\geq R)} \|f\|_2 < \varepsilon \quad \text{for } R = R(\varepsilon) \gg 1 \text{ indep of } f.$$

Equicts: $\|(Af)(x+\delta) - (Af)(x)\|_{L^2_x} \leq \|K(x+\delta, y) - K(x,y)\|_{L^2_y L^2_x} \|f\|_2 < \varepsilon$ for $\delta = \delta(\varepsilon) \ll 1$ b/c translations cts in L^2.

Def: Let A be operator on L^2 w/ cts integral kernel.

The **trace** of A is $\operatorname{tr}(A) := \int K(x,x)\, dx$

"integrate along diagonal".

Rmk: if A is HS w/ integral kernel $K(x,y)$,

the kernel of A^* (adjoint) is $\overline{K(y,x)}$: $(A^*f)(x) = \int \overline{K(y,x)}\, f(y)\, dy$.

The kernel of A^*A is: ($x \xleftarrow{z}$ "takes", $z \xleftarrow{y}$ "takes")

$$\int \overline{K(z,x)}\, K(z,y)\, dz.$$

So $\operatorname{tr}(A^*A) = \iint |K(z,x)|^2\, dz\, dx =: \|A\|_{HS}^2$

In practice this is how we compute HS norm (as trace of A^*A).

5/15/24 Recall: an operator $A: L^2 \to L^2$ is Hilbert-Schmidt (HS) (w/ integral kernel $K(x,y)$)

if $\|A\|_{HS}^2 := \|K\|_{L^2_{x,y}}^2 < \infty$.

we saw such A are cpct, and had trace finite $\text{tr}(A^*A) = \|A\|_{HS}^2$.

Thm: if A cpct operator on Hilbert space $\mathcal{H}$ (for us, L^2)

then A^*A is nonneg operator (meaning quadratic form $\langle A^*Af, f\rangle \geq 0 \ \forall f \in \mathcal{H}$)

selfadjoint

and cpct

Then by spectral thm, $\exists$ ON-basis $\{\varphi_n\}$ of $\mathcal{H}$ consisting of eigenvectors of A^*A.

Let $\sigma_n := \sigma_n(A)$ denote singular values of A

$=$ eigenvalues of A^*A, sqrt

i.e. $A^*A\varphi_n = \sigma_n^2 \varphi_n$

$\Rightarrow AA^*(A\varphi_n) = \sigma_n^2 (A\varphi_n)$

i.e. σ_n also singular values for A^* w/ eigenvector $A\varphi_n$. for $\sigma_n \neq 0$.

Observe $\langle A\varphi_n, A\varphi_m\rangle = \langle A^*A\varphi_n, \varphi_m\rangle = \langle \sigma_n^2 \varphi_n, \varphi_m\rangle = \sigma_n^2 \delta_{n=m}$.

(note $\sigma_n = 0 \Rightarrow A\varphi_n = 0$)

Discarding the $\sigma_n = 0$, we define $\psi_n := \frac{A\varphi_n}{\sigma_n}$.

These ψ_n form ON-set and $A\varphi_n = \sigma_n \psi_n$

$A^*\varphi_n = \sigma_n \varphi_n$

and $AA^*\psi_n = \sigma_n^2 \psi_n$

Thus the nonzero singular values of A are precisely the nonzero singular values of A^* (reversing argument to get reverse direction)

Using that $\{\varphi_n\}$ is a basis for $\mathcal{H}$,

$$Af = A\left(\sum \langle \varphi_n, f\rangle \varphi_n\right) = \sum \langle \varphi_n, f\rangle A\varphi_n = \sum \sigma_n \langle \varphi_n, f\rangle \psi_n.$$

i.e. can write our cpct operator A as $\sum \sigma_n \langle \varphi_n, \bullet\rangle \psi_n$ and $A^* = \sum \sigma_n \langle \psi_n, \bullet\rangle \varphi_n$.

and $\mathrm{tr}(A) := \sum \langle A e_n, e_n \rangle = \sum \sigma_n \langle \psi_n, e_n \rangle$

on basis independent property of trace!

Def: For $1 \le p < \infty$, the **Schatten class** $\mathcal{I}_p$ ("ideal") is defined as

$$\mathcal{I}_p := \{ A \text{ cpct} : \|A\|_{\mathcal{I}_p}^p := \sum |\sigma_n(A)|^p < \infty \}$$

(only nonzero σ_n contribute).

Examples: If A cpct, $\|A\|_{\mathcal{I}_2}^2 := \sum |\sigma_n(A)|^2 = \sum \langle A^* A e_n, e_n \rangle = \mathrm{tr}(A^*A) = \|A\|_{HS}^2$

recall $A^*A e_n = \sigma_n^2 e_n$

Example: Fix $1 \le p < \infty$. If $A \in \mathcal{I}_p$, B bdd operator,

$AB, BA \in \mathcal{I}_p$. (i.e. ideal in {bdd operators}).

In fact even stronger, $\sigma_n(BA) \le \|B\|_{op} \sigma_n(A)$ (*)

In ptcl, $\|BA\|_{\mathcal{I}_p} \le \|B\|_{op} \|A\|_{\mathcal{I}_p}$.

"$\ell^p \le \ell^\infty \cdot \ell^p$"

↳ there is actually a Hölder inequality, but we won't talk about it.

Note that $\sigma_n(AB) = \sigma_n(B^*A^*) \overset{(*)}{\le} \|B^*\|_{op} \sigma_n(A^*) = \|B\|_{op} \sigma_n(A)$.

How to prove (*)?

min max characterization of singular values:

$$\sigma_n(BA) = \min_{f_1 \cdots f_{n-1} \text{ LI}} \max_{g \perp \mathrm{span}\{f_1 \cdots f_{n-1}\}, \|g\|=1} \|BAg\|.$$

$$\le \|B\|_{op} \|Ag\|,$$

then use min max characterization to recover $\sigma_n(A)$

Remark: Last time we saw $\|A\|_{HS} \ge \|A\|_{op}$.

Here, $\forall 1 \le p < \infty$,

$\|A\|_{op} \le \|A\|_{\mathcal{I}_p}$. ($\uparrow$ is $p=2$).

Pf: $\|A\|_{op} = \sup_{\substack{\|f\|=1 \\ \|g\|=1}} |\langle Af, g\rangle|$. But recall $A = \sum \sigma_n \langle \varphi_n, \cdot \rangle \psi_n$.

$$= \sup_{f,g} \left| \sum \sigma_n \langle \varphi_n, f\rangle \langle \psi_n, g\rangle \right|$$

$$\leq_{\text{Hölder}} \sup_{f,g} \|\sigma_n\|_{\ell^p_n} \left\| \langle \varphi_n, f\rangle \langle \psi_n, g\rangle \right\|_{\ell^{p'}}$$

$$\leq \|\langle \varphi_n, f\rangle\|_{\ell^{2p}_{n=1}} \|\langle \psi_n, g\rangle\|_{\ell^{2p}_{n=1}}.$$

$$\leq \|\langle \varphi_n, f\rangle\|_{\ell^2} \|\langle \psi_n, g\rangle\|_{\ell^2} \leq \|f\| \cdot \|g\| = 1.$$

$$\leq \|\sigma_n\|_{\ell^p_n} \cdot 1 = \|A\|_{I_p}.$$

Lemma: Let A_j be HS operators on $L^2(\mathbb{R}^d)$ w/ integral kernels K_j:

then 1) true cyclic $\operatorname{tr}(A_1 A_2) = \operatorname{tr}(A_2 A_1)$. (caution, $[A_1, A_2] \in I_1 \not\Rightarrow \operatorname{tr}(A_1 A_2) = \operatorname{tr}(A_2 A_1)$; need $A_1 A_2$ and $A_2 A_1$ separately in trace class I_1)

2) $|\operatorname{tr}(A_1 \cdots A_\ell)| \leq \prod_{j=1}^{\ell} \|A_j\|_{I_2}$ $\forall \ell \geq 2$.

Pf: (1) follow from Fubini:

$$\operatorname{tr}(A_1 A_2) = \int \underbrace{\int K_1(x,y)}_{\text{1 end at } x} \underbrace{K_2(y,x)\, dy}_{\text{start at } x} dx \underset{\substack{K_1, K_2 \\ \text{in } L^1, \\ \text{Fubini}}}{=} \int\int K_2(y,x) K_1(x,y)\, dx\, dy = \operatorname{tr}(A_2 A_1).$$

(2) follows from CBS

$$|\operatorname{tr}(A_1 \cdots A_\ell)| = \left| \int_{\mathbb{R}^d} K_1(x_1, x_2) K_2(x_2, x_3) \cdots K_\ell(x_\ell, x_1)\, dx_1 \cdots dx_\ell \right|$$

[applies $A_2 \cdots A_{\ell-1}$ to the function $K_\ell(\cdot, x_1)$ (parameter) evaluated at x_2]

$$\leq_{\Delta\text{-ineq}} \int \left| \langle K_1(x_1, \cdot), A_2 \cdots A_{\ell-1} K_\ell(\cdot, x_1) \rangle_{x_2} \right| dx_1$$

$$\leq_{\text{CBS}} \int \|K_1(x,y)\|_{L^2_{x,y}} \|A_2 \cdots A_{\ell-1} K_\ell(\cdot, x_1)\|_{L^2_y}\, dx_1$$

$$\leq \|A_1 \cdots A_\ell\|_{op} \|K_\ell(y,x)\|_{L^2_{y,x}}$$

$$\leq_{\text{CBS}} \|K_1(x,y)\|_{L^2_{x,y}} \|K_\ell(y,x)\|_{L^2_{x,y}} \prod_{j=2}^{\ell-1} \|A_j\|_{op}$$

$\leq \|A_j\|_{HS}$

$\|K_1\|_{HS} \quad \|K_\ell\|_{HS}$

CBS requires 2 guys, everyone else can be in operator norm. ✓

Recall last time perturbation determinants

$$\det\left(\frac{-\partial^2+q+k^2}{-\partial^2+k^2}\right) = \det\left(1+\sqrt{R_0(k)}\, q\, \sqrt{R_0(k)}\right).$$

remember split $R_0(k)$ to deal w/ low regularity issue

Another issue, in order to define $\det(I+A)$ one needs eigenvalues of A summable $\Rightarrow$ $q \in L^1$.

We will address that issue later

where $R_0(k) = (-\partial^2+k)^{-1}$

$$(R_0(k) f)(x) := \frac{1}{2k}\int e^{-k|x-y|} f(y)\, dy.$$

and promised that $\|V\|_{HS} < \infty \Leftrightarrow q \in H^{-1}$.

Calculate:

$$\|\sqrt{R_0(k)}\, q\, \sqrt{R_0(k)}\|_{HS}^2 = \operatorname{tr}\left\{ \left(\sqrt{R_0(k)}\, q\, \sqrt{R_0(k)}\right)^* \left(\sqrt{R_0(k)}\, q\, \sqrt{R_0(k)}\right) \right\}$$

$$= \operatorname{tr}\left(\sqrt{R_0(k)}\, q\, \sqrt{R_0(k)}\, \sqrt{R_0(k)}\, q\, \sqrt{R_0(k)} \right)$$

self adjoint...

cycle

$$= \operatorname{tr}\left(R_0(k)\, q\, R_0(k)\, q \right)$$

$$= \iint \frac{e^{-k|x-y|}}{2k} q(y) \frac{1}{2k} e^{-k|y-x|} q(x)$$

bring back to x — end at y — start at x

so new starting pt is y.

$$= \frac{1}{k}\iint q(x) \frac{e^{-2k|x-y|}}{4k} q(y)\, dy\, dx$$

$R_0(2k)$ kernel!

$$= \frac{1}{k}\langle q, R_0(2k)\, q\rangle$$

$$= \frac{1}{k}\langle \hat{q}, \frac{1}{\xi^2+4k^2}\hat{q}\rangle \quad \text{Plancherel}$$

$$= \frac{1}{k}\int \frac{|\hat{q}(\xi)|^2}{\xi^2+4k^2}\, d\xi =: \frac{1}{k}\|q\|_{H_k^{-1}}^2$$

"vaguely rescaled H^{-1} norm"

Conclusion, $\sqrt{R_0(k)}\, q\, \sqrt{R_0(k)} \in \mathcal{I}_2 \Leftrightarrow \|q\|_{H_k^{-1}} < \infty \Leftrightarrow \|q\|_{H^{-1}} < \infty$.

Rmk. taking $q \geq 0$, $\sqrt{R_0(k)}\, q\, \sqrt{R_0(k)} \geq 0$ (in sense of quadratic forms).

$\Rightarrow$ trace of $\uparrow$ = trace class norm of $\uparrow$.

$$\operatorname{tr}\left(\sqrt{R_0(k)}\, q\, \sqrt{R_0(k)}\right) = \operatorname{tr}\left(R_0(k) q\right) = \int \frac{1}{2k} e^{-k|x-x|} q(x)\, dx = \frac{1}{2}\int q(x)\, dx$$

(end at x; start at x; cycle)

So $\det\left(1+\sqrt{R_0(k)}\, q\, \sqrt{R_0(k)}\right) := \prod(1+\lambda) < \infty \iff \sum \lambda < \infty \iff q \in L^1$.

We want to treat slower decaying potentials. Lead us to "found object" $\det_2$.

$$\det_2(1+A) := \det(1+A) \cdot e^{-\operatorname{tr}(A)}$$

recall we build up perturbation det by things left invt by KdV flow.

Adding on factor should still preserve b/c $\operatorname{tr}(A) \sim \frac{1}{2k}\|q\|_{L^1}$ ↑ preserved by KdV flow.

This makes sense for $A \in \mathcal{I}_2$ (less stringent than $A \in \mathcal{I}_1$).

$$\det_2(1+A) = \prod_\lambda (1+\lambda) e^{-\lambda}$$

$$\Rightarrow \log \det_2(1+A) = \sum_\lambda \lambda - \log(1+\lambda) \underset{\text{expand powers}}{=} \sum_\lambda \sum_{\ell \geq 2} \frac{(-1)^\ell}{\ell} \lambda^\ell = \sum_{\ell \geq 2} \frac{(-1)^\ell}{\ell} \operatorname{tr}(A^\ell).$$

This converges if $\|A\|_{HS} < 1$ b/c $|\operatorname{tr}(A^\ell)| \leq \|A\|_{\mathcal{I}_2}^\ell$ if $\ell \geq 2$.

by geom series. $\|A\|_{\mathcal{I}_2}$

Lemma

(indeed $\leq \frac{1}{k}\|q\|_{H^{-1}}^2$, so taking k large enough is < 1) (depending on q)

5/20/24

By analogy w/

$$\det_2(1+A) = \det(1+A)e^{-\mathrm{tr}(A)}$$

extend Fredholm beyond trace class operators to HS operators

$$-\log \det_2(1+A) = -\log \det(1+A)e^{-\mathrm{tr}A} = \sum_{\ell\geq 2} \frac{(-1)^\ell}{\ell}\mathrm{tr}(A^\ell) < \infty$$

convergence of series

$\|A\|_{I_2} < 1.$

$\|A\|_{HS}$

we define our central object

$$\alpha(k,q) = \sum_{\ell\geq 2} \frac{(-1)^\ell}{\ell}\mathrm{tr}\left\{\left(\sqrt{R_0(ik)}\, q\, \sqrt{R_0(ik)}\right)^\ell\right\}.$$

"power series in potential q"

we computed $\left\|\sqrt{R_0(ik)}\, q\, \sqrt{R_0(ik)}\right\|_{I_2}^2 = \frac{1}{k}\int \frac{|\hat{q}(\xi)|^2}{\xi^2+4k^2} < \frac{1}{k}\|q\|_{H^{-1}}^2 \quad \forall k \geq 1.$

So taking k suff large, $k > 1 + \|q\|_{H^{-1}}^2$, we have < 1, so

the power series defining α converged (is well defined).

---

convergent geom series proportional to dominant term; here, $\ell = 2$.

If we prove α conserved by kdV flow, gives conservation law at H^{-1} level.

Prop: Let q be Schwartz solution to kdV. If $k \geq 1 + \sup_{t\in I}\|q(t)\|_{H^{-1}}^2$

then $\frac{d}{dt}\alpha(k, q(t)) = 0 \quad \forall t \in \mathring{I}.$

correct, but if H^{-1} proved, same k keeps working, so ...

16:13

pf: $k \geq 1 + \sup_{t\in I}\|q(t)\|_{H^{-1}}^2 \Rightarrow \left\|\sqrt{R_0(ik)}\, q\, \sqrt{R_0(ik)}\right\|_{I_2}^2 < \frac{1}{k}\|q\|_{H^{-1}}^2 < 1 \quad \forall t \in I.$

so series of α converge uniformly, so to take $\frac{d}{dt}$, termwise.

$$\frac{d}{dt}\alpha(k;q(t)) = \sum_{l\geq 2} \frac{(-1)^l}{l} \frac{d}{dt} tr\left\{ \left(\sqrt{R_0}\, q\, \sqrt{R_0}\right)^l \right\}$$

↙ cyclicity of tr.

$$= \sum_{l\geq 2} \frac{(-1)^l}{l} \frac{d}{dt} tr\{ [R_0 q(t)]^l \}$$

$$= \sum_{l\geq 2} \frac{(-1)^l}{l} \sum_{k=0}^{l-1} tr\left\{ (R_0 q(t))^k R_0 \frac{d}{dt} q \underline{(R_0 q)^{l-k-1}} \right\}$$

cycle term $\frac{d}{dt}q$ in last position

$$= \sum_{l\geq 2} (-1)^l tr\left\{ (R_0 q)^{l-1} R_0 (-q'' + bqq') \right\}$$

To show this is 0, look at term which give same "#s of q's"

(1) quadratic term, only $-q''$ (gives 1 "q"), so only $l=2$ possibly give "2 q's":

i.e. WTS $tr\{R_0 q\, R_0 q''\} = 0$.

(2) WTS: ("$l+1$ many q's")

$$(-1)^l tr\{(R_0 q)^{l-1} R_0 bqq'\} + (-1)^{l+1} tr\{(R_0 q)^{(l+1)-1} R_0 (-q'')\} = 0 \quad \forall l \geq 2$$

$$\Leftrightarrow tr\{(R_0 q)^l R_0 (-q'')\} = tr\{(R_0 q)^{l-1} R_0 bqq'\}$$

notice minus sign!

Pf (1): $tr\{R_0 q\, R_0 q''\} = \iint \frac{1}{2k} e^{-k|y-x|} q(y) \frac{1}{2k} e^{-k|x-y|} q''(x)\, dy\, dx$

$e^{-k|y-x|}$: take from y back to x; $\frac{1}{2k} e^{-k|x-y|}$: taken from x to y

$$= \frac{1}{k} \iint q'''(x) \underbrace{\frac{e^{-2k|x-y|}}{4k}}_{R_0(2k) \text{ related!}} q(y)\, dy\, dx$$

$$= \frac{1}{k} \langle q''', R_0(2k)\, q \rangle$$

Plancherel $= \frac{1}{k} \int \frac{-i\xi^3 |\hat{q}(\xi)|^2}{\xi^2 + 4k^2}\, d\xi$

$$= \frac{1}{K}\int_0^\infty \frac{-i\xi^3 |\hat{q}(\xi)|^2}{\xi^2+4k^2}\,d\xi + \frac{1}{K}\int_0^\infty \frac{+i\xi^3 |\hat{q}(-\xi)|^2}{\xi^2+4k^2}\,d\xi$$

$\int_{-\infty}^{0}$ change of variable $\xi \leftrightarrow -\xi$.

$= 0$ b/c q is real valued $\Rightarrow$ $\hat{q}(-\xi) = \overline{\hat{q}(\xi)}$

PF (2): start w/ observation that $[\partial, q] = q'$

$(\partial q - q\partial)f = \partial(qf) - q\cdot(\partial f) = q'\cdot f + q\cdot(\partial f) - q\cdot(\partial f)$

So $-q''' = -[\partial, [\partial, [\partial, q]]]$.

$= -[\partial, [\partial, \partial q - q\partial]]$

$= -[\partial, \partial^2 q - \partial q\partial - \partial q\partial + q\partial^2] = -[\partial, \partial^2 q - 2\partial q\partial + q\partial^2]$

$= -\{\partial^2 q' - 2\partial^2 q\partial + 2\partial q\partial^2 + q'\partial^2\}$

we now "build" inverse of $R_0(z)$ in each of these terms. ($= -\partial^2 + k^2$)

$= (-\partial^2+k^2)q' - k^2 q' - 2(-\partial^2+k^2)q\partial + 2k^2 q\partial + 2\partial q(-\partial^2+k^2) - 2k^2\partial q + q'(-\partial^2+k^2) - k^2 q'$

$= -4k^2[\partial, q] + (-\partial^2+k^2)q' + q'(-\partial^2+k^2) - 2(-\partial^2+k)q\partial + 2\partial q(-\partial^2+k^2)$

So, plugging this in,

$\mathrm{tr}\{(R_0 q)^\ell R_0(-q'')\} = -4k^2\,\mathrm{tr}\{(R_0 q)^\ell R_0[\partial, q]\} + \mathrm{tr}\{(R_0 q)^\ell R_0 (-\partial^2+k^2) q'\} + \mathrm{tr}\{(R_0 q)^\ell R_0 q'(-\partial^2+k^2)\}$

(inverses; 1 power; $(R_0 q)^{\ell-1} R_0 q'(-\partial^2+k^2) R_0 q$, inverses)

$-2\,\mathrm{tr}\{(R_0 q)^\ell R_0(-\partial^2+k^2) q\partial\} + 2\,\mathrm{tr}\{(R_0 q)^\ell R_0 \partial q(-\partial^2+k^2)\}$

(inverses; 1 power)

$= -4k^2\,\mathrm{tr}\{(R_0 q)^\ell R_0 \partial q\} + 4k^2\{(R_0 q)^\ell R_0 q\partial\} + \mathrm{tr}\{(R_0 q)^{\ell-1} R_0 q q'\} + \mathrm{tr}\{(R_0 q)^{\ell-1} R_0 q' q\}$

(commute, look on Fourier side; must commute; same term)

$-2\,\mathrm{tr}\{(R_0 q)^{\ell-1} R_0 q^2\partial\} + 2\,\mathrm{tr}\{(R_0 q)^{\ell-1} R_0 \partial q^2\}$

$$= -4k^2 \operatorname{tr}\{(R_0 q)^\ell \partial R_0 q\} + 4k^2 \{(R_0 q)^\ell R_0 q\} \quad + \quad 2 \operatorname{tr}\{(R_0 q)^{\ell-1} R_0 q' \}$$

cycle, match up; so this whole underlined bit = 0.

$$+ 2 \operatorname{tr}\{(R_0 q)^{\ell-1} R_0 \underbrace{[\partial, q^2]}_{(q^2)' = 2qq'}\}$$

$$= 6 \operatorname{tr}\{(R_0 q)^{\ell-1} R_0 q q'\}.$$

"α is macroscopic conservation law"

Thm: Let q be Schwarz sol to KdV. Fix $-1 \le s < 0$ and let

$$k \ge 1 + 10 \|q(0)\|_{H^{-1}}^2 .$$ Then

$$\|q(t)\|_{H^s} \lesssim_s \left(1 + \|q(0)\|_{H^{-1}}^2\right)^{|s|} \|q(0)\|_{H^s} \quad \forall t \in \mathbb{R}.$$

"bootstrap, control H^{-1} at $t=0$, control it forever more".

Pf: by cty of the flow, $\exists$ small interval $I \ni 0$ s.t.

$$k \ge 1 + 4 \|q(t)\|_{H^{-1}}^2 \quad \forall t \in I.$$

Then, $\left\| \sqrt{R_0(k)}\, q(t) \sqrt{R_0(k)} \right\|_{I_2}^2 < \frac{1}{k} \|q(t)\|_{H^{-1}}^2 < \frac{1}{4} \quad \forall t \in I.$ (Monica missed the 2 oops!)

So α is well-defined and conserved on I.

Moreover, α comparable to dominant term:

$$\left| \alpha(k; q(t)) - \frac{1}{2} \left\| \sqrt{R_0(k)}\, q(t) \sqrt{R_0(k)} \right\|_{I_2} \right| \le \sum_{\ell \ge 3} \frac{1}{\ell} \left\| \sqrt{R_0(k)}\, q(t) \sqrt{R_0(k)} \right\|_{I_2}^\ell$$

$$\le \frac{1}{3} \left\| \sqrt{R_0(k)}\, q(t) \sqrt{R_0(k)} \right\|_{I_2}^2 \left(\frac{\left\| \sqrt{R_0(k)}\, q(t) \sqrt{R_0(k)} \right\|_{I_2}}{1 - \|\cdots\|_{I_2}} \right)$$ (geometric series)

$$\le \frac{1}{3} \left\| \sqrt{R_0(k)}\, q(t) \sqrt{R_0(k)} \right\|_{I_2}^2 \cdot \frac{\frac{1}{2}\|\cdots\|}{1 - \frac{1}{2}\|\cdots\|}$$ (b/c $\|\cdots\|^2$ is small by $\frac{1}{4}$.)

$= \frac{1}{10}$

$$\Rightarrow \quad \frac{7}{15} \| \sqrt{R_0(H)}\, q(H)\, \sqrt{R_0(H)} \|_{I_2}^2 \le \alpha(K, q(H)) \le \frac{8}{15} \| \sqrt{R_0(H)}\, q(H)\, \sqrt{R_0(H)} \|_{I_2}^2 \quad \forall H \in I.$$

α conserved!

$$\Longrightarrow \quad \| \sqrt{R_0(H)}\, q(H)\, \sqrt{R_0(H)} \|_{I_2}^2 \le \frac{15}{7} \alpha(K, q(H)) = \frac{15}{7} \alpha(K; q(0)) \le \frac{8}{7} \| \sqrt{R_0(H)}\, q(0)\, \sqrt{R_0(H)} \|_{I_2}^2$$

$$\le \frac{8}{7} \frac{1}{K} \| q(0) \|_{H^1}^2 < \frac{1}{2} \frac{8}{7} \frac{1}{20} = \frac{2}{35}$$

b/c $K \ge 1 + \frac{1}{20} \| q(0) \|_{H^1}^2$

and indeed $\frac{2}{35} < \frac{1}{11}$

5/22/24

Thm: Let q be a Schwartz solution to KdV. Fix $-1 \le s < 0$, $K \ge 1 + 20\|q(0)\|_{H^{-1}}^2$

Then $\sup_{t\in\mathbb{R}} \|q(t)\|_{H^s} \lesssim (1+\|q(0)\|_{H^{-1}}^2)\,\|q(0)\|_{H^s}$.

Pf: By ctty of flow

$\exists I \ni 0$ s.t. $K \ge 1 + 9\|q(0)\|_{H^{-1}}^2 \quad \forall t\in I$.

Then

$$\|\sqrt{R_0(K)}\, q(t) \sqrt{R_0(K)}\|_{I_2}^2 \le \frac{1}{K}\|q(t)\|_{H^{-1}}^2 < \frac{1}{9} \quad \forall t\in I \quad (\star)$$

Since α given by geom series in q, proportional to leading order term

$$\left|\alpha(K;q(t)) - \frac{1}{2}\|\sqrt{R_0(K)}\, q(t) \sqrt{R_0(K)}\|_{I_2}^2\right| \le \frac{1}{3}\|\sqrt{R_0(K)}\, q(t) \sqrt{R_0(K)}\|_{I_2}^2 \cdot \frac{\|\sqrt{R_0(K)}\, q(t) \sqrt{R_0(K)}\|_{I_2}}{1-\|\sqrt{R_0(K)}\, q(t) \sqrt{R_0(K)}\|_{I_2}}$$

$$< \underbrace{\frac{1}{3}\cdot\frac{\frac{1}{3}}{1-\frac{1}{3}}}_{=\frac{1}{6}} \|\ldots\|$$

so $\frac{1}{3}\|\sqrt{R_0(K)}\, q(t) \sqrt{R_0(K)}\|_{I_2}^2 \le \alpha(K;q(t)) \le \frac{2}{3}\|\sqrt{R_0(K)}\, q(t) \sqrt{R_0(K)}\|_{I_2}^2 \quad \forall t\in I$

Since α is conserved, $\|\sqrt{R_0(K)}\, q(t) \sqrt{R_0(K)}\|_{I_2}^2 \le 3\alpha(K;q(t)) = 3\alpha(K;q(0)) \le 2\|\sqrt{R_0(K)}\, q(0) \sqrt{R_0(K)}\|_{I_2}^2 < 2\cdot\frac{1}{10} < \frac{1}{9}$

bootstrap then yields $I = \mathbb{R}$.

---

above ineq

$$\frac{1}{4K^3}\int \frac{|\hat q(t,\xi)|^2}{\xi^2+1}\le \frac{1}{K}\int \frac{|\hat q(t,\xi)|^2}{\xi^2+4K^2}\,d\xi = \|\sqrt{R_0(K)}\, q(t) \sqrt{R_0(K)}\|_{I_2}^2 \le 2\|\sqrt{R_0(K)}\, q(t) \sqrt{R_0(K)}\|_{I_2}^2 = \frac{2}{K}\int \frac{|\hat q(0,\xi)|^2}{\xi^2+4K^2}\,d\xi \le \frac{2}{K}\|q(0)\|_{H^{-1}}^2$$

$(4K^2 \ge 1)$

$$\frac{1}{4K^3}\|q(t)\|_{H^{-1}}^2$$

$$\Rightarrow \|q(t)\|_{H^{-1}}^2 \le 8K^2\|q(0)\|_{H^{-1}}^2$$

Taking $K = K_0 := 1 + 20\|q(0)\|_{H^{-1}}^2$ gives the claim when $s=-1$.

"a priori bounds at level of H^{-1}"

for remaining s,

Note $\int_{k_0}^{\infty} \frac{1}{\xi^2+4k^2} k^{1+2s}\, dk \sim (\xi^2+k_0^2)^s$

"Build less fast decreasing fcn by adding fast decreasing fcn"

Pf: if $k_0 \geq |\xi|$ $\int_{k_0}^{\infty} \frac{1}{\xi^2+4k^2} k^{1+2s}\, dk \sim \int_{k_0}^{\infty} k^{-1+2s}\, dk \sim k_0^{2s} \sim (k_0+|\xi|^2)^s$

$k \geq k_0 \geq |\xi| \Rightarrow \xi^2+4k^2 \sim k^2$

if $k_0 < |\xi|$, $\int_{k_0}^{\infty} \frac{k^{1+2s}}{\xi^2+4k^2}\, dk = \int_{k_0}^{|\xi|} \dots + \int_{|\xi|}^{\infty} \dots$

$\lesssim \int_{k_0}^{|\xi|} \frac{k^{1+2s}}{|\xi|^2}\, dk + \int_{|\xi|}^{\infty} k^{-1+2s}\, dk$

$1+2s>0$ ← if integrable at 0 so extend to 0 so FTC lower endpt is 0.

$\leq \int_0^{|\xi|} \frac{k^{1+2s}}{|\xi|^2}\, dk$

$\frac{|\xi|^{2+2s}}{|\xi|^2}$

$\sim |\xi|^{2s} \sim (k_0^2+\xi^2)^s$.

integrate both $\int_{k_0}^{\infty} k^{1+2s}\, dk$.

$$\int_{k_0}^{\infty} \int \frac{|\hat{q}(t,\xi)|^2}{\xi^2+4k^2}\, d\xi\, k^{1+2s}\, dk \leq 2 \int_{k_0}^{\infty} \int \frac{|\hat{q}(0,\xi)|^2}{\xi^2+4k^2}\, d\xi\, k^{1+2s}\, dk$$

above Note $\Rightarrow$ $\frac{1}{k_0^{2s}} \int \frac{|\hat{q}(t,\xi)|^2}{(\xi^2+1)^{|s|}}\, d\xi \lesssim \int \frac{|\hat{q}(t,\xi)|^2}{(\xi^2+k_0^2)^{|s|}}\, d\xi \lesssim \int \frac{|\hat{q}(0,\xi)|^2}{(\xi^2+k_0^2)^{|s|}}\, d\xi \lesssim \int \frac{|\hat{q}(0,\xi)|^2}{(\xi^2+1)^{|s|}}\, d\xi = \|q(0)\|_{H^s}^2$

$\Rightarrow$ $\boxed{\|q(t)\|_{H^s} \lesssim k_0^{|s|} \|q(0)\|_{H^s}}$ a priori bds in H^s.

Cor: Fix $-1 \le s < 0$. The orbit of a Schwartz sol to KdV is equicts in H^s

Pf: by Plancherel, tightness on Fourier side $\Longleftrightarrow$ equicts physical side

$\Downarrow$

So suffices to show $\forall \varepsilon > 0$, $\exists K = K(\varepsilon)$ s.t.

$$\sup_{t \in \mathbb{R}} \int_{|\xi| > K} \frac{|\hat{q}(t,\xi)|^2}{(\xi^2+1)^{|s|}} \, d\xi < \varepsilon.$$

In previous results we showed that

$$\int_{|\xi| \ge K} \frac{|\hat{q}(t,\xi)|^2}{(\xi^2+1)^{|s|}} d\xi \le \int \frac{|\hat{q}(t,\xi)|^2}{(\xi^2+k^2)^{|s|}} d\xi \lesssim \int \frac{|\hat{q}(0,\xi)|^2}{(\xi^2+k^2)^{|s|}} d\xi$$

(replace $\int_{k_0}^{\infty}$ above w/ $\int_{k'}^{\infty}$)

$$\lesssim \int_{|\xi| \le h} \left(\frac{h^2}{k^2}\right)^{|s|} \frac{|\hat{q}(0,\xi)|^2}{(\xi^2+1)^{|s|}} d\xi + \int_{|\xi| \ge h} \frac{|\hat{q}(0,\xi)|^2}{(\xi^2+1)^{|s|}} d\xi.$$

$$\lesssim \left(\frac{h}{k}\right)^{2|s|} \|q(0)\|_{H^s} + \text{higher freq piece of } H^s \text{ norm of init data.}$$

If have many init data all bdd unif. in H^s can choose h just based on this unif bd, i.e. choose h uniform in choice of init data in bdd class:

Then, choose $h = h(\varepsilon)$ large to make 2nd term small (MCT and bddness of H^s norm of $q(0)$),

and then choose K large enough that 1st term small

Cor: if $Q \in S(\mathbb{R})$ is bdd and equicts in H^s for $-1 \le s < 0$,

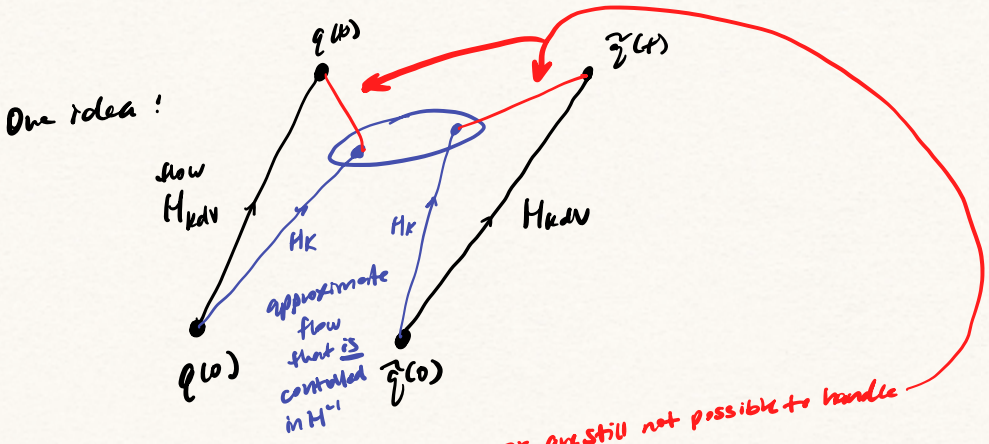
then orbits emanating from Q form a bdd and equicts in H^s.

17:29

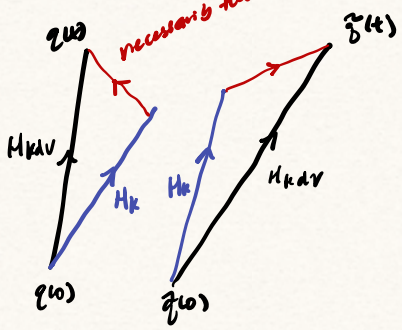
Ill posedness of KdV in H^{-1}.

contraction mapping fails that $\Rightarrow$ crossing control, but we know $s < -\frac{3}{4}$

data $\to$ sol not unif cts.



The idea is to find H_K flow commuting:



17:34

start computation.

$$\alpha(k;q) = \sum_{l\geq 2} \frac{(-1)^l}{l} \operatorname{tr}\{(R_0 q)^l\}$$

$$= \frac{1}{2k}\int \frac{|\hat{q}|^2}{\xi^2+4k^2}\,d\xi - \frac{1}{3(2\pi)^{3/2}}\iiint \frac{\hat{q}(\xi-\mu)}{\xi^2+4k^2}\frac{1}{\mu^2+k^2}\hat{q}(\mu-\eta)\frac{1}{\eta^2+k^2}\hat{q}(\eta-\xi)\,d\xi\,d\eta\,d\mu + O(k^{-7})$$

(ξ_2: $\mu-\eta$; ξ_1: $\eta-\xi$)

back to ξ ← to η ← go to η ← start at ξ

lower order in k

at least 4 R_0, each k^2, but integral gives ak.

expand in $\frac{1}{k}$ powers

$$= \underbrace{\frac{1}{2k\cdot 4k^2}\int |\hat{q}|^2 \sum_{l\geq 0}\left(\frac{-\xi^2}{4k^2}\right)^l}_{\text{Plancherel}} - \underbrace{\frac{1}{3(2\pi)^{3/2}}\iiint \frac{\hat{q}(\xi_1)\,\hat{q}(\xi_2)\,\hat{q}(-\xi_1-\xi_2)}{(\xi_1^2+k^2)\left((\xi_1+\xi_2)^2+k^2\right)\left((\xi_1+\xi_2+\xi_2)^2+k^2\right)}\,d\xi_1\,d\xi_2}_{\text{integrate wrt. } \xi_2 \text{ (easy for computer)}} + O\left(\tfrac{1}{k^7}\right)$$

$$= \frac{1}{8k^3}\int |q|^2 - \frac{1}{32k^5}\int |q'|^2 + O(k^{-7})$$

$$-\frac{1}{3(2\pi)^{1/2}K}\iint \frac{\hat{q}(\xi_1)\,\hat{q}(\xi_2)\,\hat{q}(-\xi_1-\xi_2)\,(\xi_1^2+\xi_1\xi_2+\xi_2^2+12k^2)}{(\xi_1^2+4k^2)(\xi_2^2+4k^2)((\xi_1+\xi_2)^2+4k^2)}\,d\xi_1\,d\xi_2$$

appears when we integrate

a 4 appear after integral.

$$= \frac{12k^2}{(4k^2)^3}\iint \hat{q}(\xi_1)\,\underbrace{\hat{q}(\xi_2)\hat{q}(-\xi_1-\xi_2)}_{\text{convolution}}\,d\xi_2\,d\xi_1$$

+ next term has $(k^2)^4$ in denom, and will contain $\frac{1}{k}$, indeed $O(\frac{1}{k^7})$.

$\sqrt{2\pi}\langle q, \hat{q}^{-1}(\hat{q}*\hat{q})\rangle = \sqrt{2\pi}\int q(x)^3\,dx$

$\hat{q}^{-1}$

momentum

$$= \frac{1}{4k^3}\widehat{P(q)} - \frac{1}{16k^5}\int \tfrac{1}{2}|q'|^2 + q(x)^3\,dx + O(k^{-7})$$

$= \frac{1}{4k^3} P(q) - \frac{1}{16k^5} H_{KdV} + O(k^{-7})$

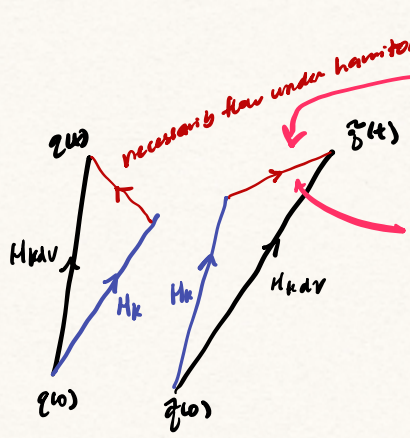
conserved quantities of Gardner... Miura!!!

(α was built from spectral data ... 17:48)

commute w/ H_K b/c $P(q)$, H_{KdV}, ... conserved by H_{KdV}. "everybody commute w everybody" 17:57.

Choose $H_K = -16k^5 \alpha(k; q) + 4k^2 P(q) = H_{KdV} + O(k^{-2}) \xrightarrow[\text{Kato-Hormander converge}]{k\to\infty} H_{KdV}$.

the hope is that these errors go to 0 as $k \to \infty$.



this arrow is not difference but 2 solutions, so we have some hope.

H_K well posed in H^{-1} "trivially"

5/24/24

Recall $H_k(q) := -16k^5 \alpha(k;q) + 4k^2 p(q) = H_{kdV} + O(k^{-2})$.

(Hamiltonians)

Claim: flow corresponding to H_k is globally well posed in H^{-1}.

The flow corresponding to $H_k(q)$ is (recalling Poisson str discussed previously for KdV)

$$\frac{d}{dt} q = \left(\frac{\delta H_k}{\delta q}\right)' = -16k^5 \left(\frac{\delta \alpha(k)}{\delta q}\right)' + 4k^2 q'$$

- $\frac{\delta H_k}{\delta q}$: functional derivative
- $\frac{\delta \alpha(k)}{\delta q}$: "forcing term"
- q': functional derivative of momentum is q.

Recall functional derivative defined to satisfy $\int \frac{\delta \alpha(k)}{\delta q}(x) f(x)\,dx = \frac{d}{ds}\Big|_{s=0} \alpha(q+sf)$.

(k omitted from notation b/c it's always fixed k.)

$$\alpha(q+sf) = \sum_{l\geq 2} \frac{(-1)^l}{l} \operatorname{tr}\left\{(R_0(q+sf))^l\right\}$$

$$\frac{d}{ds}\Big|_{s=0} \alpha(q+sf) = \sum_{l\geq 2} (-1)^l \operatorname{tr}\left\{(R_0 q)^{l-1} R_0 f\right\}$$

derivative attaches one of l copies of $(R_0 \cdot)^l$; by cyclicity can always assume this last copy, so l total copies of something, cancel $\frac{1}{l}$.

$$= -\sum_{l\geq 1} \operatorname{tr}\left\{(-1)^l (R_0 q)^l R_0 f\right\}$$

Let $R(k,q)$ denote the resolvent of $-\partial^2 + q + k^2$.

Resolvent identity says $R(k,q) - \underbrace{R_0(k)}_{\| \; R(k,0) \text{ (take potential } = 0)} = -R(k,q)\underbrace{(q-0)}_{\text{difference of potentials}} R_0(k)$

(i.e. hint: look at $\frac{1}{-\partial^2+q+k^2} - \frac{1}{-\partial^2+k^2}$ ...)

$$= \dots$$

iterating ...

$$= \sum_{l\geq 1} (-1)^l (R_0 q)^l R_0$$ "Born series"

So $\frac{d}{ds}\Big|_{s=0} d(q+sf) = -\operatorname{tr}\{(R(k,q) - R_0(k))f\} = +\frac{1}{2k}\int f(x)\,dx \; - \int G(x,x;k,q) f(x)\,dx$

(integral kernel associated to resolvent evaluated at diagonal.)

We know $\operatorname{tr}\{R_0(k)f\}$, exact computation ↺ 18:01

where $G(x,y;k,q)$ denotes the integral kernel of the resolvent $R(k,q)$

(well defined, and this so it makes sense to restrict to zero once set diagonal)

and we denote $g(x;k,q) = G(x,x;k,q)$ restrict to diagonal

we get $\frac{\delta d(k)}{\delta q} = \frac{1}{2k} - g(x)$ ("diagonal green fn associated to q")

$$\Rightarrow \boxed{\frac{d}{dt}q = 16k^5 g' + 4k^2 g'} \quad \text{this is the Hk flow.}$$

Thm: Fix $q \in H^1(\mathbb{R})$. Then $\exists!$ self adj operator L associated to the quadratic form

Lax operator $-\partial^2 + q$ ⟶ quadratic form $\int |\psi'(x)|^2 + q(x)|\psi(x)|^2\,dx$ (perturbation)

w/ domain H^1. It is semibounded.

Moreover, if $k \geq 1 + 20\|q\|_{H^1}^2$, then resolvent $R(k,q)$ given by norm convergent series

$$R(k,q) = \sum_{l\geq 0} (-1)^l (R_0(k)q)^l R_0(k)$$ exactly Born series above

$$= \sum_{l\geq 0} (-1)^l \sqrt{R_0(k)} \left(\sqrt{R_0(k)}\, q \sqrt{R_0(k)}\right)^l \sqrt{R_0(k)}$$

(recall operator norm bdd by HS norm.)

Pf: To construct L we use **KLMN Thm** — in Reed & Simon.

Thm (KLMN): Let A positive self adj operator (in our case, the Laplacian) w/ quadratic form domain $Q(A)$ (in our case H^1).

Let $\beta(f,g)$ be symm bilinear form (in our case i.e. $\beta(f,g) = \overline{\beta(g,f)}$ like Hermpod)

"perturbation of A" — quadratic form assoc to A.

s.t. $|\beta(f,f)| \le a \underbrace{|\langle f, Af\rangle|}_{= \langle f, Af\rangle \text{ b/c } A \text{ pos.}} + b\|f\|^2 \quad \forall f \in Q(A)$ for some $0 \le a < 1$, $b \in \mathbb{R}$

Then $\exists!$ self adj operator C w/ $Q(C) = Q(A)$

s.t. $\langle f, Cg\rangle = \langle f, Ag\rangle + \beta(f,g) \quad \forall f,g \in Q(C)$

($\langle f, Cg\rangle$ = quadratic form associated to C)

(And C bdd below by $-b$) ← easy, just Δ-ineq.

So by this thm, to construct L, suffices to show

$$\int q(x)\,|h(x)|^2\,dx \le a\int |\psi'(x)|^2\,dx + b\int |\psi|^2(x)\,dx. \quad \text{for some } 0 \le a < 1,\ b \in \mathbb{R}.$$

we ultimately want $q \in H^{-1}$, so only thing we can do is

$$\int q(x)\,|h(x)|^2\,dx \le \|q\|_{H^{-1}}\,\||\psi|^2\|_{H^1} \lesssim \|q\|_{H^{-1}}\,\|\psi\|_{H^1}\,\|\psi\|_{L^\infty}$$

How to show $\|\psi\|_{L^\infty}$ controlled by H^1?

$$\|\psi\|_{L^\infty} \le \|\hat\psi\|_{L^1} \le \|\underbrace{(\xi^2+\varkappa^2)^{1/2}\hat\psi}_{\text{Parseval side}}\|_{L^2_\xi}\,\|\underbrace{\frac{1}{(\xi^2+\varkappa^2)^{-1/2}}}_{\text{Mikabappa}}\|_{L^2_\xi} \quad \text{for } \varkappa \ge 1.$$

$$\lesssim \varkappa^{1/2}\|q\|_{H^{-1}} \int |\psi'(x)|^2 + \varkappa^2|\psi(x)|^2\,dx$$

$\varkappa^{1/2}\|q\|_{H^{-1}} < 1$ if $\varkappa \gg \|q\|^2_{H^{-1}}$

Claim wanted today...

$\|q\|_{H^{-1}}\,\|\psi\|^2_{H^1} \le \varepsilon\,\|\psi'\|_{L^2} + \frac{1}{\varepsilon}\|q\|^2_{H^{-1}}$ — doesn't depend on ψ.

18:24

Finally, $\left\|\sum_{\ell\ge 0}(-1)^\ell \sqrt{R_0}\,(\sqrt{R_0}\,q\,\sqrt{R_0})^\ell \sqrt{R_0}\right\|_{op} \lesssim \sum_{\ell\ge 0} \|\sqrt{R_0}\|^2_{op}\,\underbrace{\|\sqrt{R_0}\,q\,\sqrt{R_0}\|^\ell_{op}}_{\text{bdd by HS norm}} \le \sum_{\ell\ge 0}\frac{1}{k^2}\left(\frac{1}{\sqrt{k}}\|q\|_{H^{-1}}\right)^\ell \lesssim \frac{1}{k^2}$

18:25 "as expected"

Previously, we fixed q arbitrary and looked at k large ($\geq 1 + 20\|q\|_{H^1}^2$)

By scaling, equivalent to look at $k \geq 1$, and restrict to all q small

This is how we phrase the next thm.

equivalent to take q in ball B_δ in H^{-1}, and look only at $k \geq 1 + 20 \sup_{q \in B_\delta} \|q\|_{H^{-1}}$

Thm: $\exists \delta > 0$ s.t. $\forall k \geq 1$ we have the following:

1) $\forall q \in B_\delta := \{ q \in H^{-1}, \|q\|_{H^{-1}} < \delta \}$ the integral kernel $G(x,y;k,q)$ is cts in x,y.

↳ Thm we may define $g(x,k,q) = G(x,x,k,q)$.

2) This means that $q \mapsto g - \frac{1}{2k}$ and $q \mapsto k - \frac{1}{2g}$

are real analytic diffeos from B_δ to H^1!

18:29.

3) if $q \in S(\mathbb{R})$, then both $g - \frac{1}{2k}, k - \frac{1}{2g} \in S(\mathbb{R})$ (subtract $\frac{1}{2k}$ to go to 0 at ∞)

"g gains 2 degrees of regularity compared to q"

In fact Schwartz in quantitative way

equivalent $\longleftrightarrow$ $\|\partial_x^n g'\|_2 \simeq \|q\|_{H^{n-1}}$

$\|g - \frac{1}{2k}\|_{H^{n+1}} \lesssim \|q\|_{H^{n-1}}$ and $\|\langle x \rangle^n g'\|_2 \simeq \|\langle x \rangle^n q\|_{H^{-1}}$

Pf: (1) we'll show $G - G_0 \in H^1_x H^1_y$ (where G_0 = green fct for free resolvent ($q=0$))

then, in $d=1$, $H^1 \hookrightarrow C$, and G_0 cts (we already know), so G cts in x,y.

So we define $g(x) = \sum_{\ell \geq 0} (-1)^\ell \langle \delta_x, \sqrt{R_0} (\sqrt{R_0} q \sqrt{R_0})^\ell \sqrt{R_0}\, \delta_x \rangle$

(integral kernel restricted to diag.)

$$= \frac{1}{2k} + \sum_{\ell \geq 1} (-1)^\ell \langle \delta_x, \sqrt{R_0} (\sqrt{R_0} q \sqrt{R_0})^\ell \sqrt{R_0}\, \delta_x \rangle$$

("kernel of R_0 restr. to diag")

How to show)?

Test against H^{-1} fns: let $f, h \in H^{-1}$.

$$\left| \iint h(x) (G-G_0)(x,y) f(y)\, dx dy \right| \quad \text{WTS} \quad \leq \|f\|_{H^{-1}} \|h\|_{H^{-1}}.$$

$$= \left| \mathrm{tr}\{ h (R-R_0) f \} \right|$$

$$\leq \underset{\text{Born series}}{\sum_{\ell \geq 1}} \left| \mathrm{tr}\{ h (R_0 g)^\ell R_0 f \} \right|$$

$$= |\langle h, (R-R_0) f \rangle|$$

if CBS now, put h in L^2; $(R-R_0) f$ in L^2.

instead insert operators so left guy in L^2 gets $\|h\|_{H^{-1}}$.

$$= \left| \left\langle \frac{1}{\sqrt{-\partial^2 + k^2}} h,\ \sqrt{-\partial^2+k^2}\,(R-R_0) f \right\rangle \right|$$

$$\sqrt{-\partial^2+k^2}\,(R-R_0)\sqrt{-\partial^2+k^2} \cdot \frac{1}{\sqrt{-\partial^2+k^2}} f$$

$\frac{1}{\sqrt{-\partial^2+k^2}} h \in L^2$; first factor: op norm; second factor: L^2

$$\leq \|f\|_{H^{-1}} \|h\|_{H^{-1}} \sum_{\ell \geq 1} \| \sqrt{R_0}\, g \sqrt{R_0} \|_{HS}^{\ell} \leq \|f\|_{H^{-1}} \|h\|_{H^{-1}} \left(\frac{\#}{\sqrt{k}} \|g\|_{H^{-1}} \right)$$

since k large enough that op norm ≤ 1, Born series

leading order term

$\|g\|_{H^{-1}} < \delta$, $k \geq 1$

$$\leq \|f\|_{H^{-1}} \|h\|_{H^{-1}}$$

Remember forcing term is g' in H_k flow

PDE theory: g' Lipschitz $\Rightarrow$ ☺.

But g' is real analytic by (2) above

5/29/19 Last time: Gelfand-Shubin

Prop: $\exists \delta > 0$ s.t. $\forall K \geq 1$,

1) $q \in B_\delta = \{q \in H^{-1} : \|q\|_{H^{-1}} < \delta\}$, the resolvent $R(K,q) = (-\partial^2 + q + K^2)$ admits a (unique) kernel $G(x,y;K,q)$. Furthermore, may define diagonal

(proof is $M H^1$ in each component, $H^1 \hookrightarrow$ CTS)

$$g(x;K,q) = G(x,x;K,q)$$

$$= \frac{1}{2K} + \sum_{\ell \geq 1} (-1)^\ell \left\langle \delta_x, \sqrt{R_0}\,(\sqrt{R_0}\, q \sqrt{R_0})^\ell \sqrt{R_0}\, \delta_x \right\rangle \quad (1)$$

"Born series"

2) mappings $q \mapsto g - \frac{1}{2K}$, $q \mapsto K - \frac{1}{2g}$ are real analytic diffeomorphisms from B_δ into H^1.

3) If $q \in S(\mathbb{R})$, then $g - \frac{1}{2K}$, $K - \frac{1}{2g} \in S(\mathbb{R})$

Moreover $\forall n \geq 0$, q gain 2 degrees of regularity over g (in the sense of derivatives)

$$\|\partial_x^n g'\|_2 \lesssim \|q\|_{H^{n-1}}$$

$$\|\langle x \rangle^n g'\|_2 \lesssim \|\langle x \rangle^n q\|_{H^{-1}}.$$

Pf. (1) Last time.

(2) Want to prove that result actually lives in H^1.

By duality.

$f \in H^{-1}$, want to bound $\left| \int f (g - \frac{1}{2K}) dx \right|$ only in terms of $\|f\|_{H^{-1}}$.

$$\leq \sum_{\ell \geq 1} \left| \mathrm{tr} \{ f \sqrt{R_0} (\sqrt{R_0}\, q \sqrt{R_0})^\ell \sqrt{R_0} \} \right| \leq \sum_{\ell \geq 1} \|\sqrt{R_0} f \sqrt{R_0}\|_{HS} \|\sqrt{R_0}\, q \sqrt{R_0}\|_{HS}^\ell$$

$\leq$ Δ-ineq using Born series (1) above for $g - \frac{1}{2K}$

cycle

$$\leq \sum_{\ell \geq 1} \frac{1}{\sqrt{K}} \|f\|_{H^{-1}} \left(\frac{1}{\sqrt{K}} \|q\|_{H^{-1}} \right)^\ell$$

$$\leq \frac{2}{K} \|f\|_{H^{-1}} \cdot \delta.$$

$\|q\|_{H^{-1}} < \delta$, $K \geq 1$

$$\Rightarrow \|g - \tfrac{1}{2K}\|_{H^1} \leq \frac{2}{K} \delta.$$

H^1 controls L^∞

So taking $\delta \leq \frac{1}{8}$ (sufficiently small to absorb implicit constant $\|\hat{f}\|_{L^\infty} \leq \|f\|_{L^1} \lesssim \|f\|_{H^1} \left\| \frac{1}{\langle \xi \rangle^2} \right\|_{L^2}$) $\Rightarrow \frac{1}{4K} \leq g \leq \frac{3}{4K}$ (2) So g is never zero (so $K - \frac{1}{2g}$ makes sense).

Look at this argument, we have actually proven that Born series (1) converges in H^1.

$\Rightarrow g$ is real analytic function of q.

Note that we may write $K - \frac{1}{2g} = K\,\frac{2K(g-\frac{1}{2K})}{1+2K(g-\frac{1}{2K})}$ and $f \mapsto \frac{f}{1+f}$ is a diffeomorphism from small ball in H^1 into H^1

So $g \mapsto K - \frac{1}{2g}$ is a diffeo from B_S into H^1

if $g \mapsto g - \frac{1}{2K}$ is a diffeo from B_S into H^1.

Together w/ (2) this would also show that $g \mapsto K - \frac{1}{2g}$ is real analytic (reciprocal of real analytic fn never 0).

So, to prove 3), remains to show . (using inverse fn theorem)

By resolvent identity $R(K, q+sf) - R(K,q) = -R(K;q+sf)\,sf\,R(K;q)$

perturbation

numerator of Newton difference quotient

dividing by s and letting $s \to 0$, get

$$dg|_q(f) = -\int G(x,y)\, f(y)\, G(y,x)\, dy$$

$dg|_q$: derivative of g at potential q; f: "direction f"

$G(x,y)$: $x \leftarrow y$ locate to; $G(y,x)$: $y \leftarrow$ start at x

specializing $\Rightarrow$ $dg|_{q=0}(f) = -\int \frac{1}{2K} e^{-K|x-y|} f(y) \frac{1}{2K} e^{-K(y-x)}\, dy = -\frac{1}{K}\int \frac{1}{4K} e^{-2K|x-y|} f(y)\, dy$

trick from before, $R_0(2K)$ kernel!

$= -\frac{1}{K} R_0(2K) f$.

smooth 2 degrees; rough fn in H^{-1}

$\Rightarrow$ $dg|_{q=0}$ is an isom from H^{-1}_K into $H^1_K = \{f : (\xi^2+4K^2)^{\frac{1}{2}} \hat{f} \in L^2\}$

$\{f : (\xi^2+4K^2)^{-1/2} \hat{f} \in L^2\}$

Thus for δ small, $g - \frac{1}{2K}$ is diffeo from small ball $\{g : \|g\|_{H^{-1}_K} < \delta\}$ into H^1_K.

δ depending on K

As $\|g\|_{H^{-1}_K} \le \|g\|_{H^{-1}}$ and $K \ge 1 \Rightarrow \|f\|_{H^1} \le \|f\|_{H^1_K}$.

This shows the mapping $q \mapsto g^{-\frac{1}{2k}}$ is a diffeo from B_δ into H^1.

Almost fine ...

To see that δ may be chosen independently of k, we use contraction mapping pf of inverse fn thm.

To this end, suffices to show

$$\underbrace{\| dg|_q - dg|_{q=0} \|_{H^{-1}}}_{\text{map } H^{-1} \text{ to } H^1} \leq \delta \, \| (dg|_{q=0})^{-1} \|^{-1}_{H^1_k \to H^{-1}_k}$$

(recall in IFT, then the mapping $\Phi(q) := q + (dg|_{q=0})^{-1}(f - g(q))$ is a contraction $H^{-1}_k \to H^1_k$

"given f in image, find q s.t. $g(q) = f$" — want to

$$\text{Ie } \| d\Phi|_q \|_{H^1_k \to H^1_k} = \| 1 - (dg|_{q=0})^{-1} dg|_q \|_{H^{-1}_k \to H^{-1}_k} \leq \| (dg|_{q=0})^{-1} \|_{H^1_k \to H^1_k} \| dg|_{q=0} - dg|_q \|_{H^{-1}_k \to H^1_k} \leq \delta$$

pf of: by duality $f \in H^1_k$, $h \in H^{-1}_k$

$$\left| \langle (dg|_q - dg|_{q=0})(f), h \rangle \right| = \left| \operatorname{tr} \{ Rf R h - R_0 f R_0 h \} \right|$$

↓ expand R in Born series in both locations, q and $q=0$.

$$\leq \sum_{k+j \geq 1} \left| \operatorname{tr} f (R_0 q)^k R_0 f (R_0 q)^j R_0 h \right|$$

(at least one q appears.)

↓ usual trick, split $R_0 = \sqrt{R_0}\sqrt{R_0}$ and cycle $\sqrt{R_0}$ around.

$$\leq \sum_{k+j \geq 1} \| \sqrt{R_0} f \sqrt{R_0} \|_{HS} \, \| \sqrt{R_0} h \sqrt{R_0} \|_{HS} \, \| \sqrt{R_0} q \sqrt{R_0} \|^{k+j}_{HS}$$

$$\leq \frac{1}{\sqrt{k}} \| f \|_{H^1_k} \, \frac{1}{\sqrt{k}} \| h \|_{H^1_k} \cdot 2\delta \quad \text{(sum)}$$

$$\leq 2\delta \, \| f \|_{H^{-1}_k} \, \| h \|_{H^{-1}_k} \, \frac{1}{k} = \| (dg|_{q=0})^{-1} \|^{-1}_{H^1_k \to H^{-1}_k}$$

---

3) if $g \in S(\mathbb{R})$, then $g^{-\frac{1}{2k}}, k^{-\frac{1}{2g}} \in S(\mathbb{R})$

Moreover $\forall n \geq 0$, g (in the sup norms) gains 2 degrees of regularity over g $\quad \| \partial_x^n g' \|_2 \lesssim \| g \|_{H^{n-1}}$

$\| \langle x \rangle^n g' \|_2 \lesssim \| \langle x \rangle^n g \|_{H^{-1}}$.

Assume $g \in S(\mathbb{R})$. How to translate derivative of g to derivative of f?

KdV spatial translation $g(x+h;g) = g(x;g(\cdot+h))$.

$$\left|\int \partial_x^n \left(g_{-\frac{1}{2k}}\right) f\right| \leq \sum_{l\geq 1} \|\sqrt{R_0} f \sqrt{R_0}\|_{HS} \sum_{\sigma_1+\cdots+\sigma_l=n} C_{\sigma_1\cdots\sigma_l} \prod_{i=1}^{l} \|\sqrt{R_0}\, g^{(\sigma_j)} \sqrt{R_0}\|_{HS}$$

(Born series (1) for g. — derivatives on l-many copies. — at most l^n ("n choose l sort of thing"))

$$\leq \frac{1}{\sqrt{k}} \|f\|_{H^{-1}} \sum_{l\geq 1} \sum_{\sigma_1+\cdots+\sigma_l=n} l^n \prod_{j=1}^{l} \left(\frac{1}{\sqrt{k}} \|g^{(\sigma_j)}\|_{H^{-1}}\right)$$

Hölder on Fourier side / "interpolation": $\|g^{(\sigma_j)}\|_{H^{-1}} \lesssim \|g^{(n)}\|_{H^{-1}}^{\sigma_j/n} \|g\|_{H^{-1}}^{1-\sigma_j/n}$

$$\lesssim \frac{1}{\sqrt{k}} \|f\|_{H^{-1}} \sum_{l\geq 1} l^n \|g^{(n)}\|_{H^{-1}}^{\sum \sigma_j/n = n} \left(\frac{1}{\sqrt{k}} \|g\|_{H^{-1}}\right)^{l-1} \cdot \left(\frac{1}{\sqrt{k}}\right)^{1}$$

$$\lesssim \frac{1}{k} \|g\|_{H^{n-1}} \|f\|_{H^{-1}}$$

$$\Rightarrow \|g_{-\frac{1}{2k}}\|_{H^{n+1}} \lesssim \|g\|_{H^{n-1}}$$ (even a $\frac{1}{k}$ holding here.)

Rmk:

convert weights attaching g = powers sensing g into weights attaching g, need to commute weight part $\sqrt{R_0}$. $\langle x \rangle$

commutator of inverse operator can be rewritten using this identity to move inverse outside!

$$[\langle x\rangle, R_0(k)] = R_0(k)\left[-\partial^2+k^2, \langle x\rangle\right] R_0(k)$$

$$= -R_0(k)\left[\frac{x}{\langle x\rangle}\partial + \partial\frac{x}{\langle x\rangle}\right] R_0(k).$$

So commute another power of $\langle x\rangle$ means commute w/ this operator

$$\left[\langle x\rangle,\ \frac{x}{\langle x\rangle}\partial+\partial\frac{x}{\langle x\rangle}\right] \underset{\text{Leibniz rule.}}{=} \frac{x}{\langle x\rangle}[\langle x\rangle,\partial]+[\langle x\rangle,\partial]\frac{x}{\langle x\rangle} = -\frac{2x^2}{\langle x\rangle^2}$$

$\langle x\rangle$ w/ $\frac{x}{\langle x\rangle}\partial$ — both multiplication operator....

Inductively, similar computations show

$$\langle x\rangle^n R_0 = \sum_{j=0}^{n} \sqrt{R_0}\,\underbrace{A_{j,n}}_{\text{operator}}\,\sqrt{R_0}\,\langle x\rangle^j.$$

bdd on L^2 uniformly in j: $\|A_{j,n}\|_{op} \lesssim_n 1$.

$f \in H^{-1}$ for duality

so then $\left|\int \langle x\rangle^n (g-\frac{1}{2k})f\right| \le \sum_{l\ge 1} |\mathrm{tr}\, f\,\langle x\rangle^n (R_0 q)^l R_0|$

commute, using above fact. ONCE, i.e. $\langle x\rangle^n R_0^2$, and then set ∂

$$\le \sum_{l\ge 1}\sum_{j=0}^{n} \|\sqrt{R_0} f \sqrt{R_0}\|_{HS}\, \|A_{j,n}\|_{op}\, \|\sqrt{R_0}\langle x\rangle^j q \sqrt{R_0}\|_{HS}\, \|\sqrt{R_0} q \sqrt{R_0}\|_{HS}^{l-1}$$

$$\lesssim \frac{1}{k}\,\|\langle x\rangle^n g\|_{H^{-1}}.$$

(using that $\|\langle x\rangle^j g\|_{H^1} \le \|\langle x\rangle^n g\|_{H^{-1}}$)

Smooth scaling from here.

Green fn g' trace term. Now that we understand it well quantitatively, smooth scaling in H_k flow

05/31/19

H_K flow

warmup

Prop: Let q be Schwartz sol to KdV w/ $q(0) \in B_\delta := \{\|q\|_{H^{-1}} < \delta\}$ and fix $K \geq 1$

Let $g(t,x) := g(x; K, q(t))$. Then q evolve under KdV $\Rightarrow$ g evolve under equation:
(diagonal reconstruction)

$$\frac{d}{dt} g = -2q'g + 2gg' - 4k^2 g'$$

$$\frac{d}{dt} \frac{1}{2g} = \left(\frac{g}{g} - \frac{2k^2}{g}\right)'$$

$\frac{1}{2g}$: on purpose b/c we had mapping $g \mapsto g - \frac{1}{2k}$, $g \mapsto k - \frac{1}{2g}$.

Pf: chain rule - how to diff g wrt potential, which we do using resolvent identity.

$$\frac{d}{dt} g = -\int \underbrace{G(x,y)}_{\text{kernel of resolvent}} \left(\frac{d}{dt} q\right)(y) \; G(y,x)\, dy = -\int G(x,y)\left(-q''' + 6qq'\right)(y)\, G(y,x)\, dy$$

x to y; y to x

The defining property of G is kernel of resolvent of Lax operator, so applying Lax operator to left or right, better get δ function!! I.e.:

$$(-\partial_y^2 + q(y) + k^2)\, G(y,x) = G(x,y)(-\partial_y^2 + q(y) + k^2) = \delta(y-x)$$

How to take advantage of this? We did something similar when proving $\frac{d}{dt} d = 0$.

But unlike then when we had free resolvent, $q=0$, we have need to introduce a q.

We showed

$$-f''' + (-\partial^2 + k^2)f' + f'(-\partial^2 + k^2) - \lambda(-\partial^2 + k^2)f\partial + 2\partial f(-\partial^2 + k^2) - 4k^2 f'$$

$$(-\partial^2 + q + k^2) f' + f'(-\partial^2 + q + k^2) - \lambda(-\partial^2 + q + k^2) f\partial + 2\partial f(-\partial^2 + q + k^2) - 4k^2 f'$$

$$= -f''' + 2qf' \underbrace{- 2qf\partial + 2\partial fq}_{= 2[\partial, fq] = 2(fq)'} .$$

$$= -f''' + 2qf' + 2(fq)'$$

$$\Rightarrow \int G(x,y)\left(-f''' + 2qf' + 2(fq)' + 4k^2 f'\right)(y)\, G(y,x)\, dy$$

$$= \int G(x,y)\Big[\underbrace{(-\partial^2+q+k^2)f'}_{\text{annihilate to } \delta x} + f'\underbrace{(-\partial^2+q+k^2)}_{\text{annihilate to } \delta x} - 2\underbrace{(-\partial^2+q+k^2)}_{}f\delta + 2\underbrace{f(-\partial^2+q+k^2)}_{}\Big] G(y,x)$$

$$= \underbrace{f'g}_{} \quad \underbrace{f'g}_{} \quad \underbrace{-2fg'}_{} \quad \underbrace{2g'f}_{}$$

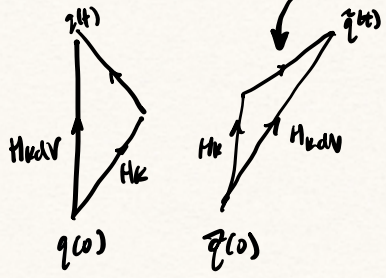
$$= 2f'g - 2fg'$$

$$g'(x) = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) G(x,y)\Big|_{x=y}$$

Taking $f = q \Rightarrow$ $\int G(x,y)\left(-q''' + 6q q' + 4k^2 q'\right)(y)\, G(y,x) = 2gg' - 2gg'$

So $\frac{d}{dt} g = -(2g'g - 2gg') + 4k^2 \underbrace{\int G(x,y)\, g'(y)\, G(y,x)\, dy}_{= -g'(x)}$

So $H_k(q) - 16k^5 a(k,q) + 4k^2 p(q) = H_{KdV} + O(k^{-2})$.



flow commute ↔ flow according to Hamiltonian $H_{KdV} - H_k$

1) show H_k globally well posed.
2) show $H_{KdV} - H_k$ more shrink as $k \to \infty$.

$\Rightarrow$ win.

Today, Prop (Hk flows): Fix $k \geq 1$. Then $\exists \delta > 0$ small s.t. H_k flow is globally well posed on B_δ (δ-neighbd in H^{-1}).

- The H_k-flow conserves $a(x)$ $\forall x \geq 1$ and commutes w/ KdV flow.
- If init data Schwartz, sol remains Schwartz $\forall$ time.
- Finally $\frac{1}{dt}\frac{1}{2g(x)} = \frac{-4k^5}{k^2 - x^2}\left(\frac{g(k)}{g(x)}\right)^2 + 4k^2\left(\frac{1}{2g(x)}\right)' \quad \forall x \neq k.$

KdV² scaling symmetry in $H^{-3/2}$, so can rescale to get H^{-1} norm smaller than δ. We are above critical regularity $-\frac{3}{2}$!

Pf: the H_k flow is

$$\frac{d}{dt} q = \left[\frac{\delta H}{\delta q}\right]' = -16k^5\left(\frac{\delta a(k)}{\delta q}\right)' + 4k^2 q' \overset{\text{recall power series in } \frac{1}{k}}{=} -16k^5\left(\frac{1}{2x} - q\right)' + 4k^2 q' = \underbrace{16k^5 q'}_{\text{forcing term is } g'} + 4k^2 q'$$

So letting $\tilde{q}(t,x) := q(t, x - 4k^2 t)$

turn $\frac{d}{dt} q(t,x) = 16k^5 g'(x-4k^2t, q(t))$

$\Rightarrow$ FTC in t

$$q(t,x) = \underbrace{q(0,x)}_{\text{"}q(0,x)} + \int_0^t 16k^5 g'(x-4k^2s, q(s))\,ds.$$

$\Rightarrow$ undoing translation on LHS

$$\boxed{q(t,x) = q(0, x+4k^2t) + 16k^5 \int_0^t g'(x+4k^2(t-s), q(s))\,ds}$$

Easy ODE, Picard Iteration, as long as g' is Lipschitz on H^{-1}.

But we've shown $q \mapsto g\frac{1}{2k}$ is real analytic map from B_δ to H^1, g' is certainly Lipschitz:

$$\| g'(q_1) - g'(q_2) \|_{H^{-1}} \lesssim \| g(q_1) - g(q_2) \|_{H^1} \lesssim \| q_1 - q_2 \|_{H^{-1}}$$

g Lipschitz $H^1 \to H^{-1}$.

So Picard thm gives local-in-time wellposedness on B_δ.

Now, must extend to global-in-time

i.e. need to show HK flow has a priori bounds on H^{-1}.

Enough to prove $\alpha(x)$ is conserved b/c $\alpha(x)$ encodes a priori bds in H^{-1}:

$$\alpha(x,q) \sim \frac{1}{\pi} \int \frac{|\hat{q}(\xi)|^2}{\xi^2 + x^2}\,d\xi.$$

leading order term in expansion

As the HK flow locally wellposed (LWP) in B_δ, suffices to prove conservation of $\alpha(x)$ for Schwartz init data.

Now if I start w/ Schwartz, obviously solution also Schwartz:

Last time proved Schwartz potential $\Rightarrow$ diagonal Green fn $g(q) - \frac{1}{2k}$ also Schwartz

So Rookie ball cap, indeed Schwartz data $\to$ Schwartz sol.

So now, $\frac{d}{dt}\alpha(x) = \{\alpha(x), H_K\} = \int \frac{\delta\alpha(x)}{\delta q} \left(\frac{\delta H_K}{\delta q}\right)' dx = \int \left(\frac{1}{2x} - g(x)\right)\left[-16k^5\left(\frac{1}{2k} - g(k)\right) + 4k\xi_0^2\right]' dx$

Poisson bracket

shouldn't matter b/c g is Schw

"Superstition" so IBP to move derivative away from g''

$$= 16k^5 \int g'(x)\, g(k)\, dx + 4k^2 \int g'(x) q\, dx.$$

Claim: $=0$ $\quad$ $=0$

by no means obvious.

A miracle of complete integrability.

$\rightsquigarrow$ diagonal Green fn satisfies elliptic eq! (first known in 1880).

$\forall$ spectral param x.

$$g'''(x) = 2q'g(x) + 4q\,g'(x) + 4x^2 g'(x).$$

rewrite $= 2q\,g'(x) + 2(q\,g(x))' + 4x^2 g'(x)$

(rewrite to involve known what's coming)

Begin w/ $\quad 2\int g'(x) q\, dx = \int g'''(x) - 2(q\,g(x))' - 4x^2 g'(x)\, dx$

all derivatives of Schwartz fn $\Rightarrow$ integral $=0$.

Next, $\quad 4x^2 \int g'(x) g(k) = \int g(k) g''(x) - 2\int q' g(x) g(k) - 4\int q\, g'(x)\, g(k)$

$$= -2\int q' g(k) g(x) - 4\int q\, g'(k)\, g(x) + 4k^2 \int g(k) g'(x) - 2\int q' g(x) g(k) - 4\int q\, g'(x)\, g(k)$$

$$\Rightarrow \quad 0 = -4\int (q\, g(k)\, g(x))' = 4(x^2 - k^2) \int g'(x) g(k).$$

(q Schwartz $\Rightarrow$; $g \in C^\infty$)

$$\Rightarrow \int g'(x) g(k) = 0 \quad \forall\, x, k \geq 1 \qquad \left(\text{if } x = k, \ = \int (g(x)^2)' \right)$$

Appendix: how to prove elliptic eq for g.

$$g'''(x) = (\partial_x + \partial_y)^3 G(x,y)\big|_{x=y}$$

Use (fundamental nature of G)

$$(-\partial_x^2 + q(x))\, G(x,y) = (-\partial_y^2 + q(y))\, G(x,y) = -k^2 G(x,y) + \delta(x,y).$$

$$\Rightarrow (\partial_x^2 - \partial_y^2)\, G(x,y) = (q(x) - q(y))\, G(x,y)$$

$$\underbrace{(\partial_x - \partial_y)(\partial_x^2 - \partial_y^2)}_{\partial_x^3 + \partial_y^3 - \partial_y \partial_x^2 - \partial_x \partial_y^2}\, G(x,y) = (q'(x) + q'(y))\, G(x,y) + (q(x) - q(y))(\partial_x - \partial_y) G(x,y)$$

$\bullet\bullet\bullet$ etc.

6/12/24 Last day: finish pf of wellposedness of KdV in H^{-1}. (Guest lecture)

Recall apriori bound:

$\exists \delta > 0$ s.t.

$\forall$ Schwartz sol $q(t)$ to KdV w/ init data $q(0) \in B_\delta := \{f : \|f\|_{H^{-1}} < \delta\}$

have a priori bound (i) $\sup_{t\in\mathbb{R}} \|q(t)\|_{H^{-1}} \lesssim \|q(0)\|_{H^{-1}}$ (proved by perturbation determinant presented $\alpha(k; q(t)) = \alpha(k; q(0))$)

orbit cpct in H^{-1} (ii) $\{q(t) : t\in\mathbb{R}\}$ is cpct in H^{-1} ($\Leftrightarrow \|q(t)\|_{H^{-1}_K} \xrightarrow{K\to\infty} 0$ ss $K \cdot \alpha(K; q(t))$)

Can replace KdV by other flow that have α preserved

like $\underbrace{tH_{KdV} + sH_K}_{\text{Hamiltonian}}$ ($\{\alpha(k), tH_{KdV} + sH_K\} = 0$ Poisson commute)

Thm: Let q_n be seq of Schwartz solution to KdV w/ $q_n(0)$ converging in H^{-1}.

Then, $\forall T > 0$, $q_n(t)$ converges in H^{-1} uniformly for $|t| \le T$

Cor (global wellposedness of KdV): the data $\to$ sol extends (and uniquely so) from Schwartz space to a jointly cts map $\Phi : \mathbb{R}_t \times H^{-1}(\mathbb{R}) \to H^{-1}(\mathbb{R})$.

Moreover, $\forall q \in H^{-1}$, the orbit $\{\Phi(t;q) : t\in\mathbb{R}\}$ is bdd and cpct

Pf of Cor: $S(\mathbb{R})$ dense in $H^{-1}(\mathbb{R})$, so extension $\exists$ and is unique.

Why joint cts? Suffices to show for each $T>0$, $\Phi : [-T,T] \times H^{-1}(\mathbb{R}) \to H^{-1}(\mathbb{R})$ is joint cts.

We show:
(A) $\Phi(t;q)$ is cts in time t.
(B) $\Phi(t;q)$ is uniformly cts in q (cpct set time interval)

→ Std argument ε-η → joint cty.

To show (A), first **Lemma:** Schwartz sol to KdV are H^{-1}-cts in t

(Pf: recall energy method from earlier classes show H^2-cts in time $\Rightarrow$ H^{-1}-cts in time. (indeed Bourgain method show H^0-cts in time))

+ unif limit of cts fn is cts.

To show (B), std arguments "1st year analysis"

Why orbit bdd & cpct? Thm says true for Schwartz sol, so "Std arguments" says preserved in extension.

Pf of Thm: first rescale to small initial data: KdV enjoys the scaling symmetry

$$q(t,x) \longmapsto q_\lambda(t,x) := \lambda^2 q(\lambda^3 t, \lambda x)$$

(is $H^{-3/2}$ critical, but we are concerned about H^{-1} regularity, so

$$\|q_\lambda(t)\|_{H^{-1}}^2 = \lambda^4 \int \frac{\lambda^{-2}\,|\hat{q}(\lambda^3 t, \frac{\xi}{\lambda})|^2}{\xi^2+1}\,d\xi \overset{\text{change of var.}}{=} \int \frac{\lambda^3}{(\lambda\xi)^2+1}\,|\hat{q}(\lambda^3 t, \xi)|^2\,d\xi$$

provided $0<\lambda<1$

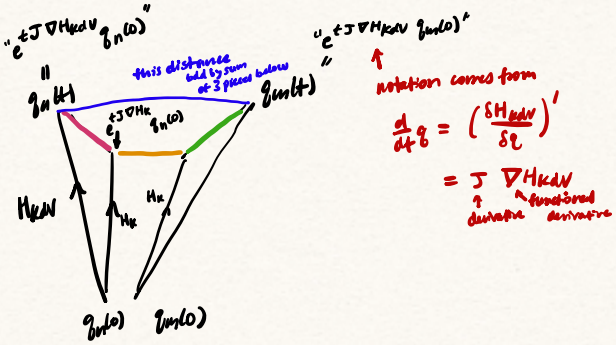
$$\lesssim \lambda \int \frac{1}{\xi^2+1}\,|\hat{q}(\lambda^3 t,\xi)|^2\,d\xi$$

$$\lesssim \lambda\, \|q(\lambda^3 t)\|_{H^{-1}}$$

So for $t=0$, λ small enough, may assume

$$\|q(0)\|_{H^{-1}} < \delta$$

Now, the commuting flow argument.



notation comes from

$$\frac{d}{dt}q = \left(\frac{\delta H_{KdV}}{\delta q}\right)' = J\nabla H_{KdV}$$

(' = functional derivative, ∇ = gradient)

Lemma: $e^{tJ\nabla H_{KdV}} = e^{tJ\nabla(H_{KdV}-H_K)} \circ e^{tJ\nabla H_K}$

Pf: comes from Poisson commute $\{H_{KdV}-H_K, H_K\} = 0$.

So,

$$\|q_n(t)-q_m(t)\|_{H^{-1}} \le \|e^{tJ\nabla H_{KdV}} q_n(0) - e^{tJ\nabla H_{KdV}} q_m(0)\|_{H^{-1}} + \|e^{tJ\nabla(H_{KdV}-H_K)} e^{tJ\nabla H_K} q_n(0) - e^{tJ\nabla H_K} q_m(0)\|_{H^{-1}}$$

$$+ \quad \text{same but } m \text{ in place of } n.$$

As H_K flow globally wellposed in H^{-1},

$$\sup_{|t|\le T} \|e^{tJ\nabla H_{KdV}} q_n(0) - e^{tJ\nabla H_{KdV}} q_m(0)\|_{H^{-1}} \to 0 \text{ as } n,m\to\infty.$$

Denote $Q := \{q_n(0)\}_{n=1}^\infty \subseteq B_\delta$

$$Q^* := \left\{ e^{J\nabla(tH_{KdV}+sH_K)} q : q\in Q,\ t,s\in\mathbb{R},\ k\ge 1 \right\}$$

(i.e. results of any combination of H_{KdV}, H_K flow acting on Q.)

It sufficies to prove

$$\sup_{q\in Q^*} \sup_{|t|\le T} \|\underbrace{e^{tJ\nabla(H_{KdV}-H_K)} q}_{=:q(t)\text{ lies in }Q^*} - q\|_{H^{-1}} \to 0 \text{ as } K\to\infty.$$

We would like $Q^* \subseteq B_S$ so we can use what we proved earlier

think of $tH_{KdV} + sH_K$ as Hamiltonian and $e^{1 \cdot J\nabla(tH_{KdV}+sH_K)}$ (time 1)

"$q(t=1)$ for $tH_{KdV}+sH_K$ flow"

(perturbation determinant α preserved,

so also get a priori bound

$\sup_{t\in\mathbb{R}} \|q(t)\|_{H^{-1}} \lesssim \|q(0)\|_{H^{-1}}$ for $q(0)\in B_S$

constant should only depend on formulation for α; as long as α conserved, should have same implicit const no matter what the flow is.

$\Rightarrow$ by choosing init data even smaller, $Q^* \subseteq B_S$

$$\left| \int \left[g(x,x;q) - \frac{1}{2k} + \frac{1}{k} R_0(2k)\, q \right] f(x)\, dx \right|$$

test $f \in L^\infty$ fn to show is in L^1 (duality!)

where recall $G(x,y;k,q)$ integral kernel of $(-\partial_x^2 + q + k^2)^{-1} = \sum_{l=0}^{\infty} (-1)^l \sqrt{R_0}(\sqrt{R_0}\, q \sqrt{R_0})^l \sqrt{R_0}$

and $g(x;k,q) = G(x,x;k,q)$, we have that $g(x;k,q) = \frac{1}{2k} - \frac{1}{k} R_0(2k) q + \sum_{l=2}^{\infty} \dots$

so

$$= \left| \sum_{l=2}^{\infty} (-1)^l \operatorname{tr}\{ \sqrt{R_0} (\sqrt{R_0}\, q \sqrt{R_0})^l \sqrt{R_0} f \} \right|$$

cycle

$$\le \sum_{l=2}^{\infty} \|R_0 f R_0\|_{op} \|\sqrt{R_0}\, q \sqrt{R_0}\|_{HS}^l \le \sum_{l=2}^{\infty} \|\sqrt{R_0}\|_{op}^2 \|f\|_\infty \left(\frac{1}{\sqrt{k}} \|q\|_{H^{-1}_k} \right)^l$$

remember $\sqrt{R_0} = (-\partial_x^2 + k^2)^{-1/2}$, on Fourier side is mult by $\frac{1}{(\xi^2+k^2)^{1/2}}$

so $\|\sqrt{R_0}\|_{op} \lesssim \frac{1}{k}$. $\leftarrow$ Lec 6 Fourier multipliers

$$\lesssim \frac{1}{k^2} \|f\|_\infty \frac{1}{k} \|q\|^2_{H^{-1}_k}$$

assuming $\|q\|_{H^{-1}_k} < 1$, geom series is $\sim$ leading term

$$\Rightarrow \left\| g - \frac{1}{2k} + \frac{1}{k} R_0(2k)\, q \right\|_{L^1} \lesssim \frac{1}{k^3} \|q\|^2_{H^{-1}_k}.$$

$H^1 \hookrightarrow L^\infty$ so $L^1 \hookrightarrow H^{-1}$, so can control

$$\Rightarrow \left\| 4k^3 \left(g - \frac{1}{2k}\right) + q \right\|_{H^{-1}} \le 4k^3 \left\| g - \frac{1}{2k} + \frac{1}{k} R_0(2k) q \right\|_{H^{-1}} + \left\| g - 4k^2 R_0(2k)\, q \right\|_{H^{-1}}$$

$$\lesssim 4k^3 \frac{1}{k^3} \|q\|^2_{H^{-1}_k} + \|q\|_{H^{-1}_k}$$

comparison Fourier side

$$\lesssim \|q\|^2_{H^{-1}_k} + \|q\|_{H^{-1}_k}$$

Return to Goal above: $\sup_{g \in \mathcal{Q}^*} \sup_{|t| \le T} \| \underbrace{e^{tJ\nabla(H_{KdV} - H_K)} g}_{=: g(t)} - g \|_{H^{-1}} \to 0$ as $K \to \infty$.

$g(t)$ lies in $\mathcal{Q}^*$

Let $x_c \ge 1$ be fixed. Take $K \ge x_c + 1$.

By diffeomorphism $z \mapsto x - \frac{1}{2g(x)}$ $\quad g \in \mathcal{Q}^* \subseteq B_S$ (space on which we defined diffeomorphism.)

$H^{-1} \supseteq B_S \longrightarrow H^1$

It suffices to work in H^1 instead of H^{-1}! :

Goal': $\sup_{g \in \mathcal{Q}^*} \sup_{|t| \le T} \| \frac{1}{2g(t)} - \frac{1}{2g(0)} \|_{H^1} \xrightarrow{x_c \to \infty} 0$ where $g(t) := g(t, x, x_c) = g(x; x_c, g(t))$

$\| \cdot \|_{L^\infty}$

$\int_0^t \left(\frac{d}{d\tau} \frac{1}{2g(\tau)} \right)$

$$\frac{d}{dt} \frac{1}{2g(t)} = -\frac{1}{2g^2(t)} \frac{d}{dt} g(t) = \left\{ \frac{1}{g(t)} \left[2 + \frac{4K^5}{K^2 - x^2} \left(g(t) - \frac{1}{2K} \right) + \frac{4x^5}{K^2 - x^2} \left(\frac{1}{2x} - g(x) \right) \right] \right\}$$

$\vdots$

$\| \frac{d}{dt} \frac{1}{2g(t)} \|_{H^{-2}} \xrightarrow{x_c \to \infty} 0.$

We are 3 derivatives short $\frac{1}{2}$.

But actually this is fine. (using equality at $\mathcal{Q}^*$)