4/1/24 Monica Visan 257B, KdV equation.

Conventions: (Fourier transform): $f \in S(\mathbb{R}, \mathbb{C})$

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} f(x)\, dx \quad \text{so that} \quad \text{FIF} \quad f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi} \hat{f}(\xi)\, d\xi.$$

and Plancherel has $\|f\|_2 = \|\hat{f}\|_2$.

(mostly homog, but define inhomog too)

often use Sobolev spaces: $H^s(\mathbb{R}) :=$ completion of $S(\mathbb{R})$ wrt. norm

$$\|f\|_{H^s} := \| \langle \xi \rangle^s \hat{f}(\xi) \|_{L^2_\xi}$$

"up to s derivatives in L^2"

defined on Fourier side

$\langle x \rangle := (1 + |x|^2)^{1/2}$ (linear growth at ∞, const at 0)

homog: $\|f\|_{\dot{H}^s} := \| |\xi|^s \hat{f}(\xi) \|_{L^2_\xi}$

Consider init-val problem for Airy eq: $\begin{cases} u_t + u_{xxx} = 0 \\ u(0,x) := u_0(x) \in S(\mathbb{R}) \end{cases}$

Here (physical considerations), only interested in real-valued fns. $u: \mathbb{R}_t \times \mathbb{R}_x \to \mathbb{R}$

How to solve? Take FT wrt. space variable!:

$$\begin{cases} \partial_t \hat{u}(t,\xi) - i\xi^3 \hat{u}(t,\xi) = 0 \\ \hat{u}(0,\xi) = \hat{u}_0(\xi) \end{cases}$$

Now an ODE in t w parameter ξ, $\Rightarrow \quad \hat{u}(t,\xi) = e^{it\xi^3} \hat{u}_0(\xi)$ (1)

unique sol to above init val problem.

Deduce immediately: from (1)

1) $u_0 \in S(\mathbb{R}) \Rightarrow \hat{u}_0 \in S(\mathbb{R}) \Rightarrow e^{it\xi^3} \hat{u}_0(\xi) \in S(\mathbb{R}_\xi) \quad \forall t \in \mathbb{R}$

$\Downarrow$ inverse FT

$u(t,\xi) \in S(\mathbb{R}_\xi) \quad \forall t \in \mathbb{R}$

2) ∞-many conservation laws.

$$\|u(t)\|_{H^s} := \|\langle\xi\rangle^s \hat{u}(t,\xi)\|_2 \underset{(1)}{=} \|\langle\xi\rangle^s \hat{u}_0(\xi)\|_2 =: \|u_0\|_{H^s} \quad \forall t \in \mathbb{R}, \ s \in \mathbb{R}$$

(1) unimodular factor, goes away

Note also that

$$\int u(t,x)\,dx = \int u_0(x)\,dx \quad \forall t \in \mathbb{R}$$ by taking ∂_t and seeing it is 0:

$$\partial_t \int u(t,x)\,dx = -\int \partial_x^3 u(t,x)\,dx = 0 \quad (\text{b/c } \lim_{x\to\infty} u_{xx} = 0)$$

integral of derivative of nice fn, FTC

u is Schwartz, discussed above

Taking inverse FT of (1),

$$u(t,x) = \frac{1}{\sqrt{2\pi}} \int e^{ix\xi} \hat{u}(t,\xi)\,d\xi = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi} e^{it\xi^3} \hat{u}_0(\xi)\,d\xi = \frac{1}{\sqrt{2\pi}} \int e^{i\xi(x+t\xi^2)} \hat{u}_0(\xi)\,d\xi.$$

compare to $u_0(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\xi x} \hat{u}_0(\xi)\,d\xi$

superposition of cosines given by $\hat{u}_0(\xi)$

superposition of wave packet travel to left b/c $+t\xi^3 > 0$.

(wave packet of wave vector ξ travels to the left w/ velocity ξ^2)

b/c high frequencies escape faster,

expect

- decay in time
- local smoothing effects (∀ cpct region of spacetime, after some time, high freq all escape, leaving behind low freq, which are smoother)

defining feature of dispersive eq: different frequencies travel at different velocities

(ex Airy eq & dispersive eq).

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we use the following notation for sol of the Airy eq:

$$u(t,x) = \left[e^{-t\partial_x^3} u_0\right](x)$$ (taking FT of both sides get (1)).

---

we write $\left[e^{-t\partial_x^3} u_0\right](x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{it\xi^3 + i\xi x} \hat{u}_0(\xi)\,d\xi.$

want to get representation in terms of u_0, not $\hat{u}_0$.

Simply replace $\hat{u}_0$ w/ def:

$$= \frac{1}{2\pi} \int_{\mathbb{R}} e^{it\xi^3 + i\xi x} \int_{\mathbb{R}} e^{-iy\xi} u_0(y)\, dy\, d\xi.$$

$$= \frac{1}{2\pi} \iint e^{it\xi^3 + i\xi(x-y)} u_0(y)\, d\xi\, dy$$

illegal b/c cont. integrate in ξ, no decay in ξ!

"Fubini"

to justify Fubini, ξ-of-room w/ $e^{-\varepsilon\xi^2}$.

$$\int e^{it\xi^3 + ix\xi} e^{-\varepsilon\xi^2} \hat{u_0}(\xi)\, d\xi$$

$\downarrow$ DCT $\varepsilon \downarrow 0$ (dominated by just $|\hat{u_0}|\in L^1$ b/c it's Schwartz)

$$\int e^{it\xi^3 + ix\xi} \hat{u_0}(\xi)\, d\xi$$

$$= \frac{1}{2\pi} \iint e^{i\frac{\eta^3}{3} + i\eta\frac{x-y}{(3t)^{1/3}}} u_0(y)\, dy\, (3t)^{-1/3}\, d\eta$$

$t\xi^3 = \frac{\eta^3}{3}$

$\xi = (3t)^{-1/3}\eta$

$$= (3t)^{-1/3} \int \mathrm{Ai}\left(\frac{x-y}{(3t)^{1/3}}\right) u_0(y)\, dy \quad (3)$$

convolution of dirty fn.

$$\mathrm{Ai}(x) := \frac{1}{2\pi} \int e^{i\frac{\xi^3}{3} + i\xi x}\, d\xi.$$

(must justify why well-defined, later)

and we understand

$$\mathrm{Ai}(x) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\pi} \int e^{i\frac{\xi^3}{3} + i\xi x} e^{-\varepsilon\xi^2}\, d\xi$$

(or improper integral at $\pm\infty$)

observe that $\overline{\mathrm{Ai}(x)} = \mathrm{Ai}(x)$ $\leftrightarrow$ ξ change var to $-\xi$

$\Rightarrow$ Ai(x) is real,

$$= \mathrm{Re}\, \frac{1}{2\pi} \int e^{i\frac{\xi^3}{3} + i\xi x}\, d\xi$$

$$= \frac{1}{\pi} \int \cos\left(\frac{\xi^3}{3} + \xi x\right) d\xi$$

We'll show that $\mathrm{Ai}(x)$ is unfortunately bad. Then (3) yields

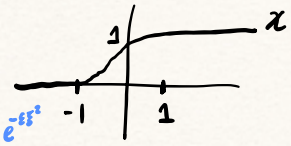
$$\| e^{-t\partial_x^3} u_0 \|_{L^\infty_x} \le (3|t|)^{-1/3} \|\mathrm{Ai}\|_{L^\infty} \|u_0\|_{L^1_x} \lesssim |t|^{-1/3} \|u_0\|_{L^1_x} \quad \forall t \neq 0.$$

This is dispersive estimate (already see smoothing: instantaneously in L^∞)

Lemma: $\sup_{x\in\mathbb{R}} |\mathrm{Ai}(x)| \lesssim 1$. (will actually show $\mathrm{Ai}(x) \to 0$ as $|x| \to \infty$)

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Pf: (need χ for "$\to 0$") split integral for Airy fn by $\chi \in C^\infty(\mathbb{R})$,



$$\chi(x) = \begin{cases} 1 & x \ge 1 \\ 0 & x \le -1 \end{cases}$$

and moreover $\chi(x) + \chi(-x) \equiv 1$.

$$\int e^{i\frac{\xi^3}{3} + i\xi x}\, dx = \int e^{i\frac{\xi^3}{3} + i\xi x} \chi(\xi)\, d\xi + \underbrace{\int e^{i\frac{\xi^3}{3} + i\xi x} \chi(-\xi)\, d\xi}_{= \int e^{-(i\frac{\xi^3}{3} + i\xi x)} \chi(\xi)\, d\xi}$$

(really $\lim_{\varepsilon\to 0}$ of $\cdot e^{-\varepsilon\xi^2}$; the first integral has $\cdot e^{-\varepsilon\xi^2}$; the second has $\cdot e^{-\varepsilon\xi^2}$)

change var $\xi \mapsto -\xi$; doesn't affect $\cdot e^{-\varepsilon\xi^2}$ b/c ξ^2 even power

$$= 2\,\mathrm{Re} \int e^{i\frac{\xi^3}{3} + i\xi x} \chi(\xi)\, d\xi.$$

$\cdot e^{-\varepsilon\xi^2}$

oscillatory integral. — understand stationary pts, ∂ phase

what derivatives (if any) unfortunately bold below

phase $\phi(\xi) = \frac{\xi^3}{3} + \xi x \Rightarrow \phi'(\xi) = \xi^2 + x$ (take derivatives until bdd ≥ 0.)

$\phi''(\xi) = 2\xi$

$\phi'''(\xi) = 2$. → ready to use van der Corput ($\lambda = 0$)

For $x \geq 1$, phase does not have stationary pts which means we do non stationary phase, Integrate By Parts:

$\phi'(\xi) = \xi^2 + x \geq x \geq 1 \quad \forall \xi \in \operatorname{supp}\chi$.

$$e^{i\phi(\xi)} = \frac{1}{i\phi'(\xi)} \partial_\xi e^{i\phi(\xi)} = \frac{1}{i(\xi^2+x)} \partial_\xi e^{i\phi(\xi)}$$

$$\Rightarrow \int e^{i\phi(\xi)} \chi(\xi)\, d\xi = \int (\partial_\xi e^{i\phi\xi}) \cdot \frac{1}{i(\xi^2+x)} \chi(\xi)\, d\xi$$

$$= \int e^{i\phi(\xi)} \partial_\xi \left[\frac{1}{i(\xi^2+x)} \chi(\xi) \right] d\xi$$

what happens at bdry?

left bdry, 0 b/c $\chi \equiv 0$ at $-\infty$

right bdry, 0 b/c $\frac{1}{\xi^2} \to 0$ as $\xi \to \infty$.

So bdy contribution is 0.

try bdry from here.

$$\left| \int e^{i\phi(\xi)} \chi(\xi)\, d\xi \right| \lesssim \left\| \frac{1}{\xi^2+x} \chi'(\xi) \right\|_1 + \left\| \frac{\xi}{(\xi^2+x)^2} \chi \right\|_1$$

(← differentiate)

$\lesssim |x|^{-1} + |x|^{\frac{1}{2}+\frac{1}{2}-2} \sim |x|^{-1} \to 0$ ✓

$\xi = |x|^{1/2}\eta$

$$= -\int e^{i\phi(\xi)} \underbrace{\partial_\xi \frac{1}{\xi^2+x} \cdot \partial_\xi \frac{1}{\xi^2+x} \cdot \chi(\xi)}_{\text{compose } \partial_\xi \text{ and mult-by-} \frac{1}{\xi^2+x} \text{ operators.}} d\xi$$

?

So $$\left| \int e^{i\phi(\xi)} \chi(\xi)\, d\xi \right| \lesssim \left\| \frac{\chi''}{(\xi^2+x)^2} \right\|_{L^1} + \left\| \frac{\chi' \cdot \xi}{(\xi^2+x)^3} \right\|_{L^1} + \left\| \frac{\chi \cdot \xi^2}{(\xi^2+x)^4} \right\|_1$$

(both derivatives hit χ; 1 derivative hit χ; both derivatives miss χ.)

χ' and χ'' cpctly supported on $[-1,1]$, both bdd by $\langle x \rangle^{-2}$

how to integrate $\int_{-1}^{\infty}$, ξ^2 in num, cancel w/ 1 factor in denom, integrate in ξ takes another ξ from denom, still end up w/ $\lesssim \langle x \rangle^{-2}$.

$\lesssim \langle x \rangle^{-2}$

For $x < 1$: we may use van der Corput on $|\phi'''(\xi)| \geq 2$ (b/c $\phi'''(\xi) = 2$)

(actually $\forall x$. But above bd for $x \geq 1$ just sharper.)

Lemma (van der Corput): For $k \geq 1$ and assume $|\phi^{(k)}(\xi)| \geq \delta > 0 \quad \forall \xi \in [a,b]$.

If $k = 1$, we assume additionally that ϕ' is monotone.

Then, $$\left| \int_a^b e^{i\phi(\xi)} \psi(\xi)\, d\xi \right| \leq C_k \delta^{-\frac{1}{k}} \left[|\psi(b)| + \|\psi'\|_1 \right]$$

C_k only depends on k, (not on δ, ψ, ϕ, and importantly not on a, b (so can take $b \to \infty$))

$$\left| \int e^{i\phi(\xi)} \chi(\xi)\, d\xi \right| \lesssim \|\chi\|_\infty + \|\chi'\|_1 \lesssim 1.$$

($k=3$, $\delta = 2$; $e^{-\varepsilon\xi^2}$; replace by $\|\chi'(\xi)\cdot e^{-\varepsilon\xi^2} + \varepsilon\xi e^{-\varepsilon\xi^2}\chi(\xi)\|_{L^1}$; L^1 bdd by $\|\varepsilon\xi e^{-\varepsilon\xi^2}\|_{L^1_\xi} \overset{\eta = \sqrt{\varepsilon}\cdot\xi}{=} \|\eta e^{-\eta^2}\|_{L^1}$, $\lesssim 1$ unif in ε. no negative powers of ε.)

Cor: $\lim_{\varepsilon \to 0} \int$ definite A_ε converges in C_0 as $\varepsilon \to 0$. b/c $\|e^{\varepsilon\xi^2} - e^{\tilde{\varepsilon}\xi^2}\|_{L^\infty}$, in fact $\lesssim \varepsilon$.

4/2/24

Recall last time, sol'n to $\begin{cases} u_t + u_{xxx} = 0 \\ u(0) =: u_0 \in \mathcal{S}(\mathbb{R}) \end{cases}$ is

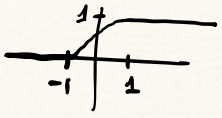
$$u(t,x) = [e^{-t\partial_x^3} u_0](x) = \frac{1}{\sqrt{2\pi}} \int e^{it\xi^3 + ix\xi} \hat{u}_0(\xi)\, d\xi$$

$$= (3t)^{-1/3} \int \mathrm{Ai}\left(\frac{x-y}{(3t)^{1/3}}\right) u_0(y)\, dy$$

where $\mathrm{Ai}(x) := \frac{1}{2\pi} \int e^{i\frac{\xi^3}{3} + i\xi x}\, d\xi$.

Lemma: $\|\mathrm{Ai}\|_{L^\infty} \lesssim 1$, and stronger, $\lim_{|x|\to\infty} \mathrm{Ai}(x) = 0$.

pf: we prove this by introducing χ



, $\chi(x) + \chi(-x) \equiv 1$

and rewrite $\mathrm{Ai}(x) = \frac{1}{\pi} \mathrm{Re} \int e^{i\frac{\xi^3}{3} + i\xi x} \chi(\xi)\, d\xi$.

above lecture

analyze using oscillatory integral.

Phase $\phi(\xi) = \frac{\xi^3}{3} + \xi x \Rightarrow \phi'(\xi) = \xi^2 + x \Rightarrow \phi''(\xi) = 2\xi$, $\phi'''(\xi) = 2$

van der Corput on $\phi''(\xi) = 2$ immediately gives $\left| \int e^{i\frac{\xi^3}{3} + i\xi x} \chi(\xi)\, d\xi \right| \lesssim \|\chi\|_{L^\infty} + \|\chi'\|_{L^1} \lesssim 1$ (uniform in x)

For $x \geq 1$, we saw that can just IBP ($\phi'(\xi) = \xi^2 + x \geq x \geq 1 \quad \forall \xi \in \mathrm{supp}\,\chi$)

last time summarized: $\left| \int e^{i\frac{\xi^3}{3} + i\xi x} \chi(\xi)\, d\xi \right| \leq \left| \int e^{i\phi(\xi)} \left(\partial_\xi \frac{1}{\xi^2 + x} \cdot\right)^2 \chi(\xi)\, d\xi \right| \lesssim |x|^{-2}$

Remains to analyze $x \leq -1$.

recall $\phi'(\xi) = \xi^2 + x = \xi^2 - |x|$.

$\exists$ a unique critical pt for ϕ on $\mathrm{supp}\,\chi$, namely $\xi_0 = |x|^{1/2}$.

Observe that $\phi''(\xi_0) = 2|x|^{1/2}$, so on small nbhd on ξ_0 we have $|\phi''(\xi)| \geq |x|^{1/2}$

(small nbhd, say $(\frac{\xi_0}{2}, \frac{3\xi_0}{2})$)

$\phi''' \equiv 2$, so nbhd can be taken of size $|x|^{1/2}$.

By van der Corput, ($k=2$, "δ" $= |x|^{1/2}$) on this nbhd, get the bound

$$\lesssim (|x|^{1/2})^{-1/2} \left[\|\chi\|_{L^\infty} + \|\chi'\|_{L^1} \right] \lesssim |x|^{-1/4} \xrightarrow{|x|\to\infty} 0$$

away from this nbhd, can IBP as we always do to get decay in x.

around ξ_0

By Young, ($\infty-1$, trivial case of Young) (see also 3/30/26 Rec 1)

$$\|e^{-t\partial_x^3} u_0\|_{L^\infty_x} \lesssim |t|^{-1/3} \|Ai\|_{L^\infty} \cdot \|u_0\|_1 \lesssim |t|^{-1/3} \|u_0\|_1$$

↳ solution to Airy eq. — uniform in $t \neq 0$.

↳ expresses 2 key ideas:
- decay
- smoothness (blc immediately $t>0$ enter L^∞ unif no matter L^∞ of orig u_0)

Interpolating w/ conservation law $\|e^{-t\partial_x^3} u_0\|_2 = \|u_0\|_2 \quad \forall t \in \mathbb{R}$

Interpolating

"Fourier multiplier, symbol unimodular"

we get generalized dispersive estimates $\forall 2 \le p < \infty$, we evaluate Airy evolution in L^p

$$\|e^{-t\partial_x^3} u_0\|_{L^p_x} \lesssim |t|^{-\frac{1}{3}(1-\frac{2}{p})} \|u_0\|_{L^{p'}_x} \quad \text{uniformly in } t \neq 0.$$

($e^{-t\partial_x^3}$: Airy propagator)

$$\|f\|_p \lesssim \|f\|_\infty^{1-\frac{2}{p}} \|f\|_2^{\frac{2}{p}}$$

intuition:
$p=\infty$ need $-\frac{1}{3}$, so get to $-\frac{1}{3}(1-\frac{2}{p})$
$p=2$ need weight to be $0 \Rightarrow$?=2.

Using a TT^* argument edits 4/8/26 Lec 5

Using this, we get the following Strichartz estimates for $2 \le q, r \le \infty$ satisfying scaling relation $\frac{3}{q} + \frac{1}{r} = \frac{1}{2}$.

$$\|e^{-t\partial_x^3} f\|_{L^q_t L^r_x} \lesssim \|f\|_{L^2_x}$$

($e^{-t\partial_x^3}$: Airy propagator; f: initial data; mixed space; unif f, implies unif in λ)

=: T

$T^* F = \int e^{-s\partial_x^3} F(s)\, ds$.

so $\Leftrightarrow$

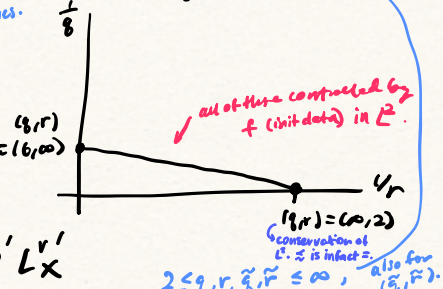
and retarded Strichartz estimate

$$\left\| \int_0^t e^{-(t-\tau)\partial_x^3} F(\tau)\, d\tau \right\|_{L^q_t L^r_x} \lesssim \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x}$$

(control backwards in time; first $\int$ in space then avg in time; whatever appear on RHS e.g. KdV nonlinearity)

RHS can in fact be $\|\cdot\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x}$ $\forall (\tilde{q}, \tilde{r})$

from $u(t,x)$ solves $\partial_t u + \partial_{xxx} u = 0$ $\Rightarrow u^\lambda(t,x) := u(\frac{t}{\lambda^3}, \frac{x}{\lambda})$ also solves.



conservation of L^2 is infact = also for $(\tilde{q}, \tilde{r})$.

$2 \le q, r, \tilde{q}, \tilde{r} \le \infty$, Littlewood principle (ct. Hausdorff-Young)

Unfortunately, all of these estimates can't help w/ KdV blc derivative in nonlinearity and we would need H^1 control to get L^∞ control, 1:15 hr min

By variation of params (basic ODE class), sol to KdV satisfies integral eq. $u(t) = e^{-t\partial_x^3} u_0 + 3\int_0^t e^{-(t-s)\partial_x^3} \cdot \partial_x(u^2)(s)\, ds$.

To build sol via contraction mapping, WTS $\phi(u)(t) = e^{-t\partial_x^3} + 3\int_0^t e^{-(t-s)\partial_x^3} \cdot \partial_x(u^2)(s)\, ds$ is self-map & contraction on suitable Banach space

but to bd $\phi(u)$, need to bd derivative of u. So to close loop need to bd $\phi'(u)$. etc. ∴
Fix: dispersive bd, gain of derivatives!

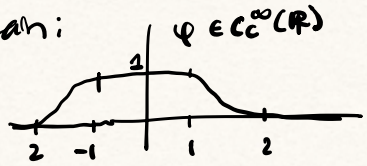
dispersive (high freqs run away very fast, ξ^2, faster than even Schrödinger running away at ξ)

↪ smoothing, gain of derivatives.

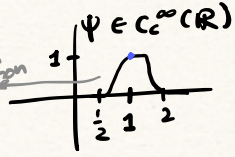
To make this rigorous, need to use language of Littlewood Paley theory. (↪ hack into LP pieces, prove each piece gains derivatives, then assemble together.)

Let us set up notation again:

$\varphi \in C_c^\infty(\mathbb{R})$



$\psi \in C_c^\infty(\mathbb{R})$ and reflection



$\psi(x) := \varphi(x) - \varphi(2x)$

dyadic $N \in 2^{\mathbb{Z}} := \{2^n : n \in \mathbb{Z}\}$, let $\psi_N(x) := \psi(\frac{x}{N})$ supp on $[\frac{N}{2}, 2N]$

$N = 1024$, super broad, "centered at 1024"

$N = \frac{1}{1024}$, super tight! "centered at $\frac{1}{1024}$"

we have partition of unity property $\sum_{N \in 2^{\mathbb{Z}}} \psi_N(x) = 1 \quad \forall x \neq 0.$

$$\varphi(x) + \sum_{N \in 2^{\mathbb{N}}} \psi_N(x) = 1 \quad \forall x \in \mathbb{R}$$

palatable bc it's obvious

LP projections defined as follows:

$\widehat{P_N f}(\xi) := \hat{f}(\xi)\,\psi(\frac{\xi}{N}) \iff P_N f = f * N\check{\psi}(N\cdot)$ — proj to freq $\xi \sim N$

$f_N := $ "f proj to freqs like N"

$P_{\leq N} f := \hat{f}(\xi)\cdot\varphi(\frac{\xi}{N}) \iff f_{\leq N} := P_{\leq N} f := f * N\check{\varphi}(N\cdot)$

$P_{>N} f = (1 - P_{\leq N}) f$

$P_{N \leq \cdot \leq M} := \sum_{N \leq k \leq M} P_k.$

$$f = f_{\leq 1} + \sum_{N \in 2^{\mathbb{N}}} f_N$$

Recall: H-Y $\|f_N\|_p \lesssim \|f\|_p \cdot \underbrace{\|N\check{\psi}(N\cdot)\|_1}_{\sim 1}$

all projs are bdd $L^p \to L^p$, unif in $N \in 2^{\mathbb{Z}}$

Recall: Bernstein Sobolev

(recall $(e^{it\partial_x^3} f) \sim \int e^{it\xi^3 + i\xi x} \hat{f}(\xi)\,d\xi$)

Lemma: we have $\varphi(\xi) \cdot \hat{f}(\xi) + \sum_{N \in 2^{\mathbb{N}}} \psi_N(\xi)\hat{f}(\xi)$ — bd in L^∞ by $\|f\|_1$

(1) $\left| \int e^{it\xi^3 + i\xi x} \varphi(\xi)\,d\xi \right| \lesssim$ "low" freq $\lesssim 1$

$$\begin{cases} \langle t \rangle^{-1/2} & \forall t, x \\ |x|^{-2} & |x| \geq 14|t| \end{cases}$$

no matter what, get $(-\frac{1}{2})$ decay

x large than t, phase ϕ no stationary pt, better decay (IBP)

(2) $\left| \int e^{it\xi^3 + i\xi x} \psi(\frac{\xi}{N})\,d\xi \right| \lesssim$ "high" focus on freq $\sim N$

$$\begin{cases} (N\langle t \rangle)^{-1/2} & \forall t, x \\ N^1 \langle |x| + N^2|t| \rangle^{-2} & |x| \not\sim N^2|t| \end{cases} \quad \leftarrow (\star)$$

$N^{-1/2}$ means (by Bernstein) $\frac{1}{2}$ derivative gain in dispersive est for freq localized fns.

Cf. Bernstein-Sobolev: $\forall 1 \leq p \leq \infty$, $\forall s \in \mathbb{R}$,

$$\| |\nabla|^s P_N f \|_p \sim N^s \|P_N f\|_p \quad \text{unif in } N \in 2^{\mathbb{Z}}$$

Using $(\star)$ + square fn estimate, we get the following Strichartz estimates — to reassemble the frequencies

$\| |\nabla|^{(s)} e^{-t\partial_x^3} f \|_{L^p_x} \lesssim |t|^{-\frac{1}{2}(\cdots)} \|f\|_{L^{p'}_x}$

at $p=\infty$, is $-\frac{1}{2}$ so $-\frac{1}{2}(1-\frac{2}{p})$

$p=2$, conservation $\Rightarrow -\frac{1}{2}(1-\frac{2}{p})$.

$p=\infty$, is $-\frac{1}{2}$, so again $-\frac{1}{2}(1-\frac{2}{p})$.

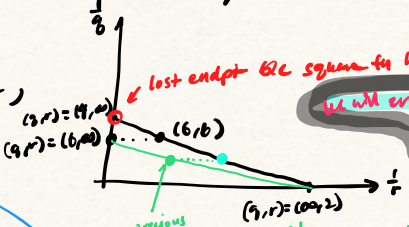
$p=2$ is 0

use this as input

$\forall 2 \leq q \leq \infty$, $\forall 2 \leq r < \infty$, $\frac{2}{q} + \frac{1}{r} = \frac{1}{2}$,

control of extra gain of $\frac{1}{q}$ derivative

$\frac{3}{q} + \frac{1}{r} - \frac{1}{q} = \frac{1}{2}$ — Airy 3 time = 1 space



we will eventually need to close contraction mapping for KdV in quadr. nonlin.

endpt, using complex interpolation method of Stein.

new Strichartz line ests. imply old ests by Sobolev embedding, except in space $L^6_t L^\infty_x$

$$\| e^{-t\partial_x^3} |\nabla|^{\frac{1}{6}} f \|_{L^6_t L^r_x} \lesssim \|f\|_2$$

gain $\frac{1}{6}$ — gain $\frac{1}{6}$ again

gain $\frac{2}{6}$ derivatives! B/c TT^*

and retarded Strichartz est: $\left\| \int_0^t e^{-(t-\tau)\partial_x^3} |\nabla|^{\frac{2}{6}} F(\tau)\,d\tau \right\|_{L^6_t L^r_x} \lesssim \|F\|_{L^{6'}_t L^{r'}_x}$

Interp! $\| |\nabla|^{\frac{1}{6}} e^{-t\partial_x^3} f \|_{L^6_{t,x}} \lesssim \|f\|_{L^2_x}$

cf. previously, $\| e^{-t\partial_x^3} f \|_{L^6_t L^\infty_x} \lesssim \|f\|_{L^2_x}$

(both these satisfy same scaling... 1:30 hr min nonlogue)

Pf: $|t|^{-1/3}$ by van der Corput, get $\langle t\rangle^{-1/3}$ by Δ-ineq.

Δ-ineq: $\left|\int e^{i\xi^3 t + i\xi x}\varphi(\xi)\,d\xi\right| \le \|\varphi\|_1 \lesssim 1$. (a)

Need to improve on Δ ineq by exploiting

van der Corput: phase $\phi(\xi) = \xi^3 t + \xi x$, $\phi'(\xi) = 3\xi^2 t + x$, $\phi''(\xi) = 6\xi t$, $\phi'''(\xi) = 6t$

$t\neq 0$ ↳ $\left|\int e^{i\xi^3 t + i\xi x}\varphi(\xi)\,d\xi\right| \lesssim |t|^{-1/3}\left[\|\varphi\|_\infty + \|\varphi'\|_1\right] \lesssim |t|^{-1/3}$ (b)

low freq $\lesssim 1$; just some #s.

(a)+(b) gives the $\langle t\rangle^{-1/3}$ decay in (1)

Note ϕ has stationary pt if:

$\phi'(\xi) = 3\xi^2 t + x = 0 \Rightarrow |x| = 3\xi^2|t| \le 12|t|$.

no stationary pts! for $\xi \in \operatorname{supp}\varphi$.

So, s/c $|x| > 12|t|$, $|x| > 14|t|$ can integrate by parts (twice, no ∂ terms b/c φ $\in C_c^\infty$ supp)

$$\left|\int e^{i\phi(\xi)}\varphi(\xi)\,d\xi\right| = \left|\int e^{i\phi(\xi)}\left(\partial_\xi \frac{1}{3\xi^2+x}\cdot\right)^2 \varphi(\xi)\,d\xi\right| \lesssim |x|^{-2}$$ (c)

$\lesssim \left\|\frac{\varphi''}{(3\xi^2+x)^2}\right\|_1 + \left\|\frac{\xi\varphi'}{(3\xi^2+x)^3}\right\|_1 + \left\|\frac{\xi^2\varphi}{(\,)^4}\right\|_1$ (same as before)

details: both ∂_ξ hit φ; both ∂_ξ hit $\frac{1}{\cdots}$

e.g. kill ξ^2 w/ 2 powers of x^2 b/c $|x| \gg |t|$.

(a)+(c) gives $\langle x\rangle^{-2}$ in (1) RHS

Now, we approach (2):

how to get $|t|^{-1/2}$? (instead of $|t|^{1/4}$ previously?) van der Corput w/ 2 derivatives.

$\phi''(\xi) = 6t\xi$ on $\operatorname{supp}\psi_N$, $|\phi''(\xi)| = 6|\xi||t| \sim N|t|$ (factor of 2)

van der Corput yields bound

$$(N|t|)^{-1/2}\left[\|\psi''(\tfrac{\cdot}{N})\|_\infty + \tfrac{1}{N}\|\psi'(\tfrac{\cdot}{N})\|_1\right] \lesssim (N|t|)^{-1/2}$$

$(\psi_N)' = \frac{1}{N}\psi'(\frac{\cdot}{N})$; ψ_N''; $\operatorname{supp}\sim N \Rightarrow$ L^1 norm $\sim N$, cancel w/ $\frac{1}{N}$

and finally, note $\phi'(\xi) = 3\xi^2 t + x = 0 \iff |x| = 3\xi^2|t| \sim 3N^2 t$ on $\operatorname{supp}\psi_N$.

So, for $x \neq N^2|t|$ (not comparable to (factor of 10 say)), IBP:

$$\left|\int e^{i\phi(\xi)}\psi\left(\tfrac{\xi}{N}\right)d\xi\right| = \left|\int e^{i\phi(\xi)}\left(\partial_\xi \frac{1}{3\xi^2+x}\cdot\right)^2 \psi\left(\tfrac{\xi}{N}\right)d\xi\right|$$

$\lesssim N^{-2}\left\|\frac{1}{(3\xi^2+x)^2}\right\|_{L^1(|\xi|\sim N)} + N^{-1}\left\|\frac{\xi t}{(3\xi^2+x)^3}\right\|_{L^1(|\xi|\sim N)} + \left\|\frac{\xi^2 t^2}{(3\xi^2+x)^4}\right\|_{L^1(|\xi|\sim N)}$

both ∂ hit φ; one ∂ hit ψ, one ∂_ξ hit $\frac{1}{3\xi^2+x}$

$\sim N^{-1}\left\|\frac{\xi t\cdot N^{-1}}{(3\xi^2+x)^3}\right\|_{L^1(|\xi|\sim N)} + \left\|\frac{\xi^3\cdot\xi^2 t^2\cdot N^{-2}}{(3\xi^2+x)^4}\right\|_{L^1(|\xi|\sim N)}$

so all terms look like 1st term.

$\lesssim N^{-1}\left(N^2|t| + |x|\right)^{-2}$

Cor: $\left\| \| e^{i\xi^3 t + ix\xi} \varphi(\xi)\, d\xi \|_{L^\infty(|t|\le T)} \right\|_{L^1_x} \lesssim \langle T \rangle$

$\left\| \| e^{i\xi^3 t + ix\xi} \psi\left(\frac{\xi}{N}\right) d\xi \|_{L^\infty(|t|\le T)} \right\|_{L^1_x} \lesssim N^{3/2} \langle T \rangle^{1/2}$

unoptimal type inf., correct price, we lose derivatives "

4/5/24

edits 4/10/26 Lec 6

Last time we proved

Lemma $$\left|\int_{\mathbb{R}} e^{it\xi^3+ix\xi}\varphi(\xi)\,d\xi\right| \lesssim \begin{cases} \langle t\rangle^{-1/3}, & \forall t,x \\ |x|^{-2}, & |x|\geq 14|t| \end{cases}$$

$$\left|\int_{\mathbb{R}} e^{it\xi^3+ix\xi}\psi_N(\xi)\,d\xi\right| \lesssim \begin{cases} (N|t|)^{-1/2}, & \forall t,x \\ N^{-1}(|x|+N^2|t|)^{-2}, & |x|\neq N^2|t| \end{cases}$$ uniformly for $N\in 2^{\mathbb{Z}}$

precisely $|x|\leq \frac{1}{2}N^2|t|$ or $|x|\geq 14N^2|t|$ (far away from stationary pt)

This provides the key ingredient in our forthcoming proof of the maximal function estimate.

Corollary
$$\left\| \left| \int_{\mathbb{R}} e^{it\xi^3+ix\xi}\varphi(\xi)\,d\xi \right|_{L^\infty(|t|\leq T)} \right\|_{L^1_x} \lesssim \langle T\rangle \qquad (1)$$

$$\left\| \left| \int_{\mathbb{R}} e^{it\xi^3+ix\xi}\psi_N(\xi)\,d\xi \right|_{L^\infty(|t|\leq T)} \right\|_{L^1_x} \lesssim N^{3/2}\langle T\rangle^{1/2} \qquad (2)$$ only care for uniformly for $1\leq N\in 2^{\mathbb{Z}}$

uniformly for $T>0$ and $1\leq N\in 2^{\mathbb{Z}}$.

Proof: Consider (1). Fix x. Split into cases based on $|x|$ large, small, med

For $|x|\geq 14\langle T\rangle$, we may use the previous lemma to estimate

$$\left\| \left| \int_{\mathbb{R}} e^{it\xi^3+ix\xi}\varphi(\xi)\,d\xi \right|_{L^\infty(|t|\leq T)} \right\|_{L^1(|x|\geq 14\langle T\rangle)} \lesssim \| |x|^{-2}\|_{L^1(|x|\geq 14\langle T\rangle)} \lesssim \langle T\rangle^{-1} \lesssim \langle T\rangle.$$

For $|x|\leq 14\langle T\rangle$, we use the previous lemma to estimate

$$\left\| \left| \int_{\mathbb{R}} e^{it\xi^3+ix\xi}\varphi(\xi)\,d\xi \right|_{L^\infty(|t|\leq T)} \right\|_{L^1(|x|\leq 14\langle T\rangle)} \lesssim \left\| \langle t\rangle^{-1/3} \right\|_{L^\infty(|t|\leq T)} \|_{L^1(|x|\leq 14\langle T\rangle)} \lesssim \langle T\rangle.$$

bring abs val inside, integrate φ in L^1, get constant of $\|\varphi\|_{L^1}$

We turn to (2). For $|x|\geq 14N^2\langle T\rangle$, we use the previous lemma to get

$$\left\| \left| \int_{\mathbb{R}} e^{it\xi^3+ix\xi}\psi_N(\xi)\,d\xi \right|_{L^\infty(|t|\leq T)} \right\|_{L^1(|x|\geq 14N^2\langle T\rangle)} \lesssim \| N^{-1}|x|^{-2}\|_{L^1(|x|\geq 14N^2\langle T\rangle)}$$

$$\lesssim N^{-3}\langle T\rangle^{-1} \lesssim N^{3/2}\langle T\rangle^{1/2}$$

$\left|\int e^{it\xi^3+ix\xi}\psi_N(\xi)\,d\xi\right| \lesssim N^{-1}|x|^{-2}$ uniformly in t.

For $|x|\leq N^{-1}$ we simply estimate

$$\left\| \left| \int_{\mathbb{R}} e^{it\xi^3+ix\xi}\psi_N(\xi)\,d\xi \right|_{L^\infty(|t|\leq T)} \right\|_{L^1(|x|\leq N^{-1})} \lesssim N^{-1}\|\psi_N\|_{L^1} \lesssim 1 \lesssim N^{3/2}\langle T\rangle^{1/2}.$$

$\left|\int e^{it\xi^3+ix\xi}\psi_N(\xi)\,d\xi\right| \leq \|\psi_N\|_{L^1}$, uniformly in t. bring abs val inside

(integrate in $L^1(|x|\leq N^{-1})$) $\leq \| \|\psi_N\|_{L^1} \|_{L^1(|x|\leq N^{-1})}$ $\lesssim N\cdot N^{-1}\lesssim 1$

Finally, for $N^{-1}\leq |x|\leq 14N^2\langle T\rangle$ we distinguish two cases:

x fixed above in "med" case: x fixed

Case 1: If $|x|\sim N^2|t|$, instead t is s.t. $|x|\sim N^2|t|$ $\leftrightarrow |t|\sim N^{-2}|x|$, then using the previous lemma we get

Assemble $\xi \sim N$ pieces together: LP sq fn: $2 \le p \le \infty$

$$\| |\nabla|^{\frac{1}{2}(1-\frac{2}{p})} e^{-t\partial_x^3} f \|_p \sim \Big\| \sqrt{\sum_{N \in 2^{\mathbb{Z}}} N^{1-\frac{2}{p}} |e^{-t\partial_x^3} f_N|^2} \Big\|_p$$

move sqrt outside

$$\sim \Big\| \sum_N N^{1-\frac{2}{p}} |e^{-t\partial_x^3} f_N|^2 \Big\|_{p/2}^{1/2}$$

Δ-ineq? $\lesssim \sqrt{\sum_N N^{1-\frac{2}{p}} \| \dots \|_p^2}$ ...

$$\lesssim \sqrt{\sum_N |t|^{-(1-\frac{2}{p})} \|f_N\|_{p'}^2}$$

$$\lesssim |t|^{-\frac{1}{2}(1-\frac{2}{p})} \sqrt{\sum_N \|f_N\|_{p'}^2}$$

$1 < p' \le 2$ equiv to $\|f\|_p \lesssim \sqrt{\sum_N \|f_N\|_p^2}$

$\lesssim \|f\|_{p'}$

unif in $t \neq 0$

$$\| e^{it\partial_x^3} f \|_{L_x^{p'}} \lesssim |t|^{-(1-\frac{2}{p})} \| f \|_{L_x^{p'}}$$

$2 \le p \le \infty$

$$\| e^{it\partial_x^3} f \|_{L_x^p} \lesssim \sum_N \| e^{it\partial_x^3} f_N \|_{L_x^p} \lesssim |t|^{-(1-\frac{2}{p})} \sum_N \| f_N \|_{L_x^{p'}}$$

$$\lesssim |t|^{-(1-\frac{2}{p})} \| f \|_{L_x^{p'}}$$

but OTOH unconditional $\| e^{it\partial_x^3} f \|_{L_x^p} = \| f_N e^{it\partial_x^3} \|_{L_x^p} \lesssim \| f_N \|_{L_x^p} \lesssim \| f \|_{L_x^p}$

$$\left| \int e^{it\xi^3 + ix\xi} \psi_N(\xi)\, d\xi \right| \lesssim (N|t|)^{-\frac{1}{2}} \simeq \left(\frac{|x|}{N}\right)^{-\frac{1}{2}} \le N^{\frac{1}{2}} |x|^{-\frac{1}{2}}$$

(for fixed x, look at region of t s.t. $|x| \sim N^2|t|$)

Case 1: If instead $|x| \not\sim N^2|t|$, then the previous lemma yields

$$\left| \int e^{it\xi^3 + ix\xi} \psi_N(\xi)\, d\xi \right| \lesssim N^{-1} |x|^{-2} \le N^{\frac{1}{2}} |x|^{-\frac{1}{2}} \text{ if } |x| \ge N^{-1}$$

As $N^{-1}|x|^{-2} \le N^{\frac{1}{2}}|x|^{-\frac{1}{2}} \iff N^{-1} \le |x|$ we find

$$\left\| \int e^{it\xi^3 + ix\xi} \psi_N(\xi)\, d\xi \right\|_{L^\infty(|t| \le T)} \lesssim N^{\frac{1}{2}} |x|^{-\frac{1}{2}} \quad \text{whenever} \quad N^{-1} \le |x| \le 14 N^2 \langle T \rangle.$$

Thus

$$\left\| \left\| \int e^{it\xi^3 + ix\xi} \psi_N(\xi)\, d\xi \right\|_{L^\infty(|t| \le T)} \right\|_{L^1(N^{-1} \le |x| \le 14N^2\langle T \rangle)}$$

$$\lesssim \| N^{\frac{1}{2}} |x|^{-\frac{1}{2}} \|_{L^1(|x| \le 14N^2\langle T \rangle)} \lesssim N^{\frac{1}{2}} \cdot (N^2 \langle T \rangle)^{\frac{1}{2}} \lesssim N^{\frac{3}{2}} \langle T \rangle^{\frac{1}{2}}.$$

(case 2 is worse bound, can use it in L^1)

This completes the proof of (2). ∎

Rmk: $\left| \int e^{it\xi^3 + ix\xi} \psi_N(\xi)\, d\xi \right| \lesssim (N|t|)^{-1/2}$ uniformly in $N \in 2^{\mathbb{Z}}$, $t \neq 0$, $x \in \mathbb{R}$

Repeating previous discussion:

$$[e^{-t\partial_x^3} f_N](x) := \frac{1}{\sqrt{2\pi}} \int e^{it\xi^3 + ix\xi} \hat{f}_N(\xi)\, d\xi = \frac{1}{\sqrt{2\pi}} \int e^{it\xi^3 + ix\xi} \tilde{\psi}_N(\xi) \hat{f}_N(\xi)\, d\xi = \frac{1}{2\pi} \int \int e^{it\xi^3 + i(x-y)\xi} \tilde{\psi}_N(\xi)\, d\xi\, f_N(y)\, dy$$

Std LP flattening trick

Using the Lemma, TT*, Hardy-Littlewood-Sobolev, and the Littlewood-Paley square function estimate, (to reassemble freq pieces) we can derive the following Strichartz estimates:

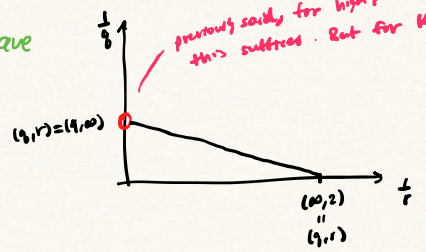
For $2 \le q \le \infty$, $2 \le r < \infty$, $\frac{2}{q} + \frac{1}{r} = \frac{1}{2}$ we have

(↑ b/c sq fn estimate fails @ ∞.)

$$\begin{cases} \| e^{-t\partial_x^3} |\nabla|^{\frac{1}{q}} f \|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R})} \lesssim \| f \|_2 \\ \left\| \int_0^t e^{-(t-\tau)\partial_x^3} |\nabla|^{\frac{2}{q}} F(\tau)\, d\tau \right\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R})} \lesssim \| F \|_{L_t^{q'} L_x^{r'}(\mathbb{R}\times\mathbb{R})} \end{cases}$$

retarded estimates

($|\nabla|^{\frac{2}{q}}$: "bk T gains $\frac{1}{3}$ der. T^* also gain $\frac{1}{3}$ derivatives")



To also obtain the endpoint estimate $\| e^{-t\partial_x^3} |\nabla|^{\frac{1}{4}} f \|_{L_t^4 L_x^\infty} \lesssim \| f \|_2$ we will rely on the following

always tool: Stein interpolation lemma (don't do freq pieces first then Σ, Σ first, then put pieces in right L^p)

Thm (Dispersive decay) (for Airy) Fix $\alpha \in [0, \frac{1}{2}]$ and $2 \le p \le \infty$. Then

($\frac{1}{2}$: best possible I can gain. $p = \infty$ endpt!!)

$$\| |\nabla|^{\alpha(1-\frac{2}{p})} e^{-t\partial_x^3} f \|_{L_x^p} \lesssim |t|^{-\frac{(\alpha+1)}{3}(1-\frac{2}{p})} \| f \|_{L_x^{p'}}$$

uniformly for $t \neq 0$.

Interpolation yields Thm for $2 \le p \le \infty$

to check, $\begin{cases} \alpha = 0, & p = \infty, \text{ get } |t|^{-1/3} \text{ exact match (van der Corput)} \\ \alpha = \frac{1}{2}, & p = \infty \text{ get } |t|^{-1/2} \text{ (which is already proved for freq } \sim N. \end{cases}$

So this "sums freqs" so understand f not just ψ_N.

Proof: The claim follows from complex interpolation for the family of operators

$$T_z(f) = |t|^{z\frac{\alpha+1}{3}} |\nabla|^{\alpha z} e^{-t\partial_x^3} f$$

$z = 1 - \frac{2}{p}$ is what we care about. $\alpha = \frac{1}{2}$.

of course analyze by looking on freq side:

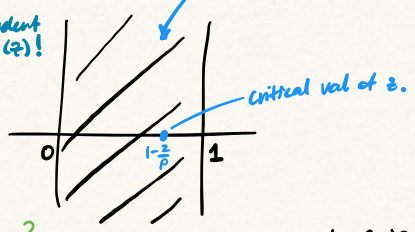
$$z \mapsto \langle T_z f, g \rangle \overset{\text{Plancherel}}{=} |t|^{\frac{\alpha+1}{3} \cdot z} \int |\xi|^{\alpha z} e^{it\xi^3} \hat{f}(\xi)\, \hat{\bar{g}}(\xi)\, d\xi$$

Schwartz f, g

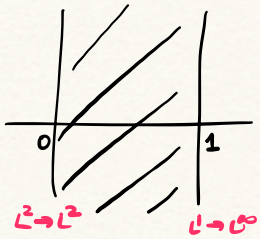
analytic in z in strip

and unif bdd: $\sup_{0 \le \mathrm{Re}\, z \le 1} |\langle T_z f, g \rangle| = \sup_{0 \le \mathrm{Re}\, z \le 1} |t|^{\frac{\alpha+1}{3} \mathrm{Re}\, z} \int |\xi|^{\alpha \mathrm{Re}\, z} |\hat{f}(\xi) \hat{g}(\xi)|\, d\xi$

Lec 7 pt 2 continue 4/13/26



defined for z in the strip $\{z \in \mathbb{C} : 0 \le \mathrm{Re}\, z \le 1\}$. want bds on bdy of strip z^ε



To apply 3-lines lemma, Note that

Left side of strip: $\|T_{i\sigma}(f)\|_{L^2_x} = \| |t|^{i\sigma\frac{\alpha+1}{3}} |D|^{i\sigma\alpha} e^{-t\partial_x^3} f\|_{L^2_x} = \|f\|_{L^2_x}$ (3) $\forall t \neq 0, \forall \sigma \in \mathbb{R}$

(unimodular; goes away by Plancherel; goes away b/c $e^{it\xi^3}$ unimodular; $\mathrm{Re}=0$)

uniformly for $t \neq 0$ and $\sigma \in \mathbb{R}$. (i.e. $\|T_{i\sigma}\|_{L^2 \to L^2} \le 1$ unif in $\sigma \in \mathbb{R}$) (unit bdd (in imag part))

To complete the proof, it suffices to show

$\|T_{1+i\sigma}(f)\|_{L^\infty_x} \lesssim \langle\sigma\rangle^2 \|f\|_{L^1_x}$ (4) unif $\forall t \neq 0, \sigma \in \mathbb{R}$ ($\|T_{1+i\sigma}\|_{L^1 \to L^\infty} \lesssim \langle\sigma\rangle^2$ unif in $\sigma \in \mathbb{R}$; poly, acceptable for 3 lines lemma)

uniformly for $t \neq 0$ and $\sigma \in \mathbb{R}$.

Interpolating between (3)左 and (4)右 we get that

$\|T_{1-\frac{2}{p}}(f)\|_{L^p_x} \lesssim \|f\|_{L^{p'}_x} \iff \| |D|^{\alpha(1-\frac{2}{p})} e^{-t\partial_x^3} f\|_{L^p_x} \lesssim |t|^{-\frac{\alpha+1}{3}(1-\frac{2}{p})} \|f\|_{L^{p'}_x}$

$\|T_{1-\frac{2}{p}}\|_{L^{p'}\to L^p} \le 1$ ($\frac{1}{p}$ is in strip; $p=2 \leftrightarrow L^2 \to L^2$, $p=\infty \leftrightarrow L^1 \to L^\infty$; $\theta = \frac{1}{p}$)

(it was originally over on LHS, can move it over.)

as desired ∎ (modulo 右 bdd) — unif in $t \neq 0$ b/c interpolate bdd bounds that are unif in t.

It remains to prove (4)右. We have

$= \left[|t|^{\frac{\alpha+1}{3}(1+i\sigma)} |D|^{\alpha(1+i\sigma)} e^{-t\partial_x^3} f\right](x)$ ← inverse FT of FT

($z = 1+i\sigma$ (focus on $\alpha = \frac{1}{2}$))

$[T_{1+i\sigma}(f)](x) = |t|^{(1+i\sigma)\frac{\alpha+1}{3}} \frac{1}{\sqrt{2\pi}} \int e^{it\xi^3 + ix\xi} |\xi|^{(1+i\sigma)\alpha} \hat{f}(\xi)\, d\xi$

expand FT $= \frac{1}{2\pi} |t|^{\frac{\alpha+1}{3}(1+i\sigma)} \iint e^{it\xi^3 + i(x-y)\xi} |\xi|^{\alpha(1+i\sigma)} f(y)\, dy\, d\xi$

do what we did in tackling Airy fn, change var so factor out front goes away, leaving just an integral.

$\xi = \frac{\eta}{(3t)^{1/3}}$ i.e. $t\xi^3 = \frac{\eta^3}{3}$ $= |t|^{(1+i\sigma)\frac{\alpha+1}{3}} \frac{1}{2\pi} \iint e^{i\frac{\eta^3}{3} + i\frac{x-y}{(3t)^{1/3}}\eta} |3t|^{-(1+i\sigma)\alpha} |\eta|^{(1+i\sigma)\alpha} (3t)^{-\frac{1}{3}} f(y)\, d\eta\, dy$

indeed leave only $|t|^{\text{imag}}$, as planned

Fubini (again $e^{-\varepsilon\eta^2}$ trick) L^∞ of $\left(\int \cdots \frac{x-y}{\cdots} dy\right)$ may call "this variable" "x", L^∞_x

$\Rightarrow \|T_{1+i\sigma}(f)\|_{L^\infty_x} \lesssim \|f\|_{L^1_x} \left\| \int e^{i\frac{\xi^3}{3} + ix\xi} |\xi|^{(1+i\sigma)\alpha} d\xi \right\|_{L^\infty_x}$ ($\cdot\frac{1}{2\pi} = \cdot[\varphi + \sum_N \psi_N]$; rename η back to ξ.)

b/c above is sensitive to f via $\|\cdot\|_{L^1}$

how to analyze? break into frequency local pieces: φ, and ψ_N w/ $N \ge 1$ (recall φ, ψ_N form partition of unity)

$\lesssim \|f\|_{L^1_x} \left\{ \left\| \int e^{i\frac{\xi^3}{3} + ix\xi} |\xi|^{(1+i\sigma)\alpha} \varphi(\xi)\, d\xi \right\|_{L^\infty_x} \right.$ (φ term)

$\left. + \left\| \sum_{N \ge 1} \left| \int e^{i\frac{\xi^3}{3} + ix\xi} |\xi|^{(1+i\sigma)\alpha} \psi_N(\xi)\, d\xi \right| \right\|_{L^\infty_x} \right\}$ (ψ_N term) (5)

do NOT Δ-ineq (b/c Stein-Tomas, endpt wt. cant use Δ-ineq thus)

φ term We have $\left\| \int e^{i\frac{\xi^3}{3} + ix\xi} |\xi|^{(1+i\sigma)\alpha} \varphi(\xi)\, d\xi \right\|_{L^\infty_x} \lesssim \| |\xi|^{i\sigma\alpha} \varphi(\xi) \|_{L^1} \lesssim \|\varphi\|_{L^1} \lesssim 1.$ (of course $\lesssim \langle\sigma\rangle$) (6)

ψ_N term Consider $\left| \int e^{i\frac{\xi^3}{3} + ix\xi} |\xi|^{(1+i\sigma)\alpha} \psi_N(\xi)\, d\xi \right|$. proceed exactly using previous osc. integral technique.

The phase $\phi(\xi) = \frac{\xi^3}{3} + x\xi$ satisfies $\phi'(\xi) = \xi^2 + x$, $\phi''(\xi) = 2\xi$, $\phi'''(\xi) = 2$.

$\sim N$ always like N on ξ support of ψ_N

On the support of ψ_N, $|\phi''(\xi)| \sim N$. By Van der Corput we get the bound

$$\lesssim N^{-\frac{1}{2}} \left[\| |\xi|^{(1+i\sigma)\alpha} \psi_N \|_\infty + \| \left(|\xi|^{(1+i\sigma)\alpha} \psi_N \right)' \|_1 \right] \lesssim N^{-\frac{1}{2}+\alpha} \cdot \langle \sigma \rangle$$

($\sim N$ for $\xi \in$ supp ψ_N; scaling will be the same; ∂_ξ; unif in x (important b/c L^∞_x); derivative hit ψ_N, get $\frac{1}{N} N^\alpha \cdot N$, or derivative hit $|\xi|^{\alpha(1+i\sigma)} \rightsquigarrow \langle\sigma\rangle N^{\alpha-1} N$; size of supp ψ_N; $0 \le \alpha \le \frac{1}{2}$; size of supp ψ_N)

Not good enough to sum in N. for $\alpha = \frac{1}{2}$ (good enough if $0 \le \alpha < \frac{1}{2}$)

Summary: van der Corput w/ ϕ'' not good for $\alpha = \frac{1}{2}$. N^α

"van der corput is good, but not as good as IBP!"

Try vdC for ϕ': Note that $\phi'(\xi) = \xi^2 + x = 0 \Rightarrow |x| \sim N^2$ for $\xi \in \text{supp}\, \psi_N$.

(fix x, N s.t. find stationary pts.)

Thus if $|x| \not\sim N^2$ (i.e. far from N s.t. $N \not\sim |x|^{1/2}$) we can integrate by parts to get

same ψ_N term as bfr.

$$\left| \int e^{i\frac{\xi^3}{3} + ix\xi} |\xi|^{(1+i\sigma)\alpha} \psi_N(\xi)\, d\xi \right| = \left| \int e^{i\phi(\xi)} \cdot \left(\partial_\xi \frac{1}{\xi^2 + x} \cdot \right)^2 \left[|\xi|^{(1+i\sigma)\alpha} \psi_N(\xi) \right] d\xi \right|$$

(cpct support $\Rightarrow$ no bdry term)

$\lesssim \| \partial_\xi [\frac{1}{\xi^2+x} |\xi|^{\alpha(1+i\sigma)} \psi_N] \|_1$; ∂_ξ hit $|\xi|^{\alpha(1+i\sigma)}$, $|1+i\sigma| = \langle\sigma\rangle$, throw away $\alpha \le \frac{1}{2}$: $\lesssim \langle\sigma\rangle \cdot \frac{1}{N^2} \cdot N^{\alpha-1} \cdot N$ (size of supp) $= \langle\sigma\rangle N^{\alpha - \frac{1}{2}} = \langle\sigma\rangle N^{-3/2}$ at $\alpha = \frac{1}{2}$.

$$\lesssim \langle\sigma\rangle^2 \cdot N^{\alpha-2} \cdot \frac{1}{(N^2+|x|)^2} \cdot N^{\alpha+1} \lesssim \langle\sigma\rangle^2 N^{\alpha-5}$$

(2 derivatives hit; coming from integral; ξ is like N)

worst possible term, highest growth in imag part. (2 derivatives attack $|\xi|^{\alpha(1+i\sigma)}$)

$$\sum_{N \ge 1} \left| \int e^{i\frac{\xi^3}{3} + ix\xi} |\xi|^{\alpha(1+i\sigma)} \psi_N(\xi)\, d\xi \right| \lesssim \sum_{N \not\sim |x|^{1/2}} \langle\sigma\rangle^2 N^{\alpha-5} + \sum_{N \sim |x|^{1/2}} N^{\alpha - \frac{1}{2}} \lesssim \langle\sigma\rangle^2$$

($N \not\sim |x|^{1/2}$: can use good IBP bd.; $N \sim |x|^{1/2}$: can only use worse vanderCorput bd.; finitely many such terms! Still works when $\alpha = \frac{1}{2}$!)

Combining this with (5,6) completes the proof of (4). ∎ QED

Dispersive est.

9/18/24

We are now ready to prove (entirely based on our Dispersive Est above.)

Theorem (Strichartz estimates) Fix $\alpha \in [0, \frac{1}{2}]$ and $2 < q < \infty$, $2 < r \le \infty$ with $\frac{1}{q} = \frac{\alpha+1}{3}\left(\frac{1}{2} - \frac{1}{r}\right)$. Then (leave blank, fill in at end based on pf (HLS).)

(1) $$\| |\nabla|^{\alpha(\frac{1}{2}-\frac{1}{r})} e^{-t\partial_x^3} f \|_{L_t^q L_x^r (\mathbb{R}\times\mathbb{R})} \lesssim \|f\|_{L^2_x}$$

(exact expo as in Dispersive Est above b/c we use it.)

(twice as many derivatives)

retarded estimate

(2) $$\left\| \int_0^t e^{-(t-\tau)\partial_x^3} |\nabla|^{2\alpha(\frac{1}{2}-\frac{1}{r})} F(\tau)\, d\tau \right\|_{L_t^q L_x^r (\mathbb{R}\times\mathbb{R})} \lesssim \|F\|_{L_t^{q'} L_x^{r'}}$$

("contribution of inhomogeneity")

Use TT^* to prove (1) (& method of pf will prove (2) for free)

Recall prop

Proposition (Dispersive decay) Fix $\alpha \in [0, \frac{1}{2}]$ and $2 \le p \le \infty$. Then $\| |\nabla|^{\alpha(1-\frac{2}{p})} e^{-t\partial_x^3} f \|_{L^p_x} \lesssim |t|^{-\frac{\alpha(1-\frac{2}{p})}{3}} \|f\|_{L^{p'}_x}$ uniformly for $t \neq 0$.

(Plancherel, $\alpha = 0$, $p = \infty$; sharp, $\alpha = \frac{1}{2}$, $p = \infty$)

Proof: If $T(f) := |\nabla|^{\alpha(\frac{1}{2}-\frac{1}{r})} e^{-t\partial_x^3} f$, (fn of space $\longmapsto$ fn of space and time)

then its adjoint T^* is determined by (want to compute what T^* is:)

$$\langle f, T^*(F) \rangle_x = \langle T(f), F \rangle_{t,x} := \int \langle |\nabla|^{\alpha(\frac{1}{2}-\frac{1}{r})} e^{-t\partial_x^3} f, F(t) \rangle_x\, dt = \int \langle f, e^{t\partial_x^3} |\nabla|^{\alpha(\frac{1}{2}-\frac{1}{r})} F(t)\, dt \rangle_x\, dt$$

(self adjoint operator; adjoint of $e^{-t\partial_x^3}$ operator (look on Fourier side))

$$= \langle f, \int |\nabla|^{\alpha(\frac{1}{2}-\frac{1}{r})} e^{t\partial_x^3} F(t)\, dt \rangle_x$$

(can move $\int dt$ inside b/c t does not depend on x.)

so: $T^*(F) = \int e^{\tau\partial_x^3} |\nabla|^{\alpha(\frac{1}{2}-\frac{1}{r})} F(\tau)\, d\tau$

Thus $T: L^2_x \to L^q_t L^r_x \iff T^*: L^{q'}_t L^{r'}_x \to L^2_x \iff TT^*: L^{q'}_t L^{r'}_x \to L^q_t L^r_x$

$$\| TT^*(F) \|_{L^q_t L^r_x} = \left\| \int e^{-(t-\tau)\partial_x^3} |\nabla|^{2\alpha(\frac{1}{2}-\frac{1}{r}) = \alpha(1-\frac{2}{r})} F(\tau)\, d\tau \right\|_{L^q_t L^r_x}$$

extra step: $TT^*(F) = T\left(\int e^{s\partial_x^3} |\nabla|^{\alpha(\frac{1}{2}-\frac{1}{r})} F(s)\, ds \right)$

Want bd of this in terms of $\|f\|_{L^{q'}_t L^{r'}_x}$

(for pair (2) which has $\int_0^t$)

TT^* eats contribute $\alpha(\frac{1}{2}-\frac{1}{r})$ derivatives, hence twice as many

first L^r_x, then L^q_t "like composing fns"

Δ ineq / Minkowski:

$$\lesssim \left\| \int \| e^{-(t-\tau)\partial_x^3} |\nabla|^{\alpha(-\frac{2}{r})} F(\tau) \|_{L^r_x}\, d\tau \right\|_{L^q_t}$$

exactly Prop: Dispersive Decay.

$$\lesssim \Big\| \int_{\mathbb{R}} |t-\tau|^{-\frac{\alpha+1}{3}\left(1-\frac{2}{r}\right)} \|F(\tau)\|_{L_x^{r'}} \, d\tau \Big\|_{L_t^{\tilde{q}}} = \big\| |s|^{-\frac{\alpha+1}{3}\left(1-\frac{2}{r}\right)} * \|F(s)\|_{L_x^{r'}} \big\|_{L_t^{\tilde{q}}}$$

convolution! But only in weak $L^\cdot$ space, so use HLS!

$$\lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{r'}} \quad \text{by} \quad \text{Hardy-Littlewood-Sobolev} \quad (= \text{"endpt Young ineq"})$$

provided $\begin{cases} 0 < \frac{\alpha+1}{3}\left(1-\frac{2}{r}\right) < 1 \iff 2 < r \le \infty \quad (r>2) \\ 1 + \frac{1}{q} = \frac{1}{\tilde{q}} + \frac{\alpha+1}{3}\left(1-\frac{2}{r}\right) \iff \frac{2}{q} = \frac{\alpha+1}{3}\left(1-\frac{2}{r}\right) \end{cases}$

- the negative exponent "isn't too singular"
- blk dim = 1
- scaling relation
- $1 - \frac{1}{\tilde{q}} = \frac{1}{q}$
- $\tilde{q}$: space in which I and $\frac{\alpha+1}{3}\left(1-\frac{2}{r}\right)$: the weak $L^\cdot$ space the $|t|^{-\cdot}$ belongs to

$\iff \frac{1}{q} = \frac{\alpha+1}{3}\left(\frac{1}{2} - \frac{1}{r}\right)$ fills in blank above

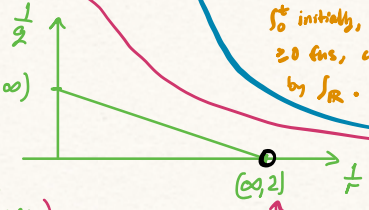
This gives (1), and (2) follows from above pf, but instead of $\int_{\mathbb{R}}$, we have $\int_0^t$ initially, which when integrate ≥ 0 fns, can be upper bdd by $\int_{\mathbb{R}}$.

QED Strichartz ests.

$\alpha=0$, $\alpha=\frac{1}{2}$

2 Special cases:

1) $\alpha = 0 \implies \left(\frac{3}{q} + \frac{1}{r} = \frac{1}{2}, \quad 2 < r \le \infty\right)$ ($\frac{1}{q} = \frac{1}{3}\left(\frac{1}{2} - \frac{1}{r}\right)$)

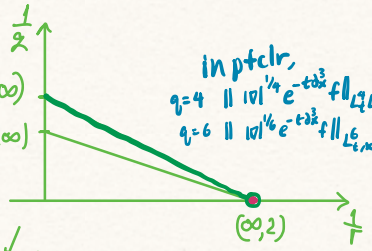


Then $\|e^{-t\partial_x^3} f\|_{L_t^q L_x^r} \lesssim \|f\|_{L_x^2}$ (*)

Note $(q,r) = (\infty, 2)$ is just the conservation of momentum (conservation of L_x^2 norm) fills in the hole left by

2) $\alpha = \frac{1}{2} \implies \left(\frac{2}{q} + \frac{1}{r} = \frac{1}{2}, \quad 2 < r \le \infty\right)$ ($\frac{1}{q} = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{r}\right)$)

$\|e^{-t\partial_x^3} |\nabla|^{\frac{1}{2}\left(\frac{1}{2}-\frac{1}{r}\right)} f\|_{L_t^q L_x^r}$



in ptclr, $q=4$ $\||\nabla|^{1/4} e^{-t\partial_x^3} f\|_{L_t^4 L_x^\infty} \lesssim \|f\|_{L^2}$

$q=6$ $\||\nabla|^{1/6} e^{-t\partial_x^3} f\|_{L_{t,x}^6} \lesssim \|f\|_{L^2}$

Then $\|e^{-t\partial_x^3} |\nabla|^{\frac{1}{2}} f\|_{L_t^q L_x^r} \lesssim \|f\|_{L_x^2}$ (**)

"gain of $\frac{1}{2}$ derivatives"

The case $(q,r) = (\infty, 2)$ is again the conservation of momentum

$\alpha = 0$ scaling relation in (*)

For $2 < r < \infty$, $\iff 6 < q < \infty$, (**) implies (*) by Sobolev embedding: For $\frac{3}{q} + \frac{1}{r} = \frac{1}{2}$,

want $\frac{1}{2}$ derivative on RHS so can apply (**)

$$\|e^{-t\partial_x^3} f\|_{L_t^q L_x^r} \lesssim \|e^{-t\partial_x^3} |\nabla|^{\frac{1}{2}} f\|_{L_t^q L_x^{\tilde{r}}}$$

where $\frac{1}{r} = \frac{1}{\tilde{r}} - \frac{1}{q} \implies \frac{2}{q} + \frac{1}{\tilde{r}} = \frac{2}{q} + \frac{1}{r} + \frac{1}{q} = \frac{1}{2}$, which is exactly the scaling relation for $\alpha = \frac{1}{2}$ in (**). (2)

$\frac{1}{\tilde{r}} = \frac{1}{r} + \frac{1}{q}$ we are in dim = 1.

$= \frac{3}{q} + \frac{1}{r}$

12:25pm

Shelve these for now, will use them later, in conjunction w 2 more bounds:

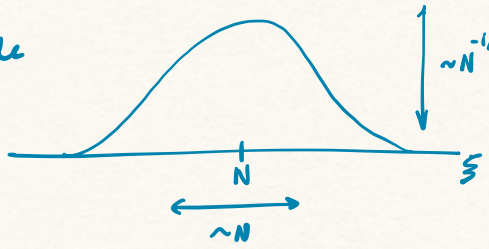
Prop (Kato smoothing): instantaneously smoothing. $\| e^{-t\partial_x^3} f \|_{L_t^4}$ ← average in time, can actually control one whole derivative

$\| \partial_x e^{-t\partial_x^3} f \|_{L_x^\infty L_t^2} \lesssim \|f\|_{L^2}$ uniformly in space!

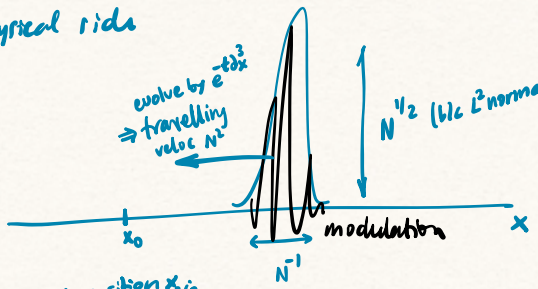
"average in time, much better regularity!"

edit 4/15/26 Lec8 **Heuristics of Kato smoothing** – "why 1 deriv ∂x"

f on Fourier side



on Physical side



Fix position x_0 in physical space

$e^{-t\partial_x^3} f(x_0)$ I see the bump for time interval of len $\frac{\text{width of bump}}{\text{velocity of bump}} \sim \frac{N^{-1}}{N^2} \sim N^{-3}$

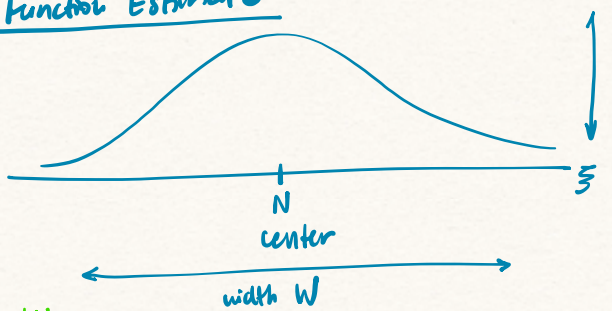
$\Rightarrow \| e^{-t\partial_x^3} f(x_0) \|_{L_t^2} \sim N^{1/2} \cdot N^{-3/2} \sim N^{-1}$ ($\cdot L_t^2$)

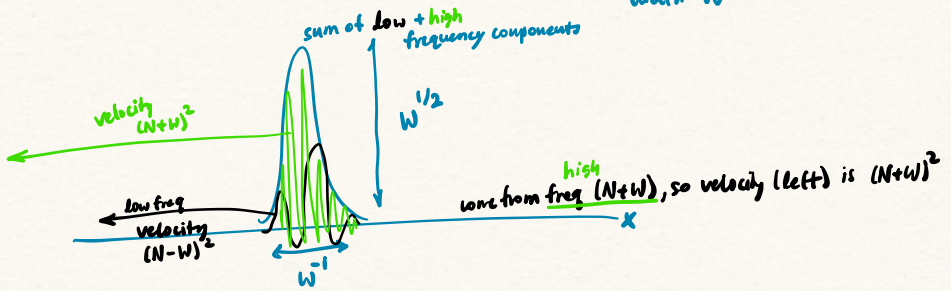
$N^{1/2}$: tallest the bump can be

means we gain 1 derivative

Heuristics for Maximal Function Estimate

f on Fourier side





physical space

Width of bump at time t is $\sim W^{-1} + tN \cdot W$.

Requiring that the bump does not disintegrate (i.e. decay & spread out) up to time T amounts to asking $T \cdot N \cdot W \sim W^{-1}$ (the stretching did not exceed orig width of bump)

$\Leftrightarrow W \sim (NT)^{-1/2}$

So $N \gg W$

Then evolution in L_t^∞: maximal height b/c not disintegrate by time t

$\| e^{-t\partial_x^3} f(x_0) \|_{L_t^\infty} \sim W^{\frac{1}{2}}$ for x_0 in interval of len $N^2 T$

$\Rightarrow \| e^{-t\partial_x^3} f \|_{L_x^1 L_t^\infty} \sim W^{1/2} \cdot (N^2 T)^{1/2} \sim (NT)^{-1/4} N T^{1/2} \sim N^{3/4} T^{1/4}$

$\frac{3}{4}$! Pay ε to sum in N, hence $s < \frac{3}{4}$ (strictly less)

warning: RHS of Max'l fn Est is $\langle T \rangle^{1/2}$ b/c at low freqs.

of Kato smoothing "$\frac{1}{2}\partial_x$"

Pf: $[\partial_x e^{-t\partial_x^3} f](x) = \frac{i}{\sqrt{2\pi}} \int e^{ix\xi + it\xi^3} \xi \hat{f}(\xi)\, d\xi$.

imagine is τ (dual of t after FT in time), set $e^{it\tau}(\ldots)$, Plancherel! ($e^{it\tau}$: FT in τ.)

"trick of Kato smoothing is Plancherel in time"

$e^{ix\xi + it\xi^3}$: inverse FT of ; $\xi \hat{f}(\xi)$: FT of $\partial_x e^{-t\partial_x^3} f$

$$= \frac{i}{\sqrt{2\pi}} \int e^{it\tau + ix\tau^{1/3}} \tau^{1/3} \hat{f}(\tau^{1/3}) \tfrac{1}{3}\tau^{-2/3} d\tau.$$

(2026 MV use variable $\xi^2 = \eta$ so I may slip up)

$\xi^3 = \tau$, $\xi = \tau^{1/3}$, $d\xi = \frac{1}{3}\tau^{-2/3} d\tau$

inv FT of

$$= \frac{i}{3} \mathcal{F}_t^{-1}\left[e^{ix\tau^{1/3}} \tau^{-1/3} \hat{f}(\tau^{1/3}) \right]$$

By Plancherel, $\|(\partial_x e^{-t\partial_x^3} f)(x)\|_{L_t^2}^2 = (\tfrac{1}{3})^2 \| e^{ix\tau^{1/3}} \tau^{-\frac{1}{3}} \hat{f}(\tau^{1/3}) \|_{L_\tau^2}^2$ (Plancherel)

$\forall$ fixed $x \in \mathbb{R}$

$$= \frac{1}{9} \int |\tau|^{-2/3} |\hat{f}(\tau^{1/3})|^2 d\tau$$

undo change of var

$\tau^{1/3} = \xi \Leftrightarrow \tau = \xi^3$, $d\tau = 3\xi^2 d\xi$

$$= \frac{1}{9} \int \xi^{-2} |\hat{f}(\xi)|^2 3\xi^2 d\xi = \frac{1}{3}\|f\|_{L^2}^2$$

Plancherel again.

change of var is just to apply Plancherel

QED Kato smoothing

"balanced Kato smoothing above" in sense that above had L_x^∞. Sometimes will have to use integrability in space

Prop (Maximal fn estimate): For $s > \frac{3}{4}$

"it one allows losing $> \frac{3}{4}$ derivatives, one can control"

$$\| e^{-t\partial_x^3} f \|_{L_x^2 L_t^\infty(\mathbb{R}\times[-T,T])} \lesssim \langle T \rangle^{1/2} \|f\|_{H_x^s}$$

$\lesssim$: implicit const indep of time

worst possible behavior in time; turn integrate in space

Sobolev embedding says to control L_t^∞ need to pay derivatives the question is how many

Not as elegant as Kato smoothing; "Bourgain said once: 'if Kato did it, it must be trivial'" MV interpret as Kato hitting nail on the head exactly, elegant

"doing time first, and then space" radical idea at the time"

Pf: here, all spacetime norm integrate over slab $\mathbb{R}\times[-T,T]$.

Break into freq local pieces

$$\| e^{-t\partial_x^3} f \|_{L_x^2 L_t^\infty} \le \| e^{-t\partial_x^3} P_{\le 1} f \|_{L_x^2 L_t^\infty} + \sum_{N \in 2^{\mathbb{Z}}} \| e^{-t\partial_x^3} P_N \tilde{P}_N f \|_{L_x^2 L_t^\infty}$$

flattened Littlewood-Paley projection corresponding to Fourier multiplier $\tilde{\psi}_N := \psi_{\frac{N}{2}} + \psi_N + \psi_{2N}$

which is $\equiv 1$ on supp ψ_N in phase

$\Rightarrow P_N \tilde{P}_N = P_N$

For low freqs

we'll use TT^*. If $T(f) := e^{-t\partial_x^3} P_{\le 1} f$, similar to above (moving all the self-adjoint ops to other side of inner prod)

(computed last time. $P_{\le 1}$ self adj, ∂_x^3 self adj w/ change in sign)

$$T^*(F) = \int e^{\tau\partial_x^3} P_{\le 1} \cdot F(\tau)\, d\tau.$$

$$\Rightarrow TT^*(F) = \int e^{-(t-s)\partial_x^3} P_{\le 1} F_{\le 1}(s)\, ds$$

$P_{\le 1}$, one I stick onto F.

one I keep around

for high freqs

and similarly, $T(f) = e^{-t\partial_x^3} P_N f \Rightarrow T^*(F) = \int e^{\tau\partial_x^3} P_N F(\tau)\, d\tau$

$TT^*(F) = \int e^{-(t-s)\partial_x^3} P_N F_N(s)\, ds.$

$T: L^2 \to L^2_x L^\infty_t \iff T^*: L^2_x L^1_t \to L^2 \iff TT^*: L^2_x L^1_t \to L^2_x L^\infty_t.$

First, low freq $P_{\le 1}$:

$$\|TT^* f\|_{L^2_x L^\infty_t} := \Big\| \int e^{-(t-\tau)\partial_x^3} P^2_{\le 1} F(\tau)\, d\tau \Big\|_{L^2_x L^\infty_t} \le \Big\| \int_{-T}^{T} \big| e^{-(t-\tau)\partial_x^3} P^2_{\le 1} F(\tau) \big|\, d\tau \Big\|_{L^2_x L^\infty_t}$$

L^∞-abs val, Δ-ineq bring abs val inside integral

on Fourier side, this multiplying $\hat{F}(\tau)$ or FT wrt x by nice things; on physical side, convolution in x

$= F(\tau) *_x [e^{i(t-\tau)\xi^3} \varphi(\xi)^2]^\vee$

$$= \frac{1}{\text{def of } \mathcal{F}} \frac{1}{(2\pi)} \int \int e^{ix\xi + i(t-s)\xi^3} \varphi(\xi)\, \hat{F}_{\le 1}(s,\xi)\, ds\, d\xi$$

$s=\tau$

$\hat{F}_{\le 1}$: $\frac{1}{(2\pi)}\int e^{-iy\xi} F_{\le 1}(s,y)\, dy$

$$= \frac{1}{2\pi} \iint\int e^{i(t-s)\xi^3} e^{i(x-y)\xi} \varphi(\xi)\, F_{\le 1}(s,y)\, d\xi\, ds\, dy$$

convolution in space & time

$$= \frac{1}{2\pi} \big[F_{\le 1}(t,x) \big] * \Big[\int e^{ix\xi + it\xi^3} \varphi(\xi) d\xi \Big]$$

$$\Rightarrow \|TT^* f\|_{L^2_x L^\infty_t} \lesssim \Big\| \big\| |F_{\le 1}(t,x)| *_x \Big| \int e^{ix\xi+it\xi^3} \varphi(\xi) d\xi \Big| \Big\|_{L^\infty_t} \Big\|_{L^2_x}$$

$$\lesssim \Big\| \|F_{\le 1}\|_{L^1_t} *_x \Big\| \int e^{ix\xi+it\xi^3} \varphi(\xi) d\xi \Big\|_{L^\infty_t} \Big\|_{L^2_x}$$

$$\lesssim \|F_{\le 1}\|_{L^1_t L^1_t} \cdot \Big\| \int e^{ix\xi+it\xi^3} \varphi(\xi) d\xi \Big\|_{L^2_x L^\infty_t} \lesssim \langle T \rangle$$

"low freqs don't move"

Young ineq

$$\lesssim \|F\|_{L^2_x L^1_t} \cdot \Big\| \int e^{ix\xi + i(t-\tau)\xi^3} \varphi(\xi)^2 d\xi \Big\|_{L^1_x L^\infty_{t,\tau}} \lesssim \langle T \rangle$$

put $\int_{-T}^{T} d\tau$ all on, remaining go in L^∞

$$\lesssim \Big\| \underbrace{\|F\|_{L^1_\tau}}_{\text{stillatn of } x} *_x \Big\| \int e^{ix\xi + i(t-\tau)\xi^3} \varphi(\xi)^2 d\xi \Big\|_{L^\infty_{t,\tau}} \Big\|_{L^2_x}$$

$$\lesssim \|F\|_{L^2_x L^1_t} \langle T \rangle$$

So (sqrt bk TT^*), $\|e^{-t\partial_x^3} P_{\le 1} f\|_{L^2_x L^\infty_t (\text{slab})} \lesssim \langle T \rangle^{1/2} \|f\|_2$ (dependent on T)

for high freqs

Similarly,

$$\Big\| \int e^{-(t-\tau)\partial_x^3} P^2_N F(\tau)\, d\tau \Big\|_{L^2_x L^\infty_t} \lesssim N^{\frac{3}{2}} \langle T \rangle^{1/2} \|F\|_{L^2_x L^1_t}$$

$\lesssim \|F_N\|_{L^2_x L^1_t} \cdot \Big\| \int e^{ix\xi+it\xi^3} \psi_N(\xi) d\xi \Big\|_{L^1_x L^\infty_t} \lesssim N^{3/2} \langle T \rangle^{1/2}$ (again with Cor!)

$$\Rightarrow \|e^{-t\partial_x^3} P_N \tilde{P}_N f\|_{L^2_x L^\infty_t (\text{slab})} \lesssim N^{3/4} \langle T \rangle^{1/4} \|\tilde{P}_N f\|_2$$

(dependent on T)

slightly changed in 2026: taking sqrt of TT^* numerology:

$\frac{1}{4}$ for high freq, $\frac{1}{2}$ from low

$$\|e^{-t\partial_x^3} f\|_{L^2_x L^\infty_t} \lesssim \langle T \rangle^{1/2} \|f\|_2 + \sum_{N \in 2^{\mathbb{N}}} N^{3/4} \langle T \rangle^{1/4} \|f_N\|_2$$

Bernstein Sobolev turn N^0 into $|\partial|^s$

$$\lesssim \langle T \rangle^{1/2} \Big[\|f\|_2 + \sum_{N \in 2^{\mathbb{N}}} N^{3/4-s} \||\partial|^s f_N\|_2 \Big] \lesssim \langle T \rangle^{1/2} \|f\|_{H^s} \text{ for } s > \tfrac{3}{4}$$

to sum $\sum_N N^{3/4-s}$

Finally,

$$\|e^{-t\partial_x^3} f\|_{L^2_x L^\infty_t} \le \|e^{-t\partial_x^3} P_{\le 1} f\|_{L^2_x L^\infty_t} + \sum_{N \in 2^{\mathbb{Z}}} \|e^{-t\partial_x^3} P_N \tilde{P}_N f\|_{L^2_x L^\infty_t}$$

$$\lesssim \langle T \rangle^{\frac{1}{2}} \|f\|_2 + \sum_{N \ge 1} N^{3/4} \langle T \rangle^{1/4} \|\tilde{P}_N f\|_2$$

(obvious)

(Bernstein on $\tilde{P}_N f$ localized)

$$\lesssim \langle T \rangle^{1/2} \Big[\|f\|_{H^s} + \sum_{N \ge 1} N^{\frac{3}{4}-s} \|\tilde{P}_N f\|_{H^s} \Big] \lesssim \langle T \rangle^{1/2} \|f\|_{H^s}$$

$\|\tilde{P}_N f\|_{H^s} \le \|f\|_{H^s}$, $\sum_{N \ge 1} N^{\frac{3}{4}-s}$ summable.

QED Max'l Fn Est

Lec 1/2/3 See Wi23 247B Week 1, osc. integrals, van der Corput, IBP, stationary / nonstationary phase

Lec 4 $Ai(x) \to 0$

Lec 5/6 Strichartz

Lec 7 Dispersive est for Airy $\|\nabla^{\alpha} e^{-t\partial_x^3}\|_{L^{p'} \to L^p} \leq |t|^{-\blacksquare}$ for endpt Strichartz
(3 lines lemma / Stein interpolation
don't bd $\sum \|\text{freq}N\text{piece}\|_p$, bd $\|\sum |\text{freq}N\text{piece}|\|_p$)

Lec 8 Kato smoothing $L^2 \to L^\infty_x L^2_t$ control of $e^{-t\partial_x^3}$
Max'l fn est $H^s \to L^2_x L^\infty_{t\in[-T,T]}$ // by $< T$
$s > \frac{3}{4}$

end of Airy propagator

Chp II: KdV energy method

does NOT rely on Airy kernel Strichartz, Kato smoothing, Max'l fn est. above; Airy propagator, oscillatory integrals.

Lec 9 H^s norms of smooth solutions (assumed KdV) $\|u(t)\|_{H^s}^2 \leq \|u(0)\|_{H^s}^2 \, e^{c\int_0^t \|u(\tau)\|_{H^2} d\tau}$ $\forall\, t \in [0, T_0]$, $\alpha = s$

10 pKdV$_\varepsilon$, contraction mapping. $T_0 = T_0(\varepsilon, \|u_0\|_{H^s})$

11 $H^{k>\sigma}$ norms of pKdV$_\varepsilon$ sols, bootstrap to T_0 indp of ε.

4/17/26 Lec 9
wk 3 Friday
4/10/24

Std undergrad ODE integral eq & contraction mapping. What's correct integral ineq?

KdV equation, finally

Consider initial value problem
$$\begin{cases} u_t + u_{xxx} = 6uu_x \\ u(0) = u_0 \end{cases} \quad (1)$$
$= 3\partial_x(u^2)$

nonlinear, linear propagative terms

real valued
$u: \mathbb{R}_t \times \mathbb{R}_x \to \mathbb{R}$

$u_0 \in H^s(\mathbb{R})$, $(s > \frac{3}{2})$ for now integer params $2 \le s \in \mathbb{N}$.

History: appears first time in footnote of textbook in 1877 Boussinesq

but more prominently, 1895 Korteweg de Vries

controversy
- water wave steepen and break
- John Scott Russell 1930 solitary waves + waves passing thru each other w/o changing shape & speed ~ soliton particle like

derive using physically motivated/compelling argument eg, admitting solitons

speed proportional to width

$$Q_c(t,x) = -\frac{c}{2}\operatorname{sech}^2\left(\frac{\sqrt{c}}{2}(x-ct)\right)$$
$c > 0$

nonlinear PDE
1st eq shown to be completely integrable.. 3:35 monologue

Standard procedure: solve ODE for instance
"undergrad knowledge"
variation of parameters, integral formulation, contraction mapping.

we seek solutions of the form $u(t,x) := \left[e^{-t\partial_x^3} v(t)\right](x)$

(if KdV RHS = 0, then $u(t,x) = [e^{-t\partial_x^3} u_0](x)$ varying coeff

$$u_t + u_{xxx} = 6uu_x$$

Product rule
$$-\partial_x^3 e^{-t\partial_x^3} v(t) + e^{-t\partial_x^3}\partial_t v + \partial_x^3 e^{-t\partial_x^3} v(t) = 6uu_x$$

moving propagator onto RHS
$$\Rightarrow \partial_t v = e^{t\partial_x^3}(6uu_x)$$

space derivative
$$\overset{\text{FTC}}{\Rightarrow} v(t) = \underbrace{v(0)}_{\text{from def: } v := [\text{Prop}]v} + \int_0^t e^{\tau\partial_x^3}(\underbrace{6uu'}_{=3\partial_x(u^2)})(\tau)\,d\tau$$

$$\Rightarrow \boxed{u(t) = e^{-t\partial_x^3}u(0) + \int_0^t e^{-(t-\tau)\partial_x^3}(6uu')(\tau)\,d\tau}$$

Duhamel formula assoc'd IVP (1)

Solutions were only first constructed in 1965

Hard b/c u' in RHS, "to control LHS (u) in H^s, need H^{s+1} control of u for RHS!" neverending loop !!.

Let's dig into this problem more in detail

Question: assume u is global smooth solution dropped from heaven. Can we control its H^s norm (for simplicity, $s \in \mathbb{N}^+$)?

w/o any extra structure, no.

KdV conserves momentum (L^2 norm)

recall we did this w Airyeq on day 1.

we compute

$$\partial_t \|u(t)\|_2^2 = 2\langle u, u_t\rangle \overset{\text{KdVeq}}{=} 2\langle u, -\partial_x^3 u + 6uu'\rangle$$

(real valued)

$$= 2\langle u, -\partial_x^3 u\rangle + 12\langle u, uu'\rangle$$

$12\int u^2 u'$

$-\partial_x^3$: anti self adjoint, telling us this term disappears (i.e. IBP thrice $\langle u, -\partial_x^3 u\rangle = \langle \partial_x^3 u, u\rangle$ something $= -$ itself $\Rightarrow$ it's 0)

also a derivative! Special structure!

$= 4\int \partial_x(u^3)$ (using u smooth)

$= 0$

Rephrasing, we basically observed that,

$u_t = \partial_x(-\partial_x^2 u + 3u^2)$

$\Downarrow$ space integral of $u_t = 0 \Rightarrow \partial_t$ of $\int_{\mathbb{R}_x} u = 0$, "total mass of wave" doesn't change.

$\Rightarrow$ conservation law $\|u(t)\|_2 = \|u(0)\|_2 \quad \forall t \in \mathbb{R}$

Warm up, $s=2$ (∂_x^2)

As $\|u(t)\|_{H^s} \le \|u(t)\|_2 + \|\partial_x^s u(t)\|_2$, remains to control $\|\partial_x^s u(t,x)\|_2$

(already controlled by conservation above)

$$\partial_t \|u''(t)\|_2^2 = 2\langle u'', \partial_x^3 u_t\rangle$$

$$= 2\langle u'', -\partial_x^3 u''\rangle + 12\langle u'', \partial_x^2(uu')\rangle$$

anti-SA $\Rightarrow 0$

∂_x^2 looks scary, but don't worry:

$$= 12\langle u'', u'u''\rangle + 24\langle u'', u u'''\rangle$$

$$:= \int u\cdot u''\cdot u''' = \tfrac{1}{2}\int u\cdot \partial_x[(u'')^2] \overset{\text{IBP}}{=} -\tfrac{1}{2}\int u'\cdot(u'')^2$$

$\Rightarrow \partial_t\|u''(t)\|_2^2 \lesssim \|u''\|_2^2 \cdot \|u'\|_\infty$

Note $\|f\|_\infty \lesssim \|f\|_{H^0} \ \forall \sigma > \frac{1}{2}$

b/c $\|f\|_\infty \le \|\hat{f}\|_1 \overset{\text{CBS}}{\le} \|\langle\xi\rangle^\sigma \hat{f}\|_2 \cdot \|\tfrac{1}{\langle\xi\rangle^\sigma}\|_2 \sim \|f\|_{H^\sigma}$

so $\|u'\|_{L^\infty} \lesssim \|u\|_{H^2}$.

By Gronwall, get $\|u''(t)\|_2^2 \le \|u_0''\|_2^2 \, e^{C\cdot\int_0^t \|u'(s)\|_{L^\infty_x} ds}$

(combine w aforementioned conservation L^2, $\|u(t)\|_{H^2}^2 \le \|u_0\|_{H^2}^2 e^{C\cdot\int_0^t \|u'(s)\|_{L^\infty_x} ds}$)

"B/c u' in L^∞ control by u in H^2 win"

$$\partial_x^s(uu') = \partial_x^s u\cdot u' + \partial^s u\cdot u' + s\,\partial_x^s u\cdot u' + \text{"in btw terms"}$$

($k=s$: u gets all ∂_x^s; $k=0$: u' gets all ∂_x^s; $k=1$: u gets 1 ∂_x, u' gets ∂_x^{s-1}; s many choices to put ∂_x^1 on u.)

$$\langle \partial_x^s u, \partial_x^s(uu')\rangle = \langle \partial_x^s u, (\partial_x^s u)\cdot u'\rangle + \langle \partial^s u, u\cdot(\partial^{s+1}u)\rangle + s\langle \partial^s u, u'\cdot\partial_x^s u\rangle + \sum_{\substack{k+j=s-1\\ 1\le k\le s-2}} c_k \langle \partial_x^s u, \partial_x^{k+1}u\cdot\partial_x^{s+1}u\rangle$$

(c_k has $\binom{s}{k}$; change of variable from previous k, to symmetrize)

$$= (1+s)\langle \partial_x^s u, (\partial_x^s u)\cdot u'\rangle + \int u\cdot\tfrac{1}{2}\partial[(\partial^s u)^2] + \sum_{\substack{k+j=s-1\\ 1\le k\le s-2}} c_k \langle \partial_x^s u, \partial_x^{k+1}u\cdot\partial^{j+1}u\rangle$$

$\int u\cdot\frac{1}{2}\partial[(\partial^s u)^2] = -\frac{1}{2}\int u'\cdot(\partial^s u)^2$ (IBP)

same! type of term

Hölder: $\partial_x^s u$ in L^2 b/c that's what's on LHS

So plugging in above,

$$\partial_t\|\partial_x^s u(t)\|_2^2 \lesssim_s \|\partial_x^s u(t)\|_2^2\cdot\|u'(t)\|_\infty + \sum_{\substack{k+j=s-1\\ 1\le k\le s-2}} \|\partial_x^s u\|_2\cdot\|\partial_x^{k+1}u\|_{\frac{2(s-1)}{k}}\cdot\|\partial_x^{j+1}u\|_{\frac{2(s-1)}{j}}$$

To continue doing, we'll use the ineq:

"Gagliardo Nirenberg type"

$$(*)\quad \|\partial^{k+1}f\|_{\frac{2(s-1)}{k}} \lesssim \|f'\|_\infty^{\theta\,(1-\frac{k}{s-1})}\cdot\|\partial^s f\|_2^{\frac{k}{s-1}} \qquad \forall\ 1\le k\le s-2$$

(1 derivative; "maximal" # of derivatives; by scaling)

$k+1$ derivatives on LHS $= 1\cdot\theta + s\cdot(1-\theta)$ derivatives on RHS $\Rightarrow \theta = \frac{s-k-1}{s-1} = 1-\frac{k}{s-1}$ (the L^p, here $p=\frac{2(s-1)}{k}$, $\frac{2(s-1)}{j}$; "change of var" params "take care of themselves")

continue (just easy plugging in)

to get $\partial_t\|\partial_x^s u(t)\|_2^2 \lesssim_s \|\partial_x^s u(t)\|_2^2\cdot\|u'(t)\|_\infty + \sum_{\substack{k+j=s-1\\ 1\le k\le s-2}} \|\partial_x^s u\|_2\cdot\|u'\|_\infty^{1-\frac{k}{s-1}}\cdot\|\partial_x^s u\|_2^{\frac{k}{s-1}}\cdot\|u'\|_\infty^{1-\frac{j}{s-1}}\cdot\|\partial_x^s u\|_2^{\frac{j}{s-1}}$

($k+j=s-1$)

$$\lesssim_s \|\partial_x^s u(t)\|_2^2\cdot\|u'(t)\|_\infty + \|\partial_x^s u\|_2\cdot\|\partial_x^s u\|_2^1\cdot\|u'\|_\infty^2$$

By Gronwall, we get + conservation of mass (L^2 norm):

$$\|u(t)\|_{H^s}^2 \le \|u(0)\|_{H^s}^2 \cdot e^{c\int_0^t \|u'(\tau)\|_\infty \, d\tau}$$

($\|u(0)\|_{H^s}^2 \le \|\partial_x^s u(0)\|_2^2$.)

(note, this # is ≥ 1, we do use this fact to go from minor $\lesssim \|\partial_x^s\|_2$ to $\lesssim \|\ \|_{H^s}$)

"In ptclr, if we control the H^2 norm of sol, then we can control all higher regularity Sobolev norms.
Concretely, if $T_0 := \inf\{t : \|u(t)\|_{H^2} = 2\|u_0\|_{H^2}\}$ then

$\forall\, t \in [0, T_0]$

Note (recalling Sobolev embedding)

$1 + \frac{1}{2} + \varepsilon$ rounded up to nearest integer is 2. (so we've been doing)

$$\|f\|_\infty \lesssim \|f\|_{H^{\frac{1}{2}+\varepsilon}} \quad \forall \varepsilon > 0$$

So $\|u(t)\|_{H^s}^2 \le \|u(0)\|_{H^s}^2 \, e^{c\int_0^t \|u(\tau)\|_{H^2} \, d\tau}$

To finish, we prove (*), or equivalently, (Bernstein Sobolev, sq fns, etcetc.)

$$\|\partial^k f\|_{\frac{2(s-1)}{k}} \lesssim \|f\|_\infty^{1-\frac{k}{s-1}} \, \|\partial^{s-1} f\|_2^{\frac{k}{s-1}} \qquad \forall\, 1 \le k \le s-2.$$

Try $k=1$ first. Break into Littlewood-Paley pieces

$$\|f'\|_{2(s-1)} \lesssim \sum_{N \in 2^{\mathbb{Z}}} \|f_N'\|_{2(s-1)} \lesssim \sum_{N \in 2^{\mathbb{Z}}} N \cdot \|f_N\|_{2(s-1)}$$

Bernstein to travel to L^2

$$\lesssim N^{\frac{1}{2} - \frac{1}{2(s-1)}} \|f_N\|_2$$

insert back in ∂_x^{s-1}

$$\lesssim N^{\frac{1}{2} - \frac{1}{2(s-1)}} N^{-(s-1)} \|\partial^{s-1} f\|_2$$

this works for high freq, but middle / low freq doesn't work?

Try Hölder: $\|f_N\|_{2(s-1)} \lesssim \|f_N\|_\infty^{1-\frac{1}{s-1}} \|f_N\|_2^{\frac{1}{s-1}}$

$$\lesssim \|f\|_\infty^{\frac{s-2}{s-1}} \, N^{-1} \, \|\partial^{s-1} f\|_2^{\frac{1}{s-1}}$$

$(s-1)\cdot\frac{1}{(s-1)}$

we can't sum this $\sum_{N \in 2^{\mathbb{Z}}}$, doesn't work.

$$\|f'\|_{2(s-1)} \lesssim \|f\|_\infty^{\frac{s-2}{s-1}} \, \|\partial^{s-1} f\|_2^{\frac{1}{s-1}}$$

L^2 exactly plays can sum freq pieces back to original.

Try square function

$$\|f'\|_{2(s-1)}^{2(s-1)} \lesssim \|f\|_\infty^{2(s-2)} \, \|\partial^{s-1} f\|_2^2$$

$$\|f'\|_{2(s-1)}^{2(s-1)} \sim \left\| \sqrt{\sum_N N^2 |f_N|^2} \right\|_{2(s-1)}^{2(s-1)} \sim \int \left(\sum_N N^2 |f_N|^2 \right)^{s-1}$$

$$\lesssim \sum_{N_1 \le \cdots \le N_{s-1}} N_1^2 \cdots N_{s-1}^2 \int |f_{N_1}|^2 \cdots |f_{N_{s-1}}|^2$$

Want to put small freq in L^∞!

(put all but biggest one in L^∞)

$$\lesssim \|f\|_\infty^{2(s-2)} \sum_{N_1 \geq \dots \geq N_{s-1}} N_1^2 \cdots N_{s-1}^2 \|f_{N_{s-1}}\|_2^2$$

summing geom series one by one

$$\lesssim \|f\|_\infty^{2(s-2)} \sum_{N_{s-1}} N_{s-1}^{2(s-1)} \|f_{N_{s-1}}\|_2^2$$

$$\lesssim \|f\|_\infty^{2(s-2)} \|\partial^{s-1} f\|_2^2$$

square fn patching pieces

How to handle higher K?

- try above approach, does not end up working
- induct? LOL.
- complex interpolation

exc: cook up the operator! Tz

$$\|\partial^k f\|_{\frac{2(s-1)}{k}} \lesssim \|\partial f\|_{2(s-1)}^{1-\theta} \|\partial^{s-1} f\|_2^{\theta} \quad (*)$$

where $k = (1-\theta) + (s-1)\theta \iff \theta = \frac{k-1}{s-2}$

$$\lesssim \left[\|f\|_\infty^{\frac{s-2}{s-1}} \|\partial^{s-1} f\|_2^{\frac{1}{s-1}}\right]^{1-\theta} \cdot \|\partial^{s-1} f\|_2^{\theta}$$

and exponents work out, as they must, by scaling.

Bona-Smith used regularization

went overboard, now we'd never do bc now we understand Airy propagator very well"

regularize ∂_x^3 as well

$$u_t = -\frac{\partial_x^3}{1-\varepsilon\partial_x^2} u + 3\frac{\partial_x}{1-\varepsilon\partial_x^2}(u^2)$$

very messy.

Instd, we use parabolic regularization

why $\varepsilon\partial_x^4$? Recall Duhamel

$u(t) = e^{-t\partial_x^3} u_0 + \int_0^t e^{-(t-s)\partial_x^3} \partial_x(u^2)$

$\|u(t)\|_{H^s} \leq \|u_0\|_{H^s} + \int_0^t \|\partial_x(u^2)\|_{H^s} ds$

can't close contraction map.

but if $\|\frac{\partial_x}{1-\varepsilon\partial_x^2}(u^2)\|$ ↑ bdd operator (bd dep on ε, $\frac{1}{\varepsilon}$)

$$u_t = -\partial_x^3 u - \varepsilon\partial_x^4 u + 3\partial_x(u^2)$$

sadly "smaller ε, shorter time of $\exists$"

we want to extend to time interval indep of ε.

say $[0,1]$

4th derivative changes propagator

$e^{-t(\partial_x^3 + \varepsilon\partial_x^4)}$

on FT, $e^{-\varepsilon\xi^4}$, exp decay

→ allows us to iterate

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Back to IVP $\begin{cases} u_t + u_{xxx} = 6uu_x \\ u(0) = u_0 \in H^s(\mathbb{R}) \end{cases}$ $2 \le s \in \mathbb{N}$ (technically, can do $\frac{3}{2} < s \in \mathbb{R}$ but $\mathbb{N}$ for simplicity.)

Ill-posedness to KdV via the energy method by Bona-Smith (1975)

They did it by regularizing equation: $\begin{cases} (1 - \varepsilon\partial_x^2) u_t + u_{xxx} = 6uu_x \\ u(0) = u_0 \end{cases}$ (∂_x^2 - elliptic diff operator) $\boxed{\varepsilon > 0}$

- prove on time interval $(-1,1)$, get bound indep of $\varepsilon > 0$ of these
- prove convergence in correct topology

(fatal mistake in blue book Monica showed on day 1, in this section b/c Bona-Smith actually quite subtle.)

In similar spirit, we'll use $\begin{cases} u_t + \varepsilon u_{xxxxx} + u_{xxx} = 6uu_x \\ u(0) = u_0 \in H^0(\mathbb{R}) \quad 2 \le s \in \mathbb{N} \end{cases}$ $\varepsilon > 0$ ($pKdV_\varepsilon$) (parabolic) (see pg. 26 below "beauty of parabolic regularization")

Solution u depends on ε of course, but we suppress dependence of solution u on ε. Later, we'll write u^ε for sol. to $pKdV_\varepsilon$ in our notation

We proceed by integral eq'n contraction mapping.

Duhamel formulation of ($pKdV_\varepsilon$) is

$$u(t) = \underbrace{e^{-t(\varepsilon\partial_x^4 + \partial_x^3)} u_0}_{\text{linear flow, RHS} = 0} + \underbrace{3\int_0^t e^{-(t-\tau)(\varepsilon\partial_x^4 + \partial_x^3)} \partial_x [u^2]'(\tau)\, d\tau}_{\text{variation of parameters tells us contribution of nonlinearity}}$$

(will take 3 pgs)

Contraction Mapping Time! We define $\Phi : B \to B$, Banach space, & distance d.

Note: may be more intuitive to think of Picard iteration $u_{n+1} := (\ ) + \int_0^t \ldots \partial_x(u_n^2)$

Let $\boxed{\Phi(u)(t) := e^{-t(\varepsilon\partial_x^4 + \partial_x^3)} u_0 + 3\int_0^t e^{-(t-\tau)(\varepsilon\partial_x^4 + \partial_x^3)} [u^2]'(\tau)\, d\tau}$

= RHS of Duhamel

We'll show Φ is contraction on Banach space

$$B = \left\{ u \in C_t H^s_x([-T,T]\times\mathbb{R}) : \|u\|_{L^\infty_t H^s_x([-T,T]\times\mathbb{R})} \le 2\|u_0\|_{H^s} \right\}$$

in 2026, defined w/ $[0,T]$ instead of $[-T,T]$, also notation σ instead of s. s instead of τ

equipped w/ the distance $d(u,v) = \|u - v\|_{L^\infty_t H^s_x([-T,T]\times\mathbb{R})}$

provided $T = T(\varepsilon, \|u_0\|_{H^s})$ is ~~still~~ small. (↓ larger init data, shorter time interval)

Let's show $\boxed{\Phi : B \to B}$ (will take 2 pages)

Let $u \in B$, $|t| \le T$ (chosen later.)

Fix t ↑ (need to prove bd unif in t)

$$\|\Phi(u)(t)\|_{H^s} \overset{\Delta\text{-ineq}}{\le} \|e^{-t(\varepsilon\partial_x^4+\partial_x^3)}u_0\|_{H^s} + 3\int_0^t \|e^{-(t-\tau)(\varepsilon\partial_x^4+\partial_x^3)}[u^2]'(\tau)\|_{H^s}\,d\tau$$

We first show 3 facts #1, #2, #3:

Looking on Fourier side (Plancherel)

(★) $\|e^{-t(\varepsilon\partial_x^4+\partial_x^3)} f\|_{H^s} := \|e^{-t(\varepsilon\xi^4 - i\xi^3)}\hat{f}\cdot\langle\xi\rangle^s\|_{L^2_\xi} \le \|f\|_{H^s}$

propagator acting on some fn f; $e^{-t(\text{positive})} \le 1$; unimodular; $\le \|\langle\xi\rangle^s \hat{f}\|_2$; take $t>0$, we reverse time later.

(★★) $\|e^{-t(\varepsilon\partial_x^4+\partial_x^3)} f'\|_{H^s} := \|\underbrace{e^{-t(\varepsilon\xi^4 - i\xi^3)}}_{\text{need to estimate in } L^\infty}\xi\,\underbrace{\hat{f}\cdot\langle\xi\rangle^s}_{\text{isolate and throw in } L^2 \text{ to get } \|f\|_{H^s}}\|_2 \le \|\underbrace{e^{-t\varepsilon\xi^4}}_{\frac{(\varepsilon t)^{-1/4}}{e^{\varepsilon t\xi^4}}}\xi\|_{L^\infty}\cdot\|f\|_{H^s}$

change of variables $\| t\varepsilon\xi^4 = \eta^4$

$$\|(t\varepsilon)^{-1/4} e^{-\eta^4}\eta\|_{L^\infty_\eta}\cdot\|f\|_{H^s}$$

$\left|\frac{\eta}{e^{\eta^4}}\right|$ always ≤ 1 (cases $|\eta|\le 1$, or $|\eta|\ge 1$ Desmos)

$$\lesssim \underbrace{\varepsilon^{-1/4}}_{\text{"huge"}}\underbrace{|t|^{-1/4}}_{\text{may be dangerous}}\|f\|_{H^s}$$

#1: (★) $\Rightarrow \|e^{-t(\partial_x^3+\varepsilon\partial_x^4)}\|_{H^s\to H^s} \le 1$ ($t>0$)

#2: (★★) $\Rightarrow \|e^{-t(\partial_x^3+\varepsilon\partial_x^4)}\partial_x\|_{H^s\to H^s} \le (\varepsilon t)^{-1/4}$

Fact #3: For $s>\frac{1}{2}$, H^s is algebra b/c $\|fg\|_{H^s} \le \|fg\|_2 + \|\,|\nabla|^s(fg)\|_2$

$$\lesssim \|f\|_2\|g\|_\infty + \left(\|\,|\nabla|^s f\|_2\|g\|_\infty + \|f\|_\infty\|\,|\nabla|^s g\|_2\right)$$

($\|f\|_2\|g\|_\infty$: chain rule; $\|\,|\nabla|^s f\|_2\|g\|_\infty + \|f\|_\infty\|\,|\nabla|^s g\|_2$: chain rule)

$$\lesssim \|f\|_{H^s}\|g\|_\infty + \|f\|_\infty\|g\|_{H^s} \lesssim \|f\|_{H^s}\|g\|_{H^s}$$

(we saw last time L^∞ control by H^s if $s>\frac{1}{2}$:)

and $\|g\|_\infty \lesssim \|\hat{g}\|_1 \le \|\langle\xi\rangle^s\hat{g}\|_2\,\|\langle\xi\rangle^{-s}\|_2 \lesssim \|g\|_{H^s}$ if $s>\frac{1}{2}$

For $u\in B$, $t\in[0,T]$ we bound

$$\|\Phi(u)(t)\|_{H^s} \overset{(★)}{\le} \|u_0\|_{H^s} + C\int_0^t \varepsilon^{-1/4}\overset{(★★)}{|t-\tau|^{-1/4}}\|u(\tau)\|^2_{H^s}\,d\tau$$

use algebra prop to pull square outside!

$\|u(\tau)\|^2_{H^s} \le (2\|u_0\|_{H^s})^2$ b/c $u\in B$! ~~using purpose of B?~~

$\int_0^t |t-\tau|^{-1/4}d\tau \le t^{3/4}$

$$\le \|u_0\|_{H^s} + \|u_0\|^2_{H^s}\cdot C\,\varepsilon^{-1/4}t^{3/4} \qquad \forall |t|\le T$$

$$\le 2\|u_0\|_{H^s}$$ provided T is chosen so that $C\,\varepsilon^{-1/4}T^{3/4}\|u_0\|_{H^s} \le 1$

i.e. $T \le c\cdot\varepsilon^{1/3}\|u_0\|_{H^s}^{-4/3}$

indeed matches intuition, $\varepsilon\downarrow 0$, $T\downarrow 0$; $\|u_0\|_{H^s}\uparrow\infty$, $T\downarrow 0$

Not done w/ $\Phi: B \to B$ yet, NEED ctty in time. $\Phi(u) \in C_t H^s_x$

Let $0 \le t' < t \le T$.

$$\|\Phi(u)(t) - \Phi(u)(t')\|_{H^s} \le \left\| \left[e^{-t(\varepsilon\partial_x^4 + \partial_x^3)} - e^{-t'(\varepsilon\partial_x^4 + \partial_x^3)} \right] u_0 \right\|_{H^s}$$

$$+ 3 \left\| \int_0^{t'} \left[e^{(t-\tau)(\varepsilon\partial_x^4 + \partial_x^3)} - e^{-(t'-\tau)(\varepsilon\partial_x^4 + \partial_x^3)} \right] (u^2)'(\tau)\, d\tau \right\|_{H^s}$$

contribution from $[t', t]$, "nothing to cancel it, it is what it is".

$$+ 3 \left\| \int_{t'}^{t} e^{(t-\tau)(\varepsilon\partial_x^4 + \partial_x^3)} (u^2)'(\tau)\, d\tau \right\|_{H^s}$$

3 summands, we bound each in turn:

$$\left\| \left[e^{-t(\varepsilon\partial_x^4 + \partial_x^3)} - e^{-t'(\varepsilon\partial_x^4 + \partial_x^3)} \right] u_0 \right\|_{H^s} \le \left\| 1 \cdot \left[e^{-(t-t')(\varepsilon\xi^4 - i\xi^3)} - 1 \right] \langle\xi\rangle^s \hat{u_0} \right\|_{L^2_\xi}$$

$e^{-t'((\xi^4 - i\xi^3)}$ term multiplied here, bounded by 1. recall $t' > 0$

$e^{-t'(\partial_x^3 + \varepsilon\partial_x^4)} \left(e^{-(t-t')(\partial_x^3 + \varepsilon\partial_x^4)} - 1 \right)$

on Fourier side ≤ 1

dominated by $2\langle\xi\rangle^s \hat{u_0}$

DCT $t' \to t$

0

$$\left\| \int_0^{t'} \left[e^{(t-\tau)(\varepsilon\partial_x^4 + \partial_x^3)} - e^{-(t'-\tau)(\varepsilon\partial_x^4 + \partial_x^3)} \right] (u^2)'(\tau)\, d\tau \right\|_{H^s}$$

$$\lesssim \int_0^{t'} \left\| \left[e^{-(t-t')(\varepsilon\partial_x^4 + \partial_x^3)} - \mathrm{Id} \right] e^{-(t'-\tau)(\varepsilon\partial_x^4 + \partial_x^3)} (u^2)'(\tau) \right\|_{H^s} d\tau \longrightarrow 0$$

Δ-ineq

as an operator, $\to 0$ as $t' \to t$

$\| \checkmark \|_{H^s} \lesssim \varepsilon^{-1/4} |t'-\tau|^{-1/4} \|u(\tau)\|^2_{H^s}$ previous estimate (***)

again DCT $t' \to t$

algebra property

$$\left\| \int_{t'}^{t} e^{(t-\tau)(\varepsilon\partial_x^4 + \partial_x^3)} (u^2)'(\tau)\, d\tau \right\|_{H^s} \lesssim \int_{t'}^{t} \varepsilon^{-1/4} |t-\tau|^{-1/4} \|u(\tau)\|^2_{H^s} d\tau$$

Duhamel + previous estimate (***)

$$\lesssim \varepsilon^{-1/4} |t-t'|^{3/4} \|u_0\|^2_{H^s} \xrightarrow[t' \to t]{\text{(quantitative)}} 0$$

QED Φ is self map $B \to B$

Done w/ $\Phi: B \to B$. Now, prove Φ contraction

Let $u, v \in B$. WTS

$$d(\Phi(u), \Phi(v)) \le \tfrac{1}{2} d(u, v).$$

Recall

$$d(u,v) = \|u - v\|_{L^\infty_t H^s_x([-T,T] \times \mathbb{R})}$$

Fix $t \in [0,T]$

$$d(\Phi(u),\Phi(v)) \le 3 \left\| \int_0^t e^{-(t-\tau)(\varepsilon\partial_x^4+\partial_x^3)} [u^2-v^2]'(\tau)\, d\tau \right\|_{L_t^\infty H_x^s}$$

same linear term, only integral remains

algebra property

$$\lesssim \left\| \int_0^t \varepsilon^{-1/4} |t-\tau|^{-1/4} \|(u-v)(\tau)\|_{H^s} \|(u+v)(\tau)\|_{H^s}\, d\tau \right\|_{C_t^0}$$

prev. estimate (Duh)

using $u,v \in B$, Δ-ineq, bd $\le 4\|u_0\|_{H^s}$.

blc d is L

$$\lesssim \varepsilon^{-1/4} T^{3/4} \|u_0\|_{H^s}\, d(u,v) < \tfrac{1}{2} d(u,v)$$

for T suff small. (same $T \le c \cdot \varepsilon^{1/3} \|u_0\|_{H^s}^{-4/3}$ as before)

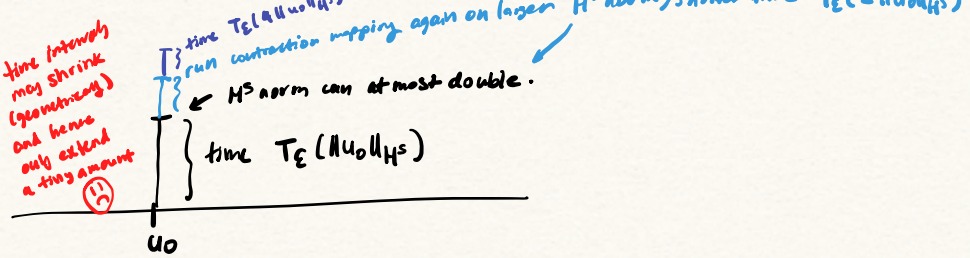
Done w/ ~~Φ contraction~~!

QED contraction mapping

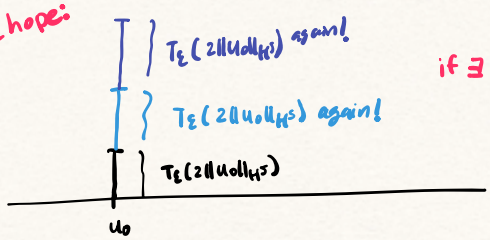
By contraction mapping thm, get unique solution in B.

Now, would like $\varepsilon \searrow 0$

Issue: ε-dependent time interval shrinking — so need to extend these sols to a time interval indep of ε.



The hope:



if $\exists$ magical T_0 s.t. H^s doesn't more than double



to do this, (ultimately to accomplish goal of extending to time interval indep of ε)

we seek a priori H^s bds for sols to pKdV$_\varepsilon$.

4/23/26. Recall $\sigma := s$.

Start w/ L^2

$$\partial_t \|u(t)\|_2^2 = 2\langle u, u_t\rangle = \underbrace{-2\langle u, \varepsilon\partial_x^4 u\rangle}_{\text{leave only this term.}} + 2\langle u, -\partial_x^3 u\rangle + 12\langle u, uu'\rangle$$

$-\partial_x^3$ antiself adjoint

$12\langle u, uu'\rangle = \frac{1}{3}\int (u^3)'$ "complete derivative"

$= 0$

$$= -2\varepsilon \langle \partial_x^2, \partial_x^2 u\rangle \le 0.$$

beauty of parabolic regularization

$$\partial_t \|\partial_x^s u(t)\|_2^2 = \underbrace{2\langle \partial^s u, \partial^s u_t\rangle}_{\text{"} 2\langle \partial^s u, -\partial^3\partial^s u - \varepsilon\partial^4\partial^s u\rangle \text{"}} = -2\varepsilon\langle \partial^s u, \partial^s\partial^4 u\rangle + 2\langle \partial^s u, \partial^s(-\partial^3)u\rangle + 12\langle \partial^s u, \partial^s(uu')\rangle$$

≤ 0 (similar to above)

anti-SA

∂_x^s spatial derivative

$$\le \|\partial_x^s u(t)\|_2^2 \cdot \|u'(t)\|_\infty$$

square bracket estimate

"Gagliardo-Nirenberg" 3-lines lemma

$= -2\varepsilon \|\partial^{s+2} u\|_2^2$, use smoothness we proved

from last lecture

So by Gronwall, (recall $\partial_t f \le f \cdot g \Rightarrow f(t) \le f(0) e^{\int_0^t g}$)

$$\|u(t)\|_{H^s}^2 \le \|u_0\|_{H^s}^2 \, e^{C \int_0^t \|u'(\tau)\|_\infty \, d\tau}$$

prime is spatial derivative

many previous lectures. $\|f\|_\infty \lesssim \|f\|_{L^r}$ if $r > \frac{1}{2}$

$\frac{1}{2}$ derivative to go from L^∞ to L^2

we use $s > \frac{1}{2}$.

$$\Rightarrow \|u(t)\|_{H^s} \le \|u_0\|_{H^s} \, e^{\hat{C} \int_0^t \|u(\tau)\|_{H^s} \, d\tau} \quad (\star)$$

Sobolev embedding

and sqrt LoL

same const.

Choosing $T_0 := T_0(\|u_0\|_{H^s})$ s.t. $e^{\hat{C} \cdot 4 T_0 \|u_0\|_{H^s}} \le 2$

very common in PDE "assume something, manifests into existence"

we can extend the solution u to $[0, T_0]$ via the usual bootstrap argument:

"always about set/interval being connected"

$$\Omega_1 = \{ t \in [0, T_0] : \|u(t)\|_{H^s} \le 4 \|u_0\|_{H^s} \}$$

$$\Omega_1 := \{ t \in [0, T_0] : \sup_{0 \le s \le t} \|u(s)\|_{H^\sigma} \le 4 \|u_0\|_{H^\sigma} \}.$$

$$\Omega_2 = \{ t \in [0, T_0] : \|u(t)\|_{H^s} \le 2 \|u_0\|_{H^s} \}$$

$$\text{2026 version: } \Omega_2 := \{ t \in [0, T_0] : \sup_{0 \le s \le t} \|u(s)\|_{H^\sigma} \le 2 \|u_0\|_{H^\sigma} \}$$

Ω_2 is clearly
- nonempty b/c $t=0$ in there,
- closed b/c closed neq.

NTS Ω_2 is open in $[0, T_0]$. Let $t_0 \in \Omega_2 \cap [0, T_0)$.

Running contraction mapping starting @ t_0, find $\delta := \delta(\varepsilon, \|u(t_0)\|_{H^\sigma})$ s.t. $\forall t \in (t_0, t_0 + \delta) \cap [0, T_0] \subseteq \Omega_1$

$$\|u(t)\|_{H^\sigma} \le \|u_0\|_{H^\sigma} \, e^{\hat{C} \int_0^t \|u(s)\|_{H^\sigma} ds} \le \|u_0\|_{H^\sigma} \, e^{C T_0 4 \|u_0\|_{H^\sigma}} \le 2 \|u_0\|_{H^\sigma}$$

(def of T_0)

$\Rightarrow t \in \Omega_2$

and then $t \in \Omega_2 \Rightarrow \exists \delta > 0$ s.t. $(t, t+\delta) \subseteq \Omega_1$

by cty of solution in time

- so Ω_2 open!

And then prove that $\Omega_1 \subseteq \Omega_2$!

Cf Bona smith $(1 - \varepsilon \partial_x^2)$, Monica says she would do something like

$$u_t + u_{xxx} = \frac{3 \partial_x}{1 - \varepsilon \partial_x^2} (u^2)$$

picture above again:

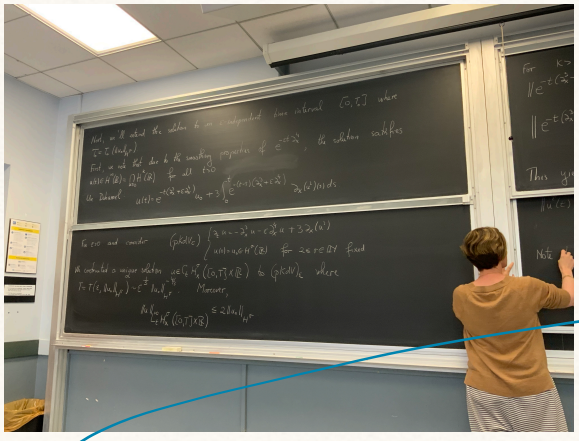
T_ε again, if $2T_\varepsilon < T_0$

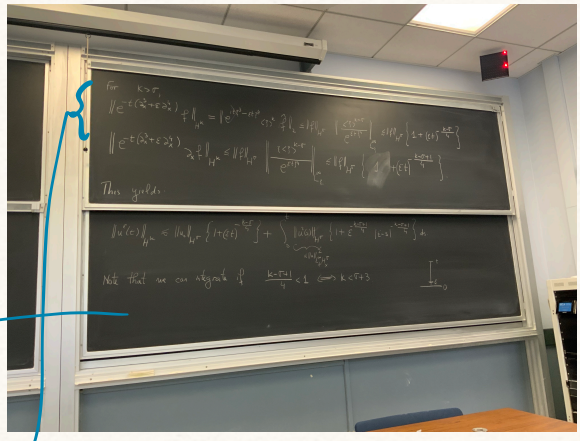
T_ε again, if $T_\varepsilon < T_0$ ← indep of ε!

$T_\varepsilon := T(\varepsilon, 2\|u_0\|_{H^\sigma})$

time 0

Lec 11
4/22/26





$f \in H^\sigma$ bound $\|e^{-t(\partial_x^3 + \varepsilon \partial_x^4)} f\|_{H^k}$ $\quad k > \sigma$

$\|e^{-t(\partial_x^3 + \varepsilon \partial_x^4)} \partial_x f\|_{H^k}$

yielding $\|u^\varepsilon(t)\|_{H^k} \lesssim$ (something $< \infty$ if $k < \sigma + 3$) i.e. $k \le \sigma + 2$, gained ≥ 2 derivatives in smoothness.

via Duhamel $u^\varepsilon(t) = e^{-t(\partial_x^3 + \varepsilon \partial_x^4)} u_0 + 3 \int_0^t e^{-(t-s)(\partial_x^3 + \varepsilon \partial_x^4)} \partial_x (u^2)\, ds$

and use $\|u^2(s)\|_{H^\sigma} \lesssim \|u\|^2_{L^\infty_t H^\sigma_x}$ (last time)

Next, we iterate the argument, using as input the Duhamel formula from time $= \delta > 0$ to time $= t$, gaining 2 derivatives at each iteration.

4/15/24 Recall last time: For $\varepsilon > 0$, we considered

regularized KdV $\begin{cases} u_t + \varepsilon u_{xxxx} + u_{xxx} = 6uu_x \\ u(0) = u_0 \in H^s(\mathbb{R}) \end{cases}$ for $2 \le s \in \mathbb{N}$. (pKdV$_\varepsilon$)

we showed (pKdV$_\varepsilon$) is locally well-posed in H^s: specifically, $\exists T_0 := T_0(\|u_0\|_{H^s})$

and a unique solution (to (pKdV$_\varepsilon$)) $u^\varepsilon \in C_t H^s_x([0,T_0]\times\mathbb{R})$ (and we used contraction mapping, ctg data $\to$ solution map)

and importantly (1) uniform in ε control

$$\|u^\varepsilon(t)\|_{H^s} \le 2\|u_0\|_{H^s} \quad \forall t \in [0,T_0]$$

i.e. $\|u^\varepsilon(t)\|_{L^\infty_t H^s_x([0,T_0]\times\mathbb{R})} \le 2\|u_0\|_{H^s}$ unif in ε

We would like to send $\varepsilon \downarrow 0$ to construct solution to IVP (KdV) (w/ same (in-data) u_0 above)

Let's show that $\{u^\varepsilon\}_{\varepsilon>0}$ is Cauchy in $C_t L^2_x([0,T_0]\times\mathbb{R})$

Claim: u^ε do converge as $\varepsilon \downarrow 0$ in weaker topology, $C_t L^2_x([0,T_0]\times\mathbb{R})$

Let $0 < \eta < \varepsilon$.

Grönwall to show "Cauchy"-ness

$$\partial_t \|(u^\varepsilon - u^\eta)(t)\|_2^2 = 2\langle u^\varepsilon - u^\eta, \partial_t(u^\varepsilon - u^\eta)\rangle$$

$$= 2\langle u^\varepsilon - u^\eta, -\varepsilon\partial_x^4 u^\varepsilon + \eta\partial_x^4 u^\eta\rangle + 2\langle u^\varepsilon - u^\eta, -\partial_x^3(u^\varepsilon - u^\eta)\rangle$$

("cubic terms"; ∂_x^3 anti-self-adjoint)

$$+ 6\langle u^\varepsilon - u^\eta, [(u^\varepsilon)^2 - (u^\eta)^2]'\rangle$$

("contributions of nonlinearity")

$$= -2(\varepsilon-\eta)\langle u^\varepsilon - u^\eta, \partial_x^4 u^\varepsilon\rangle - 2\eta\langle u^\varepsilon - u^\eta, \partial_x^4(u^\varepsilon - u^\eta)\rangle$$

≥ 0 by putting ∂_x^2 on left half

$$+ 6\langle u^\varepsilon - u^\eta, (u^\varepsilon - u^\eta)(u^\varepsilon + u^\eta)^3\rangle$$

$$+ 6\langle u^\varepsilon - u^\eta, (u^\varepsilon - u^\eta)^3(u^\varepsilon + u^\eta)\rangle$$

(general trick we always use: $f\cdot f' = \frac{1}{2}(f^2)'$) $= \frac{1}{2}\partial_x(u^\varepsilon - u^\eta)^2$, so upon integrate by parts, get term like but w/ different const.

(3^{rd} derivative)

move ∂_x^2 on LHS, CBS.

$$\le 2(\varepsilon-\eta)\|u^\varepsilon - u^\eta\|_{H^2}\|u^\varepsilon\|_{H^2} + 0 + 3\|u^\varepsilon - u^\eta\|_2^2\left(\|(u^\varepsilon)'\|_\infty + \|(u^\eta)'\|_\infty\right)$$

H^2, we don't like, we want L^2!

put in $L^\infty H^2$, bdd by init data

$\|u^\varepsilon\|_{H^2} \le \|u^\varepsilon\|_{H^\sigma} \le 2\|u_0\|_{H^\sigma}$

$$\lesssim 16(\varepsilon-\eta)\|u_0\|_{H^\sigma}^2 + \|u^\varepsilon - u^\eta\|_2^2\|u_0\|_{H^\sigma}$$

$2\cdot 2$ from u^ε, 2 from u^η, 2 from u^ε

linear growth ("FTC" if this term absent)

By Grönwall

$$\|(u^\varepsilon - u^\eta)(t)\|_2^2 \lesssim (\varepsilon-\eta)\, t\, \|u_0\|_{H^2}^2\, e^{C T_0 \|u_0\|_{H^2}} \lesssim_{T_0, \|u_0\|_{H^2}} (\varepsilon-\eta)$$

unif $t \in [0,T_0]$

$\lesssim (\varepsilon-\eta)\|u_0\|_{H^\sigma}^2 \cdot T_0 \cdot e^{C T_0 \|u_0\|_{H^\sigma}}$

$\varepsilon, \eta \to 0$ $\downarrow$ 0

uniformly for $t \in [0,T_0]$ $\Rightarrow$ convergence in $L^\infty_t L^2_x$

4/24/26

Lec 12 Summary so far: $\forall \varepsilon > 0$, we constr unique sol $u^\varepsilon \in C_t H^\sigma_x([0,T_0]\times\mathbb{R})$ to (pKdV)$_\varepsilon$ $_{u_0 \in H^\sigma}$ $_{2\le\sigma\in\mathbb{N}\text{ fixed}}$

where $T_0 = T_0(\|u_0\|_{H^\sigma})$.

Moreover, $\|u^\varepsilon\|_{L^\infty_t H^\sigma_x([0,T_0]\times\mathbb{R})} \le 2\|u_0\|_{H^\sigma}$

We showed $u^\varepsilon \xrightarrow[\varepsilon\to 0]{} u$ in $L^\infty_t L^2_x([0,T_0]\times\mathbb{R})$

In ptclr, u ots $C_t L^2_x$, and by Fatou in $L^\infty_t H^\sigma_x$ w/ $\|u\|_{L^\infty_t H^\sigma_x} \le 2\|u_0\|_{H^\sigma}$

This implies $u \in C_t H^r_x$ $\forall\ 0 \le r < \sigma$: (to apply Hölder)

recall $\|u\|_{H^r} \le \|u\|_2$ $\forall r<0$ b/c $\langle\xi\rangle^{\text{negative}} \le 1$.

Blc (look on Fourier side)

$$\|u(t)-u(t')\|_{H^r} \le \|u(t)-u(t')\|_2^{1-\frac{r}{\sigma}} \|u(t)-u(t')\|_{H^\sigma}^{r/\sigma} \le \|u(t)-u(t')\|_2^{1-\frac{r}{\sigma}} \cdot \left(2\|u\|_{L^\infty_t H^\sigma_x}\right)^{r/\sigma} \xrightarrow[|t-t'|\to 0]{} 0$$

$\frac{r}{\sigma}$ on H^σ so r derivatives total on RHS, matching LHS

Same style of arg shows $u^\varepsilon \xrightarrow[\varepsilon\to 0]{} u$ in $C_t H^r_x([0,T_0]\times\mathbb{R})$ $\forall\ 0 \le r < \sigma$.

Question: what does u have to do w/ KdV?

Let $u := \lim_{\varepsilon \searrow 0} u^\varepsilon$ in $C_t L^2_x([0,T_0]\times\mathbb{R})$

By (1), Fatou gives — unif in ε H^s bd

$$\|u(t)\|_{H^s} \le 2\|u_0\|_{H^s} \qquad \forall t \in [0,T_0]$$

So, $u \in (C_t L^2_x \cap L^\infty_t H^s_x)([0,T_0]\times\mathbb{R})$

By interpolation, in ptclr $u \in C_t H^r_x([0,T_0]\times\mathbb{R})$ $\forall\, 0 \le r < s$:

$$\|u(t) - u(t')\|_{H^r} \le \|u(t)-u(t')\|_2^{1-\frac{r}{s}} \|u(t)-u(t')\|_{H^s}^{r/s}$$

(go to Fourier side if needed)

$$\le \|u(t)-u(t')\|_2^{1-\frac{r}{s}} \left(2\|u\|_{L^\infty_t H^s_x}\right)^{r/s} \xrightarrow[|t-t'|\to 0]{} 0$$

we do Not yet know if u in $C_t H^s_x$.

Question: is u a solution to (KdV)?

Recall (Duhamel for $(\partial_t u^\varepsilon = -\varepsilon\partial_x^4 u^\varepsilon - \partial_x^3 u^\varepsilon + \partial_x(u^\varepsilon)^2)$)

$$u^\varepsilon(t) = e^{-t(\varepsilon\partial_x^4 + \partial_x^3)} u_0 + 3\int_0^t e^{-(t-\tau)(\varepsilon\partial_x^4+\partial_x^3)} (u^2)'(\tau)\,d\tau.$$

Duhamel

Know: $u^\varepsilon(t) \to u(t)$ in $C_t H^r_x$ $\forall 0 \le r < s$ as $\varepsilon \searrow 0$

converge in $C_t H^s$ ← $\varepsilon \searrow 0$ ← does it converge, in which regularity ?? Answer:

$$e^{-t\partial_x^3} u_0 + \dots \quad \text{(Duhamel fml for KdV)}$$

Will: $\|(e^{-\varepsilon t(\partial_x^4+\partial_x^3)} - e^{-t\partial_x^3})u_0\|_{H^s} \le \|(e^{-\varepsilon t\xi^4} - 1)\,\hat{u_0}\langle\xi\rangle^s\|_2$ (recall $s>5$)

↓ DCT, $\varepsilon \searrow 0$

0 want in $t \in [0,T_0]$

and similar arguments (close a derivative though)

$\| \int_0^t e^{-(t-s)(\partial_x^2+\partial_x^3)} \partial_x (u^2)^3(s) - e^{-(t-s)\partial_x^3} \partial_x (u^2)(s)\, ds \|_{H^{r-1}} \le \int_0^t \| \left[e^{-(t-s)(\partial_x^2+\partial_x^3)} - e^{-(t-s)\partial_x^3} \right] \partial_x (u^2)^3(s) \|_{H^{r-1}} ds + \| \int_0^t e^{-(t-s)\partial_x^3} \partial_x ((u^2)^3 - u^2) ds \|_{H^{r-1}}$

$\le \int_0^t \| [e^{-(t-s)\partial_x^2} - 1] (u^2)^3(s) \|_{H^r} ds + \int_0^t \| (u^2)^3 - u^2 \|_{H^r} ds \to 0$ if $r<5$

(2nd term needs $r<5$, 1st term needs $r \le 5$)

get converges to $3 \int_0^t e^{-(t-\tau)\partial_x^3} \partial_x (u^2)(\tau)\, d\tau$ in $C_t H_x^{s-1}$

above: Duhamel PCT stuff, the integrands and contour should be interpreted in C_t.

old into we already know

So u satisfies Duhamel formula in $(C_t L_x^2 \cap L_t^\infty H_x^s)([0,T_0] \times \mathbb{R})$

perhaps it should say: $u \in (C_t L_x^2 \cap L_t^\infty H_x^s)([0,T_0]\times\mathbb{R})$ satisfies the Duhamel formula

So: we constr $u \in C_t L_x^2 \cap L_t^\infty H_x^s ([0,T_0]\times\mathbb{R})$ satisfying Duhamel (*) $u(t) = e^{-t\partial_x^3} u_0 + 3\int_0^t e^{-(t-s)\partial_x^3} \partial_x (u^2)(\tau)\, d\tau$ in $C_t H_x^{r-1}$ $\forall r<s$

(!sol)

we chose parabolic reg, what if we used elliptic (Bona Smith) reg instead?

Uniqueness of sols in $C_t L_x^2 \cap L_t^\infty H_x^s$ to kdV:

Suffices to prove uniqueness in weaker topologies, b/c once agree, iterates equal

Trick "I've never seen it written down, works for any dispersive eq, use a lot"

Step1: $t \mapsto e^{t\partial_x^3} u(t)$ is $C_t^1 L_x^2$. and $\partial_t [e^{t\partial_x^3} u(t)] = 3 e^{t\partial_x^3} \partial_x (u^2)(t)$

$= u_0 + 3\int_0^t e^{\tau\partial_x^3} \partial_x (u^2)(\tau)\, d\tau$

Step2: Let $u, v \in C_t L_x^2 \cap L_t^\infty H_x^s ([0,T_0]\times\mathbb{R})$ (assume both satisfy Duhamel (*))

$\partial_t \| (u-v)(t) \|_2^2 = 2 \langle u-v, -\partial_x^3 (u-v) \rangle + 3 \langle u-v, (u^2-v^2)' \rangle$

do use $e^{t\partial_x^3}$ isom in L^2 so disappear from L^2 inner prod

similar to how we dealt w/ it before: $\frac{1}{2} \int \partial_x (u-v)^2 \cdot (u+v)$.

$= 0 + \frac{3}{2} \langle u-v, (u-v)(u+v)' \rangle$

$\le \| u-v \|_2^2 \left(\| u' \|_{\infty} + \| v' \|_{\infty} \right)$ const is 3

yes it's 0 but want general computation

const is 3

bdd by init data in H^2 choice of $T_0 \Rightarrow$ this $e^{CT_0 \cdots} \lesssim 1$

So by Grönwall, $\| (u-v)(t) \|_2^2 \le \| (u-v)(0) \|_2^2 e^{C T_0 (\| u \|_{L_t^\infty H_x^2} + \| v \|_{L_t^\infty H_x^2})} = 0$

general bd: $\| u(t) - v(t) \|_2 \lesssim \| u(0) - v(0) \|_2$.

Issue: • above only know u, v satisfy Duhamel integral eqns, not derivative eqns

(not "fatal error")

"the above derivatives were fake"

Fix: fix/mod $u \in C_t L_x^2 \cap L_t^\infty H_x^3 ([0,T_0]\times\mathbb{R})$ satisfying Duhamel eqn

Claim: $t \mapsto e^{t\partial_x^3} u(t) = u_0 + 3\int_0^t e^{\tau\partial_x^3} (u^2)'(\tau)\, d\tau$ is $C_t^1 L_x^2$

and its derivative is exactly what we thought it was $3 e^{t\partial_x^3} (u^2)'(t)$.

stop a $e^{t\partial_x^3}$ time so now has "real" derivative

turn remove the $e^{t\partial_x^3}$ const on Fourier side

once we exploited it for its real derivative.

Assuming this claim,

$\partial_t \| u(t) \|_2^2 = \partial_t \| e^{t\partial_x^3} u(t) \|_2^2 = 2 \langle e^{t\partial_x^3} u(t), 3 e^{t\partial_x^3} (u^2)'(t) \rangle$

$= 4 \int (u^3(t))'\, dx = 0$ (This was how we proved conservation of mass for NLS!!!!)

Pf of Claim: $\left\| \frac{1}{t-t'} \int_{t'}^t e^{\tau\partial_x^3} (u^2)'(\tau)\, d\tau - e^{t\partial_x^3} (u^2)'(t) \right\|_2 \le \frac{1}{t-t'} \int_{t'}^t \| (e^{\tau\partial_x^3} - e^{t\partial_x^3}) \partial_x (u^2)(\tau) \|_2\, d\tau + \frac{1}{t-t'} \int_{t'}^t \| e^{t\partial_x^3} \partial_x (u^2 - u^2)(\tau) \|_2\, d\tau$

always ∂_x

WLOG $t' < t$

propagators conv strongly $\to 0$

$\le$ "ε"

need $t-t'$ from integral bdd, so put integrand in L_t^∞.

goes along int

$\le \| u(\tau) - u(t) \|_{H_x^1} \cdot \| u \|_{L_t^\infty H_x^1}$

$|\tau - t| \to 0$

$\to 0$

$\le \frac{1}{|t-t'|} \int_{t'}^t \| e^{\tau\partial_x^3} (u^2)'(\tau) - e^{t\partial_x^3} (u^2)'(t) \|_2\, d\tau$

Δ-ineq

$\le \| (e^{\tau\partial_x^3} - e^{t\partial_x^3}) (u^2)'(t) \|_{L_\tau^\infty L_x^2 ([t',t])}$

$+ \| e^{t\partial_x^3}(u^2(\tau)-u^2(t))' \|_{L^\infty_\tau L^2_x[t',t]}$

$\downarrow$ $|t-t'| \to 0$

0

actually that notation makes it harder see the diff of squares. $\partial_x(u^2-v^2)$ instdv ✓ 4/28/26

QED ! sol KdV

(Uno) "solutions a la Hadamard"

Left to show: $u \in C_t H^s_x([0,T_0]\times\mathbb{R})$, and ctd dependence of data $\mapsto$ solution map in the H^s topology.

(Dos)

Monica calls "equicty method" "smooth init data"

Recall philosophy question "what is solution"
- classical solution. Too few of these, not useful in practice
- weak sol. Easy to constr, satisfy no good prop (lose conserve of L^2)
- Duhamel sol. Goldilocks zone!

Fix init data $u_0 \in H^s$. For dyadic freqs $N \in 2^{\mathbb{N}}$

"previous issue was init data wasn't regular enough, e.g. if $u_0 \in H^{100}$, above arguments are all good"

let $u_0^N := P_{\le N} u_0$ "smooth" init data by taking low freqs.

Note: $\|u_0^N\|_{H^s} \lesssim \|u_0\|_{H^s}$ unif in N. blc LP proj bdd.

$\Rightarrow$ kdV sol starting w/ u_0^N exist on same time interval $[0,T_0]$ as it starting u_0!!

$\|u_0^N - u_0\|_{H^s} =: \| P_{>N} u_0\|_{H^s} \xrightarrow[N\to\infty]{\text{DCT}} 0$

Let $u^N \in (C_t L^2_x \cap L^\infty_t H^s_x)[0,T_0]\times\mathbb{R}$ be the (unique) solution to (KdV) w/ data $u^N(0) := u_0^N$.

Note! To uniform in N [blc To only depend on $\|u_0\|_{H^s}$, and $\|u_0^N\|_{H^s} \lesssim \|u_0\|_{H^s}$ unif in N)

$\|u^N(t)\|_{H^s} \le 2\|u_0^N\|_{H^s} \lesssim \|u_0\|_{H^s}$ (1)

Moreover, these u^N are smoother!:

Grönwall again

$\|u^N(t)\|^2_{H^{s+1}} \le \|u_0^N\|^2_{H^{s+1}} e^{C T_0 \|u_0\|_{H^s}} \lesssim \|u_0\|^2_{H^2},\ T_0 \left(N \|u_0\|_{H^s}\right)^2$

$\le \|u_0^N\|_{H^{s+1}} e^{C\int_0^t \|(u^N)'(\tau)\|_\infty d\tau}$ ("u^N is infinitely smooth") $\le \|P_{\le N} u_0\|_{H^{s+1}} e^{C T_0 \|u_0\|_{H^s}}$ smoother!

Bernstein $\lesssim$. (Young)

So for fixed N, $u^N \in C_t L^2_x \cap L^\infty_t H^{s+1}_x([0,T_0]\times\mathbb{R})$ $\Rightarrow$ $u^N \in C_t H^s([0,T_0]\times\mathbb{R})$

for fixed N, unif bdd (bd gets worse as $N\to\infty$)

interpolation from before (Hölder on Fourier side from beginning of this lecture)

If we can show $u^N \xrightarrow[N\to\infty]{L^\infty_t H^s_x} u$, then unif limit of ctd fns is ctd $\Rightarrow u \in C_t H^s_x([0,T_0]\times\mathbb{R})$.

end of Lec 12

4/27/26 Recall, we constructed a (Duhamel) sol $u \in C_t L^2_x \cap L^\infty_t H^s_x([0,T_0]\times\mathbb{R})$ to IVP $\begin{cases}\partial_t = -\partial_x^3 + 3\partial_x(u^2)\\ u(0) = u_0 \in H^s(\mathbb{R})\end{cases}$ $2 \le s \le N$ fixed

$T_0(\|u_0\|_{H^s})$

Moreover $\|u\|_{C_t H^s_x([0,T_0]\times\mathbb{R})} \le 2\|u_0\|_{H^s_x}$.

WTS $u \in C_t H^s_x([0,T_0]\times\mathbb{R})$. Recall $u_0^N := P_{\le N} u_0 \xrightarrow[N\to\infty]{H^s} u_0$.

$\|u_0^N\|_{H^s} \le \|u_0\|_{H^s}$

First step: prove convergence to u in weaker topology, e.g. L^2_x, so can identify a limit fn of (u_N) at least. Gronwall arg, similar to last lecture

$\|u_0^N - u_0\|_{L^2} = \|P_{>N} u_0\|_{L^2} \lesssim N^{-\sigma} \|u_0\|_{H^\sigma}$ Bernstein

add ∂ at high freq gain neg powers of N

quantitative conv in weaker topology!

$$\partial_t \|(u^N - u)(t)\|_{L^2}^2 = 3\langle u^N - u, [(u^N - u)(u^N + u)]' \rangle$$

must use "Fix" above, run same as above

$$\lesssim \|u^N - u\|_{L^2}^2 \left[\|u^N\|_{H^2} + \|u\|_{H^2} \right] \lesssim_{\|u_0\|_{H^s}, T_0} \|u^N - u\|_{L^2}^2$$

see (1sol) step 2 above

trick insert propagator b/c has better regularity prop

$= \partial_t \|e^{t\partial_x^2}(u^N(t) - u(t))\|_{L^2}^2$

$= 2\langle e^{t\partial_x^2}(u^N - u)(t), \partial e^{t\partial_x^2}((u^N)^2 - u^2)'(t) \rangle$

isometry in L^2

∂ hits sum or -

So by Grönwall, $\|u^N(t) - u(t)\|_{L^2} \lesssim_{\|u_0\|_{H^s}, T_0} \|u_0^N - u_0\|_{L^2} \lesssim \|P_{>N} u_0\|_{L^2} \lesssim N^{-s} \|u_0\|_{H^s}$

Gronwall exponential absorbed here

$N \to \infty$ ↓ 0 unif in $t \in [0, T_0)$

$= 6 \int (u^N - u)(u^N - u)'(u^N + u) + \cdots$

$= 3 \int (u^N - u)^2 (u^N + u)'$

$\lesssim \|u^N - u\|_{L^2}^2 (\|(u^N)'\|_{L^\infty} + \|u'\|_{L^\infty})$

$\lesssim \|u^N - u\|_{L^2}^2 (\|u^N\|_{L_t^\infty H_x^\sigma} + \|u\|_{L_t^\infty H_x^\sigma}) \lesssim \|u_0\|_{H^\sigma} \cdot \|u^N - u\|_{L^2}^2$. Grönwall $\Rightarrow \|u^N(t) - u(t)\|_{L^2}^2 \le \|u^N - u_0\|_{L^2}^2 e^{CT\|u_0\|_{H^\sigma}}$

$\lesssim \|P_{>N} u_0\|_{L^2}^2 \lesssim N^{-2\sigma} \|u_0\|_{H^\sigma}^2$

So, $u^N \xrightarrow[N \to \infty]{C_t L^2_x} u$.

So $\|u^N(t) - u(t)\|_{L_t^\infty L_x^2} \xrightarrow{N \to \infty} 0$.

Analogous computation shows similarly $\|u^N(t) - u^M(t)\|_{L^2} \lesssim \|P_{M \le \cdot \le N} u_0\|_{L^2}$ unif in $M < N$.

Step 2 of (Uno)

The only thing left to show: $\{u^N\}_N$ Cauchy in $L_t^\infty H_x^s$, i.e. "converge to something", and by Step 1 & uniqueness of limit in weaker topology, must converge to u".

4/17/24 Last time, concluded:

Given $s \in \mathbb{N}$, $s \geq 2$ (regularity), $u_0 \in H^s(\mathbb{R})$ (init data), $\exists!$ sol. $u \in (C_t L^2_x \cap L^\infty_t H^s_x)([0,T_0] \times \mathbb{R})$

to (KdV) $\begin{cases} u_t + u_{xxx} = 6uu_x \\ u(0) = u_0 \in H^s(\mathbb{R}) \end{cases}$ where $T_0 = T_0(\|u_0\|_{H^s})$.

Moreover, $\|u(t)\|_{H^s} \leq 2\|u(0)\|_{H^s}$ $\forall t \in [0,T_0]$ "doesn't more than double"

Remains to show $u \in C_t H^s_x([0,T_0] \times \mathbb{R})$ and, the map $u_0 \mapsto u$ is cts. $H^s \to C_t H^s$

For $N \in 2^{\mathbb{N}}$, we defined $u_0^N := P_{\leq N} u_0$

then $u_0^N \xrightarrow[n\to\infty]{H^s} u_0$ by DCT, and moreover, $\sup_N \|u_0^N\|_{H^s} \leq \|u_0\|_{H^s}$ (easily seen on Fourier side)

Thus, bc unif in N bd, can choose uniform time interval on which solutions w/ init data u_0 $\exists$:

Let $u^N \in (C_t L^2_x \cap L^\infty_t H^s_x)([0,T_0] \times \mathbb{R})$ be sols to KdV w/ $u^N(0) = u_0^N$ obtained by our construction

(indep. of N)

$\|u^N(t)\|_{H^s} \leq 2\|u_0^N\|_{H^s} \leq 2\|u_0\|_{H^s}$ $\forall N \in 2^{\mathbb{N}}$, $\forall t \in [0,T_0]$.

By Grönwall arg, $\|u^N(t)\|_{H^{s+1}} \leq \|u_0^N\|_{H^{s+1}} e^{CT_0\|u_0\|_{H^2}} \lesssim N$ (from now on, all implicit constants may depend on $\|u_0\|_{H^s}$ which we treat.)

"more regular": $u^N \in C_t L^2_x \cap L^\infty_t H^{s+1}_x \Rightarrow u^N \in C_t H^s_x$ (interpolation from beginning of last lec)

Thus, if we show that $u^N \xrightarrow[N\to\infty]{L^\infty_t H^s_x} u$, then $u \in C_t H^s_x$ (unif limit of cts fns is cts)

Recall: $\|u^N(t) - u(t)\|_2 \lesssim \|u_0^N - u_0\|_2 \lesssim \|P_{>N} u_0\|_2 \lesssim N^{-s}\|u_0\|_{H^s} \xrightarrow[n\to\infty]{} 0$ $\forall t \in [0,T_0]$

so we have convergence in weaker topology $u^N \xrightarrow[N\to\infty]{C_t L^2_x} u$. (1)

Similarly, $\|u^N(t) - u^M(t)\|_2 \lesssim \|P_{M<\cdot\leq N} u_0\|_2$ $\forall t \in [0,T_0]$

It's enough to show that $\{u^N\}_N$ is Cauchy in $L^\infty_t H^s_x$ (will converge in that topology bc something, that something must agree w/ (1) i.e. converge to u)

(Step 2 of (Uno)).

For simplicity, will show $\{u^N\}_N$ is Cauchy in $L^\infty_t H^2_x$. $s=2$

For $M<N$,

$$\partial_t \| \partial_x^2 (u^N-u^M)(t)\|_{L^2}^2 = 2\langle (u^N-u^M)'', -\partial_x^3 (u^N-u^M)''\rangle + 6\langle (u^N-u^M)'', [(u^N-u^M)(u^N+u^M)]'''\rangle$$

(spatial derivative; antiself adjoint; using that u^N, u^M are both solutions to KdV; $uu_x = (u^2)'/2$; $(u^2)'' - ((u')^2)'$ difference of squares)

$$= 2\langle (u^N-u^M)'', \partial_x(-\partial_x^3)(u^N-u^M) + 3\partial_x^2((u^N)^2-(u^M)^2)\rangle$$

(∞ smooth so know ∂_x hit it is finite. But antiSA means actually 0; product rule)

$$= 6\langle (u^N-u^M)'', (u^N-u^M)''(u^N+u^M)'\rangle \quad \text{(I)}$$

(2 derivatives grouped w 3 derivative = derivative $((u^N-u^M)'')^2$..)

$$+18\langle (u^N-u^M)'', (u^N-u^M)'(u^N+u^M)''\rangle \quad \text{(II)}$$

(IBP..? $= -\langle ((u^N-u^M)'')^2, (u^N+u^M)'\rangle$, L^1, L^∞ controlled by H^1)

(I) $(\text{diff})''\cdot(\text{diff})'''$ IBP $\frac{1}{3}(\text{diff}'')^3$ trick to get term like (II). Looks like LHS.

$$+18\langle (u^N-u^M)'', (u^N-u^M)'(u^N-u^M+2u^M)''\rangle \quad \text{(III)}$$

(write sum as difference plus lower-reg parts; $\langle (u^N-u^M)'', (u^N-u^M)'(u^N-u^M)''\rangle$, L^1, L^∞ controlled by H^1; 3 derivatives fall on u^N-u^M, deal w/ like (I))

$$+6\langle (u^N-u^M)'', (u^N-u^M)(u^N-u^M+2u^M)'''\rangle \quad \text{(IV)}$$

$$\lesssim_{(I),(II)} \|(u^N-u^M)''\|_{L^2}^2 \left[\|u^N\|_{H^2}+\|u^M\|_{H^2}\right]^2$$

(actually stronger bd $\|(u^N)'\|_{L^\infty}+\|(u^M)'\|_{L^\infty}$, which is baddy)

$$+\|(u^N-u^M)''\|_{L^2}\|(u^M)''\|_{L^2}\|(u^N-u^M)'\|_{L^\infty}$$ ← Hölder

$$+\|(u^N-u^M)''\|_{L^2}\|(u^M)'''\|_{L^2}\|u^N-u^M\|_{L^\infty}$$ ← Hölder

(more derivatives than the problem allows, Bernstein $\lesssim M\|u_0\|_{H^2}$. Need M to help, get neg powers of M to help fight; scary bc note taking $M\to\infty$!)

We use the following Gagliardo-Nirenberg type ineqs:

(recall pf: Bernstein, low/high freq regimes. LP pieces. L^∞ control by $\partial^{1/2}$. i.e. $\|P_N f\|_{L^\infty} \lesssim N^{1/2}\|f\|_{L^2}$, $\lesssim N^{1/2-2}\|f''\|_{L^2}$)

$$\|f\|_{L^\infty} \lesssim \|f\|_{L^2}^{\frac{3}{4}} \|f''\|_{L^2}^{\frac{1}{4}}$$ ← figure out its $\frac{1}{4}$ by scaling argument $f(\lambda x)$, LHS unchanged; $f(\lambda x)'' = \lambda^2 f''(\lambda x)$, $\int(\lambda^2 f''(\lambda x))^2 = \int \lambda^3 f''(u)\,du$ ($u=\lambda x$) $\rightsquigarrow \lambda^{3/2}$

so LHS has $\partial^{1/2}$. RHS has ∂^2 so need $\frac{1}{4}$ power.

$$\|f'\|_{L^\infty} \lesssim \|f\|_{L^2}^{1/4}\|f''\|_{L^2}^{3/4}$$

$\frac{2}{b}-\frac{1}{a}=1$, $a+b=1$. LHS has $1+\frac{1}{2}\partial$. RHS has 2, so power $\frac{3}{4}$ b/c $\frac{3}{2}=2\cdot\frac{3}{4}$.

$\int f(\lambda x)^2 \overset{u=\lambda x}{=} \lambda^{-1}\int f(u)^2 \rightsquigarrow \lambda^{-1/2}$ ← must be negated 3:1, so $\frac{3}{4}, \frac{1}{4}$.

in L^p, λ - always start w $-1/p$ + 1 for every derivative

Pf: $f=\sum_N f_N$ (sum converges to f), take f_N to be $P_N f$ (Littlewood-Paley projection), take f_N 1-1, telescope.

$$\|f\|_{L^\infty} \le \sum \|f_N\|_{L^\infty} \lesssim \sum \min\{N^{\frac{1}{2}}\|f\|_{L^2}, N^{\frac{1}{2}-2}\|f''\|_{L^2}\}$$ optimize and sum.

Bernstein: $P_N f \in L^q(\mathbb{R})$, $q\le p$, $\|f_N\|_{L^p} \lesssim N^{d(\frac{1}{q}-\frac{1}{p})}\|f_N\|_{L^q}$

$$\|f'\|_{L^\infty} \lesssim \sum N\|f_N\|_{L^\infty} \lesssim \sum \min\{N^{\frac{3}{2}}\|f\|_{L^2}, N^{-\frac{1}{2}}\|f''\|_{L^2}\}$$ — —

continuing

So, $\partial_t \|(u^N-u^M)''\|_{L^2}^2 \lesssim \|(u^N-u^M)''\|_{L^2}$ + $\|(u^N-u^M)''\|_{L^2}^{1+\frac{3}{4}}\|u^N-u^M\|_{L^2}^{\frac{1}{4}}$ + $M\|(u^N-u^M)''\|_{L^2}^{1+\frac{1}{4}}\|u^N-u^M\|_{L^2}^{3/4}$

($\|(u^N-u^M)''\|_{L^2} \lesssim M\|u_0\|_{H^2}$)

(other factors $\lesssim \|u^N(t)\|_{H^2}$ unif in N $\lesssim \|u_0(x)\|_{H^2}$)

"A" want $\|(u^N-u^M)\|_{L^2}^2$ for Gronwall. Need "AM-GM"

$\lesssim \|P_{M<\cdot\le N}u_0\|_{L^2}^{1/4}$ $\lesssim M^{-1/2}\|u_0\|_{H^2}$

$\lesssim \|P_{M<\cdot\le N}u_0\|_{L^2}^{3/4}$ $\lesssim M^{-3/2}$

Young ineq, $ab\le a^p+b^q$: A raised to $\frac{8}{7}$, so $\frac{7}{4}\cdot\frac{8}{7}=2$; B raised to dual of $\frac{8}{7}$, means $\frac{8}{1}$

again Young ineq. "A" raise to $\frac{8}{5}$, "B" raise to $\frac{8}{3}$.

Get $\partial_t\|(u^N-u^M)''\|_{L^2}^2 \lesssim M\|(u^N-u^M)''\|_{L^2}^2 + M^{-\frac{4}{3}}$ "$(M^{1-3/2})^{8/3}$"

correct final RHS. $\lesssim \|u_0\|_{H^2}\{\|(u^N-u^M)''\|_{L^2}^2 + M^{-4}\|u_0\|_{H^2}^2 + M^{-4/3}\|u_0\|_{H^2}^2\}$

$\Rightarrow$ Gronwall

$$\|(u^N - u^M)''\|_2^2 \lesssim \|(u_0^N - u_0^M)''\|_2^2 + C T_0 M^{-\frac{4}{3}} \lesssim \{\|(u_0^N - u_0^M)''\|_2^2 + M^{-4/3}\|u_0\|_{H^2}^2\} e^{Ct\|u_0\|_{H^2}}$$

$$\lesssim \|P_{>M} u_0\|_{H^2}^2 + C T_0 M^{-\frac{4}{3}}$$

$\downarrow$ DCT, $M \to \infty$

0

unif in $t \in [0, T_0]$

only "win" by DCT, not quantitatively; i.e. rate of this convergence depend on u_0, not on anything we can control.

Lec 14

4/24/26 summary so far.

We showed $\begin{cases} \partial_t u = -\partial_x^3 u + 3\partial_x u^2 \\ u(0) = u_0 \in H^\sigma \end{cases}$ $\sigma \geq 3$, $\sigma \in \mathbb{N}$ admits ! sol $u \in C_t H_x^\sigma([0,T_0] \times \mathbb{R})$ for $T_0 = T_0(\|u_0\|_{H^\sigma})$. Remains to show (DOS): cts dependence of sol on data

(DOS)

Finally, let's show cty of data $\mapsto$ sol map in H^s.

Let $u_{0,n} \xrightarrow[n\to\infty]{H^s} u_0$. WTS $u_n(t) \xrightarrow[n\to\infty]{H^s} u(t)$ unif in t.

Let $u_n \in C_t H_x^s([0,T_0] \times \mathbb{R})$ be unique sols to (KdV) w/ init data $u_n(0) = u_{0,n}$; $u(0) = u_0$

and $T_0 = T_0(2\|u_0\|_{H^s})$

eventually $\|u_{0,n}\|_{H^s} \leq 2\|u_0\|_{H^s}$ for n large enough,

so time interval uniform in n.

Again, LP theory.

For $N \in 2^{\mathbb{N}}$, u_n^N be ! sol to KdV initdata $u_n^N(0) = P_{\leq N} u_{0,n}$ (regularizating from before, sol to KdV w/ init data $u_{0,n}^N$)

u^N ——— $u^N(0) = P_{\leq N} u_0$

$$\|u_n(t) - u(t)\|_{H^s} \leq \underbrace{\|u_n^N(t) - u_n(t)\|_{H^s}}_{(I)} + \underbrace{\|u^N(t) - u(t)\|_{H^s}}_{(II)} + \underbrace{\|u_n^N(t) - u^N(t)\|_{H^s}}_{(III)}$$

(I) $:= \|u_n^N(t) - u_n(t)\|_{H^s} = \lim_{M\to\infty} \|u_n^N(t) - u_n^M(t)\|_{H^s} \lesssim \lim_{M\to\infty} \|u_n^N(0) - u^M(0)\|_{H^s} + N^{-c(s)}$

b/c $u_n(t) = H^s\text{-}\lim_{M\to\infty} u_n^M(t)$ — last lecture

We know for fixed n, $\to 0$ as $N \to \infty$. Danger: dep on n? Fix: prev lec, expl rate of conv

(II) we already proved $\to 0$ as $N \to \infty$

(III) $\to 0$ as $n \to \infty$

$u_n^N(0) - u^M(0)$: diff in initdata

$N^{-c(s)}$: some negative exponent depending on s; smaller of 2 things b/c $M \to \infty$.

$$\lesssim \|P_{>N} u_{0,n}\|_{H^s} + N^{-c(s)}$$

$$\lesssim \|P_{>N} u_0\|_{H^s} + \|u_{0,n} - u_0\|_{H^s} + N^{-c(s)}$$

again maybe problem of n, N "suff large" chasing each others' tails.

N large make this small. n large make this small

$\downarrow$ DCT $N \to \infty$

0

Similarly, (even easier!):

$$(\mathrm{II}) := \|u^N(t) - u(t)\|_{H^s} = \lim_{M\to\infty} \|u^N(t) - u^M(t)\|_{H^s}$$

$\therefore$ above steps

$$\lesssim \|P_{>N} u_0\|_{H^s} + N^{-(\sigma-s)}$$

$\downarrow$ DCT, $N\to\infty$

0

2026 alternative pfs:

idea 1: LP again $\|u_n^N - u^N\|_{\sigma} \le \|P_{\le M}(u_n^N - u^N)\|_{\sigma} + \|P_{>M}\|_{\sigma}$

Bernstein $\lesssim M^{\sigma}\|u_n^N - u^N\|_2 + M^{-1}\|u_n^N - u^N\|_{\sigma+1}$

$\lesssim N(\|u_{n,0}\|_{\sigma} + \|u_0\|_{\sigma})$

b/c $\|u^N\|_{\sigma+1} \lesssim N\|u^N\|_{\sigma}$

choose N large, $M = M(N)$ even larger, $n = n(M)$ even larger ✓

$$(\mathrm{III}) := \|u_n^N(t) - u^N(t)\|_{H^s}$$

idea 2:

$$\partial_t \|\partial_x^s(u_n^N(t) - u^N(t))\|_{H^s}^2 \lesssim \|\partial^s(u_n^N - u^N)\|_{L^2}^2 + \|\partial^s(u_n^N - u^N)\|_{L^2}\left[\|u_n^N\|_{H^{s+1}} + \|u^N\|_{H^{s+1}}\right]\|u_n^N - u^N\|_{\infty}$$

same long argument as above

worst possible term

$\lesssim N$ up to const of init data in H^s

$\lesssim \|u_n^N - u^N\|_2^{\frac{2s-1}{2s}} \cdot \|\partial^s(\cdots)\|_2^{\frac{1}{2s}}$

$\lesssim N \cdot \|P_{>N}(u_{0,n} - u_0)\|_2^{\frac{2s-1}{2}}$

$\lesssim N \cdot N^{-\frac{2s-1}{2}}$

$\|\partial^s(u_n^N - u^N)\|_2^{\frac{2s+1}{2s}}$

raise to $\frac{4s}{2s+1}$ → raise to $\frac{4s}{2s-1}$

Young ineq.

$$\lesssim N^{\frac{(3-2s)2s}{2s-1}}$$

($s=2$, agree w/ our $N^{-\frac{4}{3}}$ above!)

idea 2: interpolation.

$\|u_n^N - u^N\|_{\sigma} \lesssim \|\cdot\|_{\sigma+1}^{\frac{\sigma}{\sigma+1}} \|\cdot\|_2^{\frac{1}{\sigma+1}}$ — throw away proj

$\lesssim \left(\|u_n^N\|_{\sigma+1} + \|u^N\|_{\sigma+1}\right)^{\frac{\sigma}{\sigma+1}} \cdot \|u_{0,n} - u_0\|_2^{\frac{1}{\sigma+1}}$

$\lesssim \left(N\|u_0\|_{\sigma+1}\right)^{\frac{\sigma}{\sigma+1}} \|u_{0,n} - u_0\|_2^{\frac{1}{\sigma+1}}$

choose N large enough, then $n = n(N)$ larger. ✓

$$\lesssim \|\partial^s(u_n^N - u^N)\|_2^2 + N^{\frac{(3-2s)2s}{2s-1}}$$

$\le$ Gronwall $\Rightarrow$

$$(\mathrm{III}) := \|u_n^N(t) - u^N(t)\|_{H^s} \lesssim \underbrace{\|P_{\le N}(u_{0,n} - u_0)\|_{H^s}}_{\le \|u_{0,n} - u_0\|_H} + CT_0 N^{\frac{(3-2s)2s}{2s-1}} \xrightarrow[\text{then } n\to\infty]{N\to\infty} 0.$$

Note: above pf works for $\sigma > \frac{3}{2}$, just need embedding from L^{∞}

(we wrote for $\frac{3}{2} < \sigma \in \mathbb{N}$)

4/28/26 Abbreviations to keep pace w/ lecture

Airy kernel $e^{i\frac{\xi^3}{3}+i\xi x} = e^{\phi_x} = e^{\phi_x(\xi)}$ (Airy phase, function of ξ, x.)

Overdots for inverse $\dot{2} = \frac{1}{2}$ — maybe vertical line

absolute values also annoying.

Absolute gradient $|\nabla|$. Just ∇ ∇ ∇ ∇ ∇ ∇ ∇f ∇ ∇ ∇ ∇ ∇ $\nabla

Chapter III

(using ChpI) Kenig - Ponce - Vega
Osc Int method

2026 Lec 14
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2024 ⋮
Lec 14

Well posedness for KdV: The Oscillatory Integral Method

Kenig-Ponce-Vega

Consider $\begin{cases} u_t + u_{xxx} = 6uu_x \\ u(0) = u_0 \in H^s(\mathbb{R}), \end{cases}$ $\frac{3}{4} < s \le 1$ (suffices to treat fractional part b/c it >1, use ordinary product rule, reduce to studying fractional)

we constr norm encapsulating Strichartz est, max'l fn ests, & local smoothing ests of Airy propagator.

Let $\Phi(u)(t) := e^{-t\partial_x^3} u_0 + 3\int_0^t e^{-(t-\tau)\partial_x^3} (u^2)'(\tau)\, d\tau$.

will show Φ is contraction on Banach ball / cplt metric space

$$B := \left\{ u \in C_t^0 H_x^s([-T_0, T_0] \times \mathbb{R}) \ : \ \|u\|_{X([-T_0,T_0])} \le C_0 \|u_0\|_{H^s} \right\}$$

as before insist on cty in time

universal / absolute constant subsuming all consts

equipped w metric $d(u,v) := \|u - v\|_{X([-T_0,T_0])}$

provided $T_0 = T_0(\|u_0\|_{H^s}) \le 1$ small enough

(i.e. take min of what contraction mapping require, & 1).

C_0 is absolute const,

and

$$\|u\|_{X([-T_0,T_0])} := \underbrace{\|u\|_{L_t^\infty H_x^s(I\times\mathbb{R})}}_{\text{(Basic)}} + \|\,|\nabla|\, u\|_{L_t^4 L_x^\infty} + \|u\|_{L_x^2 L_t^\infty} + \|\langle\nabla\rangle^s \partial_x u\|_{L_x^\infty L_x^2}$$

- $\|u\|_{X([-T_0,T_0])}$: or general $I \subseteq \mathbb{R}$
- $L_t^\infty H_x^s(I\times\mathbb{R})$: "the very least needed"
- $\|\,|\nabla|\, u\|_{L_t^4 L_x^\infty}$: use lower bd of $\frac{3}{4} < s$ here, but need σ exactly. In 2026, had ∂_x instd. recall Strichartz has $\frac{1}{4}$ derivative, but here we have $s > \frac{3}{4}$, hence full derivative ∂_x. "Strichartz norm"
- $\|u\|_{L_x^2 L_t^\infty}$: In (Strich) term, can put $\partial_x^{s-\frac{1}{4}}$, everything works but we don't need it. We do need it on (Kato) term; need to control $\|\partial_x^2 u\|_{L_x^\infty}$ somehow. b/c Monica likes writing $u(t,x)$ not $u(x,t)$. max'l fn this is main limitation on regularity
- $\|\langle\nabla\rangle^s \partial_x u\|_{L_x^\infty L_x^2}$: "Kato smoothing norm"

here and below, all spacetime norms are on $[-T_0, T_0] \times \mathbb{R}$.

Show $\Phi : B \to B$ — 2 requirements: bd, & cty in time. Fix $u \in B$.

$\|Iu\|_{XI} \overset{\text{always implicitly } I=[-T_0,T_0]}{\le} \|a^t u_0\|_{XI} + \|\int_0^t a^{t-s} \partial_x u^2(s)\, ds\|_{XI}$

$\|e^{-t\partial_x^3} u\|_{L_t^\infty H_x^s} = \|u_0\|_{H_x^s}$ (Basic) — spatial interest first, at last "conservation law" $e^{-t\partial_x^3}$ disappears unimodular. all of spacetime, for now

(Strichartz norm) $\|\,|\nabla| e^{-t\partial_x^3} u_0\|_{L_t^4 L_x^\infty} = \|\,|\nabla|^{\frac{1}{4}} e^{-t\partial_x^3} |\nabla|^{\frac{3}{4}} u_0\|_{L_t^4 L_x^\infty}$

Fourier multipliers commutative

$\lesssim_{\text{Strich}} \|\,|\nabla|^{\frac{3}{4}} u_0\|_2 \lesssim \|u_0\|_{H_x^s}$ if $s \ge \frac{3}{4}$

$|a^t \partial u_0|_{4\infty\, tx}$ ↑ naked derivative, want to match $|\partial|^{1/4}$ in Strichartz

recall $\partial = -|\partial| \cdot H$ ← Hilbert transf

$i\xi = -|\xi| \cdot -i\,\mathrm{sgn}(\xi)$

$= |a^t \partial^{\frac{1}{4}} \partial^{\frac{3}{4}} H u_0|_{4\infty\, tx}$

$\lesssim_{\text{Strich}} |\partial^{\frac{3}{4}} H u_0|_2$

$\lesssim_{H \text{ bdd on } L^2} |u_0|_{\sigma x}$ for $\sigma > \frac{3}{4}$.

(Max'l Fn) first local in spacetime bd so far. b/c max'l fn bd gets worse as I grows.

$\|e^{-t\partial_x^3} u_0\|_{L_x^2 L_t^\infty(I\times\mathbb{R})} \lesssim_{\text{maximal fn inequality}} \langle T_0\rangle^{1/2} \|u_0\|_{H_x^s} \le \|u_0\|_{H_x^s}$ for $s > \frac{3}{4}$.

assumed $T_0 \le 1$

("Kato smoothing") $\| \langle\nabla\rangle^s \partial_x e^{-t\partial_x^3} u_0 \|_{H^s} \lesssim \| \langle\nabla\rangle^s u_0 \|_2 \not\lesssim \|u_0\|_{H^s}$.

Kato smoothing

we got all 4 (...) in Xnorm, ↓ abs const to eat up consts in above ests.

Sr outcome is free evolution $\| e^{-t\partial_x^3} u_0 \|_{X([-T_0,T_0])} \le C \|u_0\|_{H^s_x}$.

Continuity ($\exists: B \to B$, preserve bd) Turn to nonlinear term. Now, **all spacetime norms on $I \times \mathbb{R} := [-T_0, T_0] \times \mathbb{R}$.**

(Basic) $\| \int_0^t e^{-(t-\tau)\partial_x^3} (u^2)'(\tau)\, d\tau \|_{L_t^\infty H_x^s}$

$\lesssim$ Δ-ineq to move H^s norm inside, kill propagator, increase $\int_0^t \|\cdot\|$ to $\int_{-T_0}^{T_0} \|\cdot\|$ — in H^s, unimodular constant

$\lesssim \int_{-T_0}^{T_0} \| (u^2)'(\tau) \|_{H^s_x} d\tau$

$\lesssim$ (2,2) Hölder in τ $T_0^{1/2} \| (u^2)' \|_{L_t^2 H_x^s}$

we now bd all the X-terms of nonlinear term by this quantity which we bd at end

---

(Strich) $\| \int_0^t e^{-(t-\tau)\partial_x^3} |\nabla| (u^2)'(\tau)\, d\tau \|_{L_t^4 L_x^\infty}$

using strichartz, $\lesssim \| |\nabla|^{\frac{1}{4}} (uu') \|_{L_t^{4/3} L_x^1}$

Unfortunately, not make us happy b/c Hölder would have to put u or u' in L^2, and $\frac{1}{2}+1$ derivatives in L^2 is too many for us to handle.

Instead, **By duality,**

$\| \int_0^t e^{-(t-\tau)\partial_x} \underbrace{|\nabla|}_{\text{in 2026, } \partial} (u^2)'(\tau)\, d\tau \|_{L_t^4 L_x^\infty} = \sup_{\|G\|_{L_t^{4/3} L_x^1} \le 1} \left\langle \int_0^t e^{-(t-\tau)\partial_x^3} \underbrace{|\nabla|}_{\text{2026, } |\partial|\cdot H \text{ (Hilbert transf)}} (u^2)'(\tau)\, d\tau,\ G(t) \right\rangle_{t,x}$

$= \sup_G \int_{-T_0}^{T_0} \int_0^t \int\int_{0<\tau<t} \langle e^{\tau\partial_x^3} |\nabla|^{\frac{3}{4}} (u^2)'(\tau),\ e^{t\partial_x^3} |\nabla|^{\frac{1}{4}} G(t) \rangle_x \, dt\, d\tau$

Fubini $= \sup_G \left\langle \underbrace{e^{\tau\partial_x^2} |\nabla|^{\frac{3}{4}} (u^2)'(\tau)}_{L^2},\ \underbrace{\int e^{t\partial_x^3} |\nabla|^{\frac{1}{4}} \chi_{[\tau,\infty)}(t)\, G(t)\, dt}_{L^2 \text{ (2,2) Hölder}} \right\rangle_{\tau,x}$

H^s, in 2026 version

$|\nabla|^{3/4}$ controlled by H^s, $s \ge \frac{3}{4}$. and in H^s $e^{\tau\partial_x^3}$ goes away. (And in 2026, H bdd in L^2)

adjoint $\| |\nabla|^{\frac{1}{4}} e^{-t\partial_x^3} f \|_{L_t^4 L_x^\infty} \lesssim \|f\|_2$

$\Leftrightarrow$

$\| \int e^{t\partial_x^3} |\nabla|^{\frac{1}{4}} F(t)\, dt \|_2 \lesssim \|F\|_{L_t^{4/3} L_x^1}$

$\leq \sup_t \left\| \|(u^2)'(\tau)\|_{H^3_x} \right\| \cdot \left\| \chi_{[\tau,\infty)} G \right\|_{L^{4/3}_t L^1_x}$ $\Big\|_{L^1_\tau}$

$\lesssim \|(u^2)'\|_{L^1_t H^3_x}$ $\lesssim \|G\|_{L^{4/3}_t L^1_x} \lesssim 1$

with $\|G\|_{L^{4/3}_t L^1_x} \leq 1$

$\lesssim T_0^{1/2} \|(u^2)'\|_{L^2_t H^3_x}$ (end of Strich) II

CBS

Lec 15
5/2/26

$\left\| \int_0^t e^{-(t-\tau)\partial_x^3} (u^2)'(\tau)\, d\tau \right\|_{L^2_x L^\infty_t} \leq \left\| \int_0^t \left| e^{-(t-\tau)\partial_x^3} (u^2)'(\tau) \right| d\tau \right\|_{L^2_x L^\infty_t}$

(Max^L Fn) III

$\leq \left\| \int_{-T_0}^{T_0} \left| e^{-(t-\tau)\partial_x^3} (u^2)'(\tau) \right| d\tau \right\|_{L^2_x L^\infty_t}$ get rid of dep on t!

$\leq \int_{-T_0}^{T_0} \underbrace{\left\| e^{-(t-\tau)\partial_x^3} (u^2)'(\tau) \right\|_{L^2_x L^\infty_t}}_{\text{maximal fn ineq to free evolution}} d\tau \lesssim \left\| e^{\tau\partial_x^3} (u^2)'(\tau) \right\|_{H^3_x} \underbrace{\langle T_0 \rangle^{1/2}}_{\leq 1}$

(again invisible to H^3_x)

$\lesssim T_0^{1/2} \|(u^2)'\|_{L^2_t H^3_x}$.

again CBS in time

Last term.

Similarly to rid of dep on t

(Kato) IV

$\left\| \int_0^t e^{-(t-\tau)\partial_x^3} \partial_x \langle\nabla\rangle^3 (u^2)'(\tau)\, d\tau \right\|_{L^\infty_x L^2_t} \leq \int_{-T_0}^{T_0} \underbrace{\left\| e^{-(t-\tau)\partial_x^3} \partial_x \langle\nabla\rangle^3 (u^2)'(\tau) \right\|_{L^\infty_x L^2_t}}_{\text{"Airy flow gain 1 deriv"}} d\tau$

Kato smoothing.

$\lesssim \int_{-T_0}^{T_0} \left\| e^{\tau\partial_x^3} \langle\nabla\rangle^3 (u^2)'(\tau) \right\|_2 d\tau$

$\lesssim T_0^{1/2} \|(u^2)'\|_{L^2_t H^3_x}$

CBS

So we have shown

$3 \left\| \int_0^t e^{-(t-\tau)\partial_x^3} (u^2)'(\tau)\, d\tau \right\|_{X([-T_0,T_0])} \leq C T_0^{1/2} \|uu'\|_{L^2_t H^3_x}$

thus $\Rightarrow \|\Phi u\|_{X_{I_0}([-T_0,T_0])} \lesssim \|u_0\|_{H^3_\sigma} + T_0^{1/2} \|uu'\|_{L^2_t H^3_x}$

If we can show $\|uu'\|_{L^2_t H^3_x} \lesssim C_0^2 \|u_0\|^2_{H^3_x}$, ($\lesssim \|u\|^2_{X_I}$ b/c $u \in B$; we'll actually show)

then we accomplish $\|\Phi(u)\|_{X([-T_0,T_0])} \lesssim C\|u_0\|_{H^3} + C T_0^{1/2} \|u_0\|^2_{H^3}$

$\leq C_0 \|u_0\|_{H^3}$ provided $C_0 T_0^{1/2} \|u_0\|_{H^3}^1 \leq 1$ and $T_0 \leq 1$.

($C_0 = 2C_1$ to absorb everything else?)

Let's look closer: $\|uu'\|_{L^2_{t,x}} \lesssim \|uu'\|_{L^2_{t,x}} + \|\partial_x^3 (uu')\|_{L^1_{t,x}}$

recall $u \in B_I$, Strichartz norm can control 1 derivative

$\|uu'\|_{L^2_{t,x}} \lesssim \|u'\|_{L^4_t L^\infty_x} \cdot \|u\|_{L^\infty_t L^2_x} \cdot T_0^{1/4}$ (remaining time)

Hölder

$\lesssim \|u'\|_{L^4_t L^\infty_x}$ (Strich) b/c control 1 2 $\cdot \|u\|_{L^4_t L^2_x}$

Hölder the time

using $T_0 \leq 1$

$\lesssim \|u\|^2_{X([-T_0,T_0])} \lesssim \|u_0\|^2_{H^3_\sigma}$

must exhaust all of space available since only control u in L^2

All impl consts in Harmonic Analysis Bds we used in I..III

$C_0 \geq 1$, i.e. $T_0 \lesssim \|u_0\|_{H^3_\sigma}^{-2}$

$\|$ s derivative of $uu'\|_{L^2_{t,x}}$

If $s=1$, (so don't worry about fractional product rule)

$$\|(uu')'\|_{L^2_{t,x}} \le \|uu''\|_{L^2_{t,x}} + \|u'\cdot u'\|_{L^2_{t,x}}$$

$$\lesssim \|u''\|_{L^\infty_x L^2_t}\,\|u\|_{L^2_x L^\infty_t} + \|u'\|_{L^q_t L^\infty_x}\,\|u'\|_{L^\infty_t L^2_x}\,T_0^{1/q}$$

"maximal fn estimate in space" — Sobolev — on before

?

$$\lesssim \|u\|^2_{X([-T_0,T_0])}$$

$\|\partial^s uu'\|_{L^2_{t,x}} \lesssim \|\partial^s uu'\|_{L^2_{t,x}} \lesssim \|u'u'\|_{L^2_{t,x}} + \|uu''\|_{L^2_{t,x}}$ (bdd orthog transf)

$\lesssim$ Hölder $\|u'\|_{L^\infty_t L^2_x}\|u'\|_{L^2_t L^\infty_x} + \|u''\|_{L^2_t L^\infty_x}\cdot\|u\|_{L^\infty_t L^2_x}$ (can switch ordering of $\int$)

$\lesssim$ Hölder $\|u\|_{X^1_T}\,T_0^{1/4}\,\|u\|_{L^\infty_t H^1_x} + \|u\|^2_{X^1_T}$

$\lesssim \|u\|^2_{X^1_T}$

time-spt Hölder but space not flexible.

"smoothing effect if you reverse time & space" saw extra "max'l fn" contribution and they then proved ↑ to control.

try $\|\partial^s(uu')\|^2_{L^2_{t,x}} \lesssim \|\partial^s u\|^2_{L^2_x}\|u'\|^2_{L^\infty_x} + \|\partial^s u'\|^2_{L^p_x}\|u\|^2_{L^q_x}\|_{L^2_t}$ problem! To use Kato smoothing, need to average in time first, not after

For $\frac{3}{4}<s<1$, we'll show (the result actually holds for $0<s<1$)

"pretend that s derivative either hit f, or g"

bound as if derivatives fell on either guy.

$$\||\nabla|^s(fg) - f|\nabla|^s g\|_2 \lesssim \||\nabla|^s f\|_2\,\|g\|_\infty$$

"peel away most problematic term"

"Monica believes the paper in which this is isolated, Ponce-Kato actually did it wrong"

Choosing $f:=u$, $g:=u'$, get

$$\||\nabla|^s(uu')\|_{L^2_{t,x}} \le \||\nabla|^s(uu') - u|\nabla|^s u'\|_{L^2_{t,x}} + \|u|\nabla|^s u'\|_{L^2_{t,x}}$$

"take nasty term away" . . . "put it back"

Kato smoothing kills 1 more derivative

crucial that same exp in t,x to Tonelli!

$\le \||\nabla|^s u\|_{L^\infty_t L^2_x}\,\|u'\|_{L^q_t L^\infty_x}\,T_0^{1/q}$ (again "maximal in space")

$\lesssim \|u\|^2_{X([-T_0,T_0])}$ ✓ "all things balanced"

$\lesssim \||\nabla|^s u'\|_{L^\infty_t L^2_x}\cdot\|u\|_{L^2_t L^\infty_x}$

recall 2026 had ∂^s instead of $|\nabla|^s$ Oops in 2026 there is issue. Proposed fix: change Kato-Ponce blackbox above replace $|\nabla|$ w/ $\langle\nabla\rangle$

To show $\Phi(u)\in B$, it remains to show $\Phi(u)$ cts in time.

For $-T_0 < t' < t < T_0$,

$$\|\Phi(u)(t) - \Phi(u)(t')\|_{H^s_x} \le \|[e^{-(t-t')\partial_x^3} - 1]\,u_0\|_{H^s_x} \xrightarrow[|t-t'|\to 0]{\text{DCT}} 0$$

(Fourier side; linear terms)

$$+ 3\left\|\int_0^t [e^{-(t-t')\partial_x^3} - 1]\,e^{\tau\partial_x^3}(u^2)'(\tau)\,d\tau\right\|_{H^s_x} \lesssim T_0^{1/2}\,\|(u^2)'\|_{L^2_t H^s_x} \xrightarrow[|t-t'|\to 0]{\text{DCT}} 0.$$

operator strongly conv to 0

move abs val inside, move to $\int_{-T_0}^{T_0}$ . . .

$$+ 3\left\|\int_{t'}^t e^{-(t-\tau)\partial_x^3}(u^2)'(\tau)\,d\tau\right\|_{H^s_x} \lesssim |t-t'|^{1/2}\,\|(u^2)'\|_{L^2_t H^s_x} \xrightarrow{|t-t'|\to 0} 0.$$

Hölder

use small-ness ← "smallness" at interval, i.e. DCT ↝ $1_{[t',t]}$

$\lesssim \|u\|^2_{X^1_T} \lesssim \|u_0\|^2_{L^2}$

4/22/24

Remains to show Φ is contraction.

QED $\Phi: B \to B$.

Module Kato Ponce blackbox

WTS $\Phi(u)(t) = e^{t\partial_x^3} u_0 + 3\int_0^t e^{-(t-\tau)\partial_x^3} (u^2)'(\tau)\, d\tau$

is contraction on Banach space

$$B = \{ u \in C_t H^3_x : \|u\|_{X([-T_0,T_0])} \le C_0 \|u_0\|_{H^s} \}$$

where

$$\|u\|_{X([-T_0,T_0])} = \|u\|_{L^\infty H^s_x} + \||\nabla|^s u\|_{L^4_t L^\infty_x} + \|u\|_{L^3_x L^\infty_t} + \|\langle\nabla\rangle^s \partial_x u\|_{L^\infty_x L^2_t}$$

all spacetime norms on $[-T_0,T_0]\times\mathbb{R}$

and $\frac{3}{4} < s \le 1$, $T_0 = T_0(\|u_0\|_{H^s}) \le 1$.

Last time we showed

$$\|\Phi(u)\|_{X([-T_0,T_0])} \lesssim \|u_0\|_{H^s} + \underbrace{T_0^{1/2}}_{\text{source of smallness}} \|uu'\|_{L^2_t H^s_x}$$

will show "can pretend all derivatives either hit f or g"

$$\||\nabla|^s(fg) - f|\nabla|^s g\|_2 \lesssim \||\nabla|^s f\|_2 \|g\|_\infty.$$

using this, $\|uu'\|_{L^2_t H^s_x} \lesssim \|uu'\|_{L^2_{t,x}} + \|\,|\nabla|^s(uu') - u|\nabla|^s u'\|_{L^2_{t,x}} + \|u|\nabla|^s u'\|_{L^2_{t,x}}$

worst amount of derivatives

only tool to est derivatives is Kato smoothing but loses a $\frac{1}{2}$ derivative & switches time & space

$$\lesssim \|u'\|_{L^4_t L^\infty_x} \|u\|_{L^\infty_t L^2_x} \cdot T_0^{1/4} + \||\nabla|^s u\|_{L^2_{t,x}} \cdot \|u'\|_{L^4_t L^\infty_x} \cdot T_0^{1/4}$$

pretend $|\nabla|^s$ hit u instead

$$+ \|u\|_{L^2_x L^\infty_t} \||\nabla|^s u\|_{L^\infty_x L^2_t}$$

$$\lesssim_{(\text{using } T_0 \le 1)} \|u\|^2_{X([-T_0,T_0])} \lesssim_{(u \in B)} \|u_0\|^2_{H^s}$$

So, taking $T_0 \lesssim \min\{1, \|u_0\|_{H^s}^{-2}\}$ and $C_0 \gg 1$, we get $\|\Phi(u)\|_X \le C_0 \|u_0\|_{H^s}$.

Last time also showed cty in time, so indeed Φ maps B to B.

Φ contraction

for $u, v \in B$, may bound

$$\|\Phi(u) - \Phi(v)\|_X \lesssim \|(u^2 - v^2)'\|_{L^2_t H^s_x} \cdot \underbrace{T_0^{1/2}}_{\text{source of smallness}}$$

fractional product rule

$$\||\nabla|^s(fg)\|_2 \lesssim \||\nabla|^s f\|_2 \|g\|_\infty + \|f\|_\infty \||\nabla|^s g\|_2$$

important that $L^2_{t,x}$ same exponent so can switch order of integration (time $\leftrightarrow$ space) so first product rule not good enough ☹.

"either no derivatives, or s derivatives" → in which case can pull out in front ∂_x fall on one or other

$$\|\nabla[(u-v)(u+v)]'\|_{L_t^2 H_x^s} \lesssim \underbrace{\|(u-v)'(u+v)\|_{L_{t,x}^2}}_{(I)} + \underbrace{\|(u-v)(u+v)'\|_{L_{t,x}^2}}_{(II)} + \underbrace{\|\nabla^s((u-v)'(u+v)) - (u+v)\nabla^s(u-v)'\|_{L_{t,x}^2}}_{(III)}$$

$$+ \underbrace{\|(u+v)\nabla^s(u-v)'\|_{L_{t,x}^2}}_{(IV)} + \underbrace{\|\nabla^s((u-v)(u+v)') - \nabla^s(u+v)'\cdot(u-v)\|_{L_{t,x}^2}}_{(V)} + \underbrace{\|(u-v)\nabla^s(u+v)'\|_{L_{t,x}^2}}_{(VI)}$$

$$(I) \lesssim \|(u-v)'\|_{L_t^4 L_x^\infty} \|u+v\|_{L_t^\infty L_x^2} \quad T_0^{1/4}$$

$$(II) \lesssim \|(u+v)'\|_{L_t^4 L_x^\infty} \|u-v\|_{L_t^\infty L_x^2} \quad T_0^{1/4}$$

$$(III) \lesssim \|\nabla^s(u+v)\|_{L_t^\infty L_x^2} \cdot \|(u-v)'\|_{L_t^4 L_x^\infty} \quad T_0^{1/4}$$

Hölder-ize L^2 into L^4 and L^4

$$(IV) \lesssim \|\nabla^s(u-v)'\|_{L_x^\infty L_t^2} \|u+v\|_{L_x^2 L_t^\infty}$$

switch order! → to Kato smooth

$$(V) \lesssim \|\nabla^s(u-v)\|_{L_t^\infty L_x^2} \|(u+v)'\|_{L_t^4 L_x^\infty} \quad T_0^{1/4}$$

$$(VI) \lesssim \|\nabla^s(u+v)'\|_{L_x^\infty L_t^2} \|(u-v)\|_{L_x^2 L_t^\infty}$$

all of these $\lesssim \|u-v\|_{X([-T_1,T_0])} \left[\|u\|_X + \|v\|_X\right]$

$$\lesssim_{(u,v\in B)} \|u_0\|_{H^s} \|u-v\|_X$$

Choosing T_0 small enough that $\|u_0\|_{H^s} T_0^{1/2} < \frac{1}{2}$

$T_0 \lesssim \min\{1, \|u_0\|_{H^s}^{-2}\}$ we get

$$\|\Phi(u) - \Phi(v)\|_X \leq \tfrac{1}{2}\|u-v\|_X.$$

Lemma: for $0<s<1$,

$$\|\nabla^s(fb) - f\nabla^s b\|_2 \lesssim \|\nabla^s f\|_2 \|b\|_\infty.$$

Pf: begin as we began for fractional product rule in 2370...

Decompose $fb = \sum_N f_N b_{\ll N} + \sum_N f_{\ll N} b_N + \sum_N f_{\sim N} b_N$

so we need to understand/bound

$$|\nabla|^s(fb) - f|\nabla|^s b = \sum_N |\nabla|^s(f_N b_{\ll N}) - \sum_N f_N |\nabla|^s b_{\ll N} + \sum_N |\nabla|^s(f_{\ll N} b_N) - \sum_N f_{\ll N} |\nabla|^s b_N$$

$$+ \sum_N |\nabla|^s(f_{\sim N} b_N) - \sum_N f_{\sim N} |\nabla|^s b_N$$

long this term needs tools beyond 2D NS, invented for Carleson measures.

$$\Big\| \sum_N |\nabla|^s(f_N b_{\ll N}) \Big\|_2 \lesssim \Big\| \sqrt{\sum_N N^{2s} |\tilde{P}_N(f_N b_{\ll N})|^2} \Big\|_2$$

sq fn estimate, $N + (\ll N)$ is like tailored $\tilde{P}_N$

$|f_N| \lesssim [Mf]$ — f convolve at rapidly decayed fn, easily proven to be controlled by MAXimal fn.

$$\lesssim \Big\| \sqrt{\sum_N N^{2s} |M[f_N \cdot Mb]|^2} \Big\|_2$$

$= |M(N^s f_N \cdot Mb)|^2$ — vector valued maximal fn!

$$\lesssim \Big\| \sqrt{\sum_N N^{2s} |f_N|^2 |Mb|^2} \Big\|_2$$

$$\lesssim \underbrace{\|Mb\|_\infty}_{\lesssim \|b\|_\infty} \Big\| \underbrace{\sqrt{\sum_N N^{2s} |f_N|^2}}_{\sim \text{sq fn} \sim \||\nabla|^s f\|_2} \Big\|_2 \lesssim \|b\|_\infty \||\nabla|^s f\|_2.$$

$$\Big\| \sum_N f_N |\nabla|^s b_{\ll N} \Big\|_2^2 \lesssim \Big\| \sqrt{\sum_N \tilde{P}_N[f_N |\nabla|^s b_{\ll N}]^2} \Big\|_2^2$$

sq fn estimate

$\tilde{P}_N$ bdd by M, as before

$$\lesssim \Big\| \sqrt{\sum_N |M[f_N |\nabla|^s b_{\ll N}]|^2} \Big\|_2^2$$

$$\lesssim \Big\| \sqrt{\sum_N |f_N|^2 \, \||\nabla|^s b_{\ll N}\|^2} \Big\|_2^2 \sim \int \sum_N |f_N|^2 \, \||\nabla|^s b_{\ll N}\|^2 dx$$

$$\leq \sum_N \|f_N\|_2^2 \, \||\nabla|^s b_{\ll N}\|_\infty^2 \lesssim_{\text{Bernstein}} \sum_N N^{2s} \|f_N\|_2^2 \|b\|_\infty^2 \lesssim \|b\|_\infty^2 \||\nabla|^s f\|_2^2.$$

sq fn pull back together

$$\Big\| \sum_N |\nabla|^s(f_{\sim N} b_N) \Big\|_2^2 \lesssim_{\text{sq fn est.}} \Big\| \sqrt{\sum_K K^{2s} |P_K(\sum_N f_{\sim N} b_N)|^2} \Big\|^2$$

2 high freq can give low freq by destructive interference but 2 low can never get 2 high

$$\sim P_K\Big(\sum_{N \gtrsim K} f_{\sim N} b_N\Big)$$

P_K bdd by M,

$$\lesssim \Big\| \sqrt{\sum_K |M[K^s \sum_{N \gtrsim K} f_{\nu N} b_N]|^2} \Big\|_2^2$$

vector valued max inq

$$\lesssim \Big\| \sqrt{\sum_K K^{2s} \Big| \sum_{N \gtrsim K} f_{\nu N} b_N \Big|^2} \Big\|_2^2$$

$(\sum_{N_1})(\sum_{N_2})$, pay factor of 2, can assume $N_1 \gtrsim N_2$

$$\lesssim \int \sum_K K^{2s} \sum_{N_1 \gtrsim N_2 \gtrsim K} |f_{\nu N_1}| \cdot |b_{N_1}| \cdot |f_{\nu N_2}| \cdot |b_{N_2}|$$

(sum in k, geom series, $s>0$, gets to biggest value N_2.)

$$\lesssim \sum_{N_1 \gtrsim N_2} N_2^{2s} \|f_{\nu N_1}\|_2 \|f_{\nu N_2}\|_2 \|b_{N_1}\|_\infty \|b_{N_2}\|_\infty$$

L-P proj bdd in L^∞, $\lesssim \|b\|_\infty^2$

Bernstein:

$$\lesssim \|b\|_\infty^2 \sum_{N_1 \gtrsim N_2} N_2^s \cdot \frac{1}{N_1^s} \| |\nabla|^s f_{\nu N_1}\|_2 \cdot \| |\nabla|^s f_{\nu N_2}\|_2$$

$$\lesssim \|b\|_\infty^2 \Big(\sum_N \| |\nabla|^s f_{\nu N}\|_2^2 \Big)$$

Schur test

$$\lesssim \|b\|_\infty^2 \| |\nabla|^s f\|_2^2$$

sqfn est.

For orange pieces, must exploit cancellations, i.e. can't handle each orange term separately.

$$\Big\| \sum_N |\nabla|^s (f_{\ll N} b_N) - \sum_N f_{\ll N} |\nabla|^s b_N \Big\|_2 \lesssim \Big\| \sqrt{\sum_N |\tilde{P}_N [|\nabla|^s (f_{\ll N} b_N) - f_{\ll N} |\nabla|^s b_N]|^2} \Big\|_2$$

$\tilde{P}_N$ bdd by M, M is bdd

$$\lesssim \Big\| \sqrt{\sum_N |\,|\nabla|^s (f_{\ll N} b_N) - f_{\ll N} |\nabla|^s b_N|^2} \Big\|_2$$

So want to understand:

$$|\nabla|^s (f_{\ll N} b_N) - f_{\ll N} |\nabla|^s b_N$$

$$= |\nabla|^s \tilde{P}_N (f_{\ll N} b_N) - f_{\ll N} |\nabla|^s \tilde{P}_N b_N = [|\nabla|^s \tilde{P}_N, f_{\ll N} \cdot] b_N$$

think of as a commutator; $f_{\ll N}\cdot$ multiplication operator

$K(x,y)$ third kernel, i.e. thing that is convolve by to produce this operator

here it is just almost convolution

$$\Big\{ [|\nabla|^s \tilde{P}_N, f_{\ll N} \cdot] b_N \Big\}(x) = \int N^{s+1} \check{\varkappa}(N(x-y)) [f_{\ll N}(y) - f_{\ll N}(x)] b_N(y)$$

on Fourier side it multiplies by

$$|\xi|^s \tilde{\psi}\Big(\frac{\xi}{N}\Big) = N^s \underbrace{\Big(\frac{|\xi|^s}{N^s}\Big) \tilde{\psi}\Big(\frac{\xi}{N}\Big)}_{=: \varkappa(\frac{\xi}{N})}$$

4/24/24

Thm: for $0<s<1$, bdd

$$\| |\nabla|^s(fb) - f|\nabla|^s b\|_2 \lesssim \| |\nabla|^s f\|_2 \|b\|_\infty .$$

Recall: decompose:

$$|\nabla|^s(fb) - f|\nabla|^s b = \sum_N |\nabla|^s(f_N b_{<cN}) - \sum_N f_N |\nabla|^s b_{<cN} + \sum_N |\nabla|^s(f_{<cN} b_N) - \sum_N f_{<cN} |\nabla|^s b_N$$

$$+ \sum_N |\nabla|^s(f_{\sim N} b_N) - \boxed{\sum_N f_{\sim N} |\nabla|^s b_N}$$

only this term needs tools beyond 24A, invented for Carleson measures.

we showed red terms, 1st blue term acceptable, $\lesssim \| |\nabla|^s f\|_2 \cdot \|b\|_\infty$.

↑ using sq fn estimates, Bernstein, vector valued max'l fn estimates (and 1st blue term, Schur test)

For orange pieces, must exploit cancellations, i.e. can't handle each orange term separately.

$$\| \sum_N |\nabla|^s(f_{<cN} b_N) - \sum_N f_{<cN} |\nabla|^s b_N \|_2 \lesssim \| \sqrt{\sum_N |\tilde{P}_N[|\nabla|^s(f_{<cN} b_N) - f_{<cN}|\nabla|^s b_N]|^2} \|_2$$

($\tilde{P}_N$ bdd by M, M is bdd)

$$\lesssim \| \sqrt{\sum_N |\,|\nabla|^s(f_{<cN} b_N) - f_{<cN}|\nabla|^s b_N|^2} \|_2$$

$$|\nabla|^s(f_{<cN} b_N - f_{<cN}|\nabla|^s b_N) = |\nabla|^s \tilde{P}_N(f_{<cN} b_N) - f_{<cN}|\nabla|^s \tilde{P}_N b_N = [|\nabla|^s \tilde{P}_N, \underbrace{f_{<cN}\cdot}_{\text{mult. operator}}] b_N$$

on Fourier side,

$$\widehat{|\nabla|^s \tilde{P}_N g}(\xi) = |\xi|^s \tilde{\psi}(\tfrac{\xi}{N}) \hat{g}(\xi) = N^s \underbrace{\left[(\tfrac{|\xi|}{N})^s \tilde{\psi}(\tfrac{\xi}{N})\right]}_{=:\chi(\frac{\xi}{N})} \hat{g}(\xi)$$

χ cptly supp away from 0 like ψ.

$$\Rightarrow \widehat{|\nabla|^s \tilde{P}_N g} = N^{s+1} \underbrace{\check{\chi}(N\cdot)}_{\text{just another L-P projection to frequencies like } N} * g.$$

$[|\nabla|^s \tilde{P}_N, f_{<cN}\cdot] b_N$ (convolution operator; mult. operator)

So $\left| |\nabla|^s(f_{<cN} b_N) - f_{<cN}|\nabla|^s b_N\right|(x) = \left| \int N^{s+1} \underbrace{\check{\chi}(N(x-y))}_{\text{fun convolve}} \underbrace{f_{<cN}(y)}_{\text{multiply}} b_N(y)\, dy - \int N^{s+1} \underbrace{\check{\chi}(N(x-y))}_{\text{convolve}} b_N(y) \underbrace{f_{<cN}(x)}_{\text{then multiply}} dy \right|$

$$\lesssim \int \frac{N^{s+1}}{\langle N(x-y)\rangle^{10}} |b_N(y)| \cdot |f_{<cN}(x) - f_{<cN}(y)|\, dy$$

$$\lesssim \|b\|_\infty \sum_{K<cN} \int \frac{N^{s+1}}{\langle N(x-y)\rangle^{10}} |f_K(x) - f_K(y)|\, dy.$$

If $|x-y| \geq \frac{1}{K}$,

$$|f_K(x) - f_K(y)| \leq |f_K(x)| + |f_K(y)| \lesssim [Mf_K](x) + [Mf_K](y)$$

— Mf_K show up in next bound, and also M play well w/ ℓ^2 norms!

If $|x-y| < \frac{1}{K}$, want to exploit cancellations:

$$|f_K(x) - f_K(y)| = |\widetilde{P}_K[f_K(x) - f_K(y)]| = \left| \int K \check{\Psi}(Kz) \left[f_K(x-z) - f_K(y-z) \right] dz \right|$$

want to FTC, but first change var to put difference in $\check{\Psi}$ and do FTC there

$y-z = x-t$, $z = y-x+t$

$$= \left| \int f_K(x-z)\, K \left[\check{\Psi}(Kz) - \check{\Psi}(Kz - K(x-y)) \right] dz \right|$$

↓ FTC

$$\lesssim \int |f_K(x-z)| \cdot K \cdot \underbrace{K|x-y|}_{\text{difference of arguments}} \int_0^1 \left| \check{\Psi}'(Kz - \theta K(x-y)) \right| d\theta\, dz$$

$\check{\Psi}$ decay to any order, say 10,

$$\lesssim \frac{1}{\langle Kz - \theta K(x-y) \rangle^{10}} \sim \frac{1}{\langle Kz \rangle^{10}} \quad \text{for } |x-y| < \tfrac{1}{K}$$

$$\lesssim K|x-y| \int \frac{K}{\langle Kz \rangle^{10}} |f_K(x-z)|\, dz$$

$$\lesssim K|x-y| \, [Mf_K](x).$$

Combining both cases above:

So, $|f_K(x) - f_K(y)| \lesssim K|x-y| \left\{ [Mf_K](x) + [Mf_K](y) \right\}$.

So then continuing above

$$\left| |D|^s (f_{<cN} b_N) - f_{<cN} |D|^s b_N \right|(x) \lesssim \|b\|_\infty \sum_{K<cN} \int \frac{N^{s+1}}{\langle N(x-y) \rangle^{10}} |f_K(x) - f_K(y)|\, dy.$$

$$\lesssim \|b\|_\infty \sum_{K<cN} \int \underbrace{\frac{N^{s+1} K|x-y|}{\langle N(x-y) \rangle^{10}}}_{N^{s-1} \cdot K \cdot \frac{N|x-y|}{\langle N(x-y) \rangle^{10}}} \left\{ [Mf_K](x) + [Mf_K](y) \right\} dy$$

$\lesssim \frac{N}{\langle N(x-y) \rangle^9}$ — comes normalization factor for next step.

$$\lesssim \|b\|_\infty \sum_{K<cN} N^{s-1} \cdot K \underbrace{\left\{ [Mf_K](x) + [M[Mf_K]](x) \right\}}_{=: c_K}$$

So we are now at.

$$\left\| \sum_N |D|^s (f_{<cN} b_N) - f_{<cN} |D|^s b_N \right\|_2 \lesssim \left\| \left\| \sum_{K<cN} N^{s-1} K c_K \right\|_{\ell^2_N} \right\|_{L^2_x} \cdot \|b\|_\infty.$$

$\sum_N \left| \sum_{K \ll N} N^{s-1} K C_K \right|^2 \lesssim \sum_N \sum_{K \le L \ll N} N^{2(s-1)} K L^{s-1} C_K C_L$ (∞ usual, $\sum_K \sum_L$, pay factor of 2 to assume $K \le L$)

K, L entangled w/ C_K, C_L, and can sum in N b/c $s<1$!

$\lesssim \sum_{K \le L} L^{2s-1} K C_K C_L$

$\lesssim \sum_{K \le L} \underbrace{\left(\frac{K}{L}\right)^{1-s}}_{\text{good kernel: (low freq / high freq) positive}} \underbrace{K^s C_K \; L^s C_L}_{\text{correct normalization for maximal fns ??}}$

& Schur test

$\lesssim \sum_K K^{2s} C_K^2 .$

So

$$\left\| \left\| \sum_{K \ll N} N^{s-1} K C_K \right\|_{\ell^2_N} \right\|_{L^2_x} \lesssim \left\| \left\| K^s C_K \right\|_{\ell^2_K} \right\|_{L^2_x}$$

Now remember what C_K is:

$$\lesssim \left\| \left\| M[K^s f_K] \right\|_{\ell^2_K} \right\|_{L^2_x} + \left\| \left\| M[M[K^s f_K]] \right\|_{\ell^2_K} \right\|_{L^2_x}$$

(M bdd; one more application of vector-valued max'l fn yields something bdd by; one more application of vector-val)

$$\lesssim \left\| \sqrt{\sum_K K^{2s} |f_K|^2} \right\|_{L^2_x} \sim \| |\nabla|^s f \|_2$$

One last term: 2nd blue term.

$$\left\| \sum_N f_{\sim N} |\nabla|^s b_N \right\|_2 \lesssim \| |\nabla|^s f \|_2 \cdot \|b\|_{BMO}$$

Proceed by duality. For $g \in L^2$

$$\left\langle \sum_N f_{\sim N} |\nabla|^s b_N , g \right\rangle = \sum_N \left\langle \underbrace{P_{\lesssim N}}_{\text{self adj operator}} (f_{\sim N} |\nabla|^s b_N), g \right\rangle$$

$$\lesssim \sum_N \left\langle N^s f_{\sim N}, \frac{(|\nabla|^s b_N)}{N^s} \cdot g_{\lesssim N} \right\rangle$$

CBS in N, CBS in x:

$$\lesssim \underbrace{\left\| \sqrt{\sum_N N^{2s} |f_{\sim N}|^2} \right\|_2}_{\sim \| |\nabla|^s f \|_2} \cdot \left\| \sqrt{\sum_N \left| \frac{|\nabla|^s}{N^s} P_{\sim N} b \right|^2 |g_{\lesssim N}|^2} \right\|_2$$

essentially just again $P_{\sim N} b$ proj to freq like N, so will just relate as b_N.

also ignore squiggle $\lesssim$.

WTS $\int \sum_N |b_N|^2 |g_{\le N}|^2 \, dx \lesssim \|g\|_2^2 \|b\|_\infty^2$ (1)

low freq annoying b/c sum them over and over again, so need "b" terms to have good tail decay.

"weights" so b_N must have special property so that (1) possible but not contradict

cf. $\| \sqrt{\sum_N |g_{\le N}|^2} \|_2 \not\lesssim \|g\|_2$

The special property satisfied by b_N that makes (1) possible is:

$$\sup_{x_0 \in \mathbb{R}} \sup_N N \int_{|x-x_0| \le \frac{1}{N}} \sum_{k \ge N} |b_k|^2(x) \, dx \lesssim \|b\|_\infty^2 \quad (2)$$

avg.

$\frac{1}{N}$ about the size of an interval at one ?

Interlude. on Carleson measure.

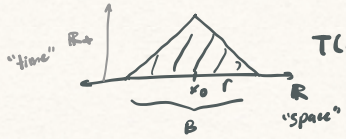
Given $d\mu$ a Borel msr on $\mathbb{R} \times \mathbb{R}_+$ (could be $\mathbb{R}^d \times \mathbb{R}_+$) we define

$$C(d\mu)(x_0) := \sup_{B \ni x_0} \frac{1}{|B|} \int_{T(B)} |d\mu|$$

↑ Carleson

where: sup taken over balls B containing x_0

and $T(B)$ = tent above B where if $B = B(x_0, r)$ then

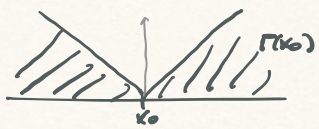


$T(B) = \{(x,t) : |x - x_0| \le r - t\}$.

we say $d\mu$ is Carleson msr if $\sup_{x_0 \in \mathbb{R}} C(d\mu)(x_0) < \infty$. (3)

Given $F : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{C}$, its nontangential maximal fn is

$$F^*(x_0) = \sup_{(x,t) \in \Gamma(x_0)} |F(x,t)| \quad \text{where } \Gamma(x_0) = \{(x,t) : |x - x_0| \le t\}$$



the relation is:

Thm: If $d\mu$ is Carleson msr and $F^* \in L^1(\mathbb{R})$, then

$$\left| \int_{\mathbb{R} \times \mathbb{R}_+} F(x,t) \, d\mu(x,t) \right| \lesssim \|F^*\|_{L^1(\mathbb{R})}$$

We think of $\{b_N\}$ as defining a msr $\int h(x,N) \, d\mu := \int \sum_N |b_N|^2 h(x,N) \, dx$ and $\frac{1}{N}$ as the "vertical" variable

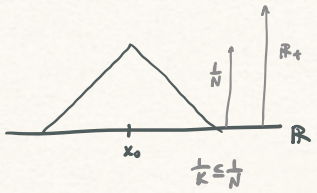
so that $\int |g_{\le N}|^2 d\mu = \int \sum_N |g_{\le N}|^2 |b_N|^2 dx$

the object we want to study.

Then (3) is similar to (2).

$$\sup_{x_0}\sup_{N}\frac{1}{|B(x_0,\frac{1}{N})|}\int_{T(B(x_0,\frac{1}{N}))}d\mu<\infty \quad\Longleftrightarrow\quad \sup_{x_0}\sup_{N} N\int_{|x-x_0|\le N}\sum_{k\ge N}|b_k|^2(x)\,dx<\infty.$$

$$\sup_{x_0}\sup_{N}\ \Updownarrow\ \sup_{B(x_0,\frac{1}{N})\ni x_0}\ \Updownarrow\ \sup_{B\ni x_0}$$



$\frac{1}{k}\le\frac{1}{N}$

4/26/24

final lecture on

Thm: for $0<s<1$, and

$$\| |\nabla|^s (fb) - f|\nabla|^s b\|_2 \lesssim \| |\nabla|^s f\|_2 \, \|b\|_\infty .$$

One last term — 2nd blue term.

$$\Big\| \sum_N f_{\sim N} |\nabla|^s b_N \Big\|_2 \lesssim \| |\nabla|^s f\|_2 \cdot \|b\|_\infty$$

(difficulty is our demand to put b in L^∞.)

If we didn't have to put b in L^∞, could very easily estimate:

CBS $\hookrightarrow$ $$\Big| \sum_N f_{\sim N} |\nabla|^s b_N \Big| \underset{CBS}{\lesssim} \sqrt{\sum_N N^{2s} |f_{\sim N}|^2} \sqrt{\sum_N \Big| \tfrac{|\nabla|^s}{N^s} P_N b \Big|^2}$$

$$\Big\| \sum_N f_{\sim N} |\nabla|^s b_N \Big\|_2 \underset{CBS}{\lesssim} \Big\| \sqrt{\sum_N N^{2s} |f_{\sim N}|^2} \Big\|_{L^p} \Big\| \sqrt{\sum_N \Big| \tfrac{|\nabla|^s}{N^s} P_N b \Big|^2} \Big\|_{L^q}$$

(sq fn est)

$$\lesssim \| |\nabla|^s f\|_p \, \|b\|_q$$

provided $\frac{1}{2} = \frac{1}{p} + \frac{1}{q}$ and $2 < p, q \le \infty$ ☹.

Pf. Last time: Arguing by duality, we said it sufficed to show

$$\int \sum_N |b_N|^2 \, |g_{\le N}|^2 \, dx \lesssim \|g\|_2^2 \, \|b\|_\infty^2 \qquad (1)$$

Write $\int \sum_N |b_N|^2(x) \, |g_{\le N}|^2(x) \, dx =: \int |g_{\le N}|^2(x) \, d\mu(x,N)$

(a nicer measure; b_N are nice)

Let's show b_N satisfy Carleson condition in the sense that

$$\sup_{x_0} \sup_N N \int_{|x - x_0| \le \frac{1}{N}} \sum_{K \ge N} |b_K|^2(x) \, dx \lesssim \|b\|_\infty^2 \qquad (2)$$

By symmetry (i.e. left/right shift b, doesn't change L^∞ norm), we may assume $x_0 = 0$.

We decompose $b = \underbrace{b 1_{\{|x| \le \frac{10}{N}\}}}_{b_1} + \underbrace{b 1_{\{|x| > \frac{10}{N}\}}}_{b_2}$

b_1:

$$N \int_{|x| \le \frac{1}{N}} \sum_{K \ge N} |P_K b_1|^2(x) \, dx \lesssim N \int \sum_K |P_K b_1|^2(x) \, dx \overset{\text{sq fn est}}{\lesssim} N \|b_1\|_2^2 \lesssim N \cdot \|b\|_\infty^2 \, N^{-1} = \|b\|_\infty .$$

$|P_K b_1|^2(x)$: blurring b_1 at scale $\frac{1}{K}$, still mostly supp on $|x| \le \frac{10}{N}$, so is "distracted"

b2: For $|x| \le \frac{1}{N}$, $k \ge N$,

$$|P_k b_2|(x) = \left| \int K \check{\psi}(K(x-y))\, b_2(y)\, dy \right| \lesssim \int_{|y| \ge \frac{10}{N}} \frac{K}{\langle K(x-y)\rangle^{10}} |b(y)|\, dy$$

$|y| \ge \frac{10}{N}$, $|x| \le \frac{1}{N}$ $\Rightarrow$ Japanese bracket can be replaced by abs val

$$\lesssim \|b\|_\infty \cdot \int_{|x-y| \ge \frac{9}{N}} \frac{K}{(K|x-y|)^{10}}\, dy$$

$$\frac{1}{K^9} \int_{|y| \ge \frac{9}{N}} \frac{1}{|y|^{10}}\, dy$$

$$\lesssim \|b\|_\infty \left(\frac{N}{K}\right)^9$$

So

$$N \int_{|x| \le \frac{1}{N}} \sum_{K \ge N} |P_k b_2|^2(x)\, dx \lesssim N \int_{|x| \le \frac{1}{N}} \sum_{K \ge N} \|b\|_\infty^2 \left(\frac{N}{K}\right)^{18} dx$$

$$\lesssim \|b\|_\infty.$$

sum geom series in K

We now use (2) to prove (1).

Let $\check{\varphi}$ denote kernel corresponding to L^1 proj to freq ≤ 1, $P_{\le 1}$.

Take slight majorization $\Phi(x) = \sup_{|y| \le 1} |\check{\varphi}(x-y)|$

(so $\Phi(x)$ is bigger than $\check{\varphi}$ on 1-ball around x).

Define the maximal fn "glorified proj to low freq"

$$[\mathcal{M} f](x) := \sup_N \int |f(y)|\, N\, \Phi(N(x-y))\, dy.$$

Clearly

$$|f_{\le N}| \le \mathcal{M} f \lesssim Mf$$

$\mathcal{M}$ b/c $\sup$ over N; LHS is one N.

M = H-L max^l fn

If fact

$$|f_{\leq N}|(x) \leq \int |f(y)| \cdot N\,|\check{\varphi}(N(x-y))|\,dy \leq \inf_{|\theta|\leq \frac{1}{N}} \int |f(y)|\,N\,|\check{\Phi}(N(x-\theta-y))|\,dy$$

all "smudging" of φ, since Φ majorizes 1-ball smudging of $\check{\varphi}$.

$$\leq \inf_{|\theta|\leq \frac{1}{N}} [\mathcal{M}f](x-\theta)$$

$$\int \sum_N |b_N|^2(x)\,|\mathbb{S}_{\leq N}|^2(x)\,dx = \int (g_{\leq N})^2(x)\,d\mu(x,N) = \int_0^\infty \mu\{(x,N) : (g_{\leq N})^2(x) > \lambda\}\,d\lambda$$

so if $|g_{\leq N}|^2(x) > \lambda \Rightarrow [\mathcal{M}g]^2(y) > \lambda \quad \forall\, |y-x| \leq \frac{1}{N}$

i.e. $y \in \overline{B(x,\frac{1}{N})}$, interval

so $\{(x,N) : |g_{\leq N}|^2(x) > \lambda\} \subseteq \{(x,N) : x \in \bigcup_j I_j \text{ s.t. } [\mathcal{M}g]^2(y) > \lambda \ \forall y \in \bigcup_j I_j \text{ and } |I_j| \geq \frac{2}{N}\}$

took maximal interval

thus $\int_0^\infty \mu\{(x,N) : (g_{\leq N})^2(x) > \lambda\}\,d\lambda \lesssim \int_0^\infty \sum_j \left(\int_{I_j} \sum_{k \geq \frac{2}{|I_j|}} |b_k|^2(x)\,dx \right) d\lambda$

i.e. integrate indicator of $\to d\mu$

cut off in "vertical direction"

integrating indicator of on (x,N) space

vertical

b/c $|I_j| \leq \frac{2}{N}$.

$$\lesssim_{(2)} \int_0^\infty \sum_j \left(|I_j| \cdot \|b\|_{\ell^2}^2 \right) d\lambda$$

using I_j disjoint

$$\lesssim \|b\|_{\ell^\infty}^2 \cdot \int_0^\infty |\{x : [\mathcal{M}g]^2(x) > \lambda\}|\,d\lambda$$

$$\lesssim \|\mathcal{M}g\|_2^2$$

$$\lesssim \|Mg\|_2^2$$

$$\lesssim \|g\|_2^2$$

so we have completed (1)

Chapter IV

Bourgain's $X^{s,b}$

We have thus completed the proof of well-posedness of KdV using oscillatory integral approach.

3/4/26
Lec 16

Well-posedness for KdV, the $X^{s,b}$ method

(sometimes called "Bourgain spaces")

pinnacle of marriage between harmonic analysis and PDE

Consider $\begin{cases} u_t + u_{xxx} = 6uu_x \\ u(0) = u_0 \in H^s(\mathbb{R}) \quad \text{for } 0 \le s \le 1. \end{cases}$

can go all the way down to $-\frac{3}{4}$, but we'll stop at 0

Later, Monica will present her own work going down to -1 anyway.

We denote by $X^{s,b}$ the Banach space of fns on $\mathbb{R}\times\mathbb{R}$ defined by norm

$$\|u\|_{X^{s,b}} := \| \langle \xi \rangle^s \langle \tau - \xi^3 \rangle^b \hat{u}(\tau,\xi) \|_{L^2_{\tau,\xi}}$$

twisted derivative

unlike before, $\hat{u}$ denote FT wrt. BOTH time & space

$$\hat{u}(\tau,\xi) = \frac{1}{2\pi} \iint e^{-it\tau - ix\xi} u(t,x)\, dx\, dt.$$

Some words: if u solved Airy eq $u_t + u_{xxx} = 0$,

Fourier transf in both t, x

$$i\tau \hat{u}(\tau,\xi) + (i\xi)^3 \hat{u}(\tau,\xi) = 0$$

then $\hat{u}(\tau,\xi)[\tau - \xi^3] = 0$

Explanation 1.

Tells us that when (τ,ξ) don't live on surface $\tau = \xi^3$, $\hat{u}$ must be 0

i.e. $\hat{u}$ is supported on $\{(\tau,\xi) : \tau = \xi^3\}$.

So twist $\langle \tau - \xi^3 \rangle^b$ penalizes τ, ξ off the surface.

"$X^{s,b}$ norm penalizes the failure of $\hat{u}$ to be supp on surface" - mandated by linear Airy "

But $X^{s,b}$ still perturbative approach, treat nonlinearity as perturbation of Airy eq.

Explanation 2: "interaction variables".

The Duhamel formulation of KdV is as we know

$$u(t) = e^{-t\partial_x^3} u_0 + 3\int_0^t e^{-(t-\tau)\partial_x^3} \partial_x(u^2)(\tau)\, d\tau$$

("in some space let's be vague")

Switch to interaction variables:

"bic twisted derivative becomes actual derivative in interaction var"

$$e^{t\partial_x^3} u(t) = u_0 + 3\int_0^t e^{\tau\partial_x^3} \partial_x(u^2)(\tau)\, d\tau$$

$f(t)$??

"pull sol u of KdV via Airy operator back down to $t=0$."

the interaction variable

$\Rightarrow$ eq becomes $f(t) = u_0 + 3\int_0^t e^{\tau\partial_x^3}\partial_x\left(\left[e^{-\tau\partial_x^3} f(\tau)\right]^2\right) d\tau.$

↓ on Fourier side, square $\rightsquigarrow$ convolution, $e^{-\tau\partial_x^3} \rightsquigarrow$ phase

Observe $\hat{f}(\tau,\xi) = \widehat{e^{t\partial_x^3} u(t)}\,(\tau,\xi)$

$= \frac{1}{\sqrt{2\pi}} \int e^{-it\tau} \mathcal{F}_x\left[e^{it\partial_x^3} u(t)\right](\xi)\, dt$

$= \frac{1}{\sqrt{2\pi}} \int e^{-it\tau - it\xi^3} \left[\mathcal{F}_x(u(t))\right](\xi)\, dt$

$= \hat{u}(\tau + \xi^3, \xi)$

Horrest derivatives of interaction variable versus $X^{s,b}$ norm.

$$\|f\|_{H_t^b H_x^s} = \|\langle\tau\rangle^b \langle\xi\rangle^s \hat{f}(\tau,\xi)\|_{L^2_{\tau,\xi}} = \|\langle\tau-\xi^3\rangle^b \langle\xi\rangle^s \hat{u}(t,\xi)\|_{L^2_{\tau,\xi}}.$$

in 2026 did all calc here: $= \left\| \langle\xi\rangle^s e^{-it\xi^3} (\mathcal{F}_x u)(t,\xi) \right\|_{H_t^b L_\xi^2} = \frac{1}{\sqrt{2\pi}} \left\| \langle\tau\rangle^b \langle\xi\rangle^s \int e^{-it\tau} e^{-it\xi^3} (\mathcal{F}_x u)(t,\xi)\, dt \right\|_{L^2_{\tau,\xi}} = \left\| \langle\tau\rangle^b \langle\tau\rangle^3 \hat{u}(\tau+\xi^3,\xi) \right\|_{L^2_{\tau,\xi}}$

For $T_0 > 0$, we define the restricted $X^{s,b}$ norm via

$$\|u\|_{X^{s,b}_{T_0}} := \inf\left\{ \|v\|_{X^{s,b}} : v \equiv u \text{ on } |t| \le T_0 \right\}$$

all possible extensions of u outside interval.

We'll construct solutions via contraction mapping w/

$$\Phi[u](t) = \eta(t)\, e^{-t\partial_x^3} u_0 + 3\eta(t) \int_0^t e^{-(t-\tau)\partial_x^3} \partial_x(u^2)(\tau)\, d\tau$$

↑ unlike before, need smooth truncation (H_t^b, $b > \frac{1}{2}$ to embed into cts-in-time)

sensitive to such subtleties?

where $\eta \in C_c^\infty([-2,2])$, $\eta \equiv 1$ on $[-1,1]$

we'll take $T_0 = T_0(\|u_0\|_{H^s}) \le 1$.

implied fixed pt of Φ satisfies Duhamel fmla on $[-T_0, T_0]$

≤ 1 small enough

The Banach ~~space~~ for contraction will be:

ball/complete metric space

≥ 1 is large abs const and $T_0 = T_0(\|u_0\|_{H^s_x})$

$$B := \left\{ u \in X^{s,b}_{T_0} : \|u\|_{X^{s,b}_{T_0}} \le C_0 \|u_0\|_{H^s} \right\}.$$

(note did not write cty in time.)

we will prove that $B \hookrightarrow$ cts in time fns

lemma immediately below

w/ $0 \le s \le 1$ and $\frac{1}{2} < b < \frac{3}{4}$ fixed.

→ to embed into cts fn. sol that are cts in time!

$H_t^{\frac{1}{2}+} \hookrightarrow C_t$.

4/29/24 10:40 Either cases about $X^{s,b}$ space "taking" being away from $\tau = \xi^3$ surface.
Monica saying $X^{s,b}$ norm "engineered" to suit the problem.

$$\|u\|_{X^{s,b}} := \| \langle \xi \rangle^s \langle \tau - \xi^3 \rangle^b \hat{u}(\tau,\xi) \|_{L^2_{\tau,\xi}} = \| e^{t\partial_x^3} u(t) \|_{H^b_t H^s_x}$$

"regular sobolev norm in interaction variable"

$$\|u\|_{X^{s,b}_{T_0}} := \inf \{ \|v\|_{X^{s,b}} : v \equiv u \text{ on } [-T_0, T_0] \}$$

fixed pt. via contraction mapping — will be a Duhamel sol to KdV on $[-T_0,T_0]$ in $X^{s,b}_{T_0}$ norm

We'll show that $\Phi[u](t) = \eta(t) e^{-t\partial_x^3} u_0 + 3\eta(t) \int_0^t e^{-(t-\tau)\partial_x^3} \partial_x (u^2)(\tau)\, d\tau$

where $\eta \in C_c^\infty([-2,2])$, $\eta \equiv 1$ on $[-1,1]$

we'll take $T_0 = T_0(\|u_0\|_{H^s}) \le 1$. (so $[-T_0,T_0] \subseteq [-1,1]$, where $\eta \equiv 1$)

The Banach space for contraction will be:

$$B := \{ u \in X^{s,b}_{T_0} : \|u\|_{X^{s,b}_{T_0}} \le C_0 \|u_0\|_{H^s} \}$$

for some large enough $C_0 \ge 1$

$T_0 = T_0(\|u_0\|_{H^s})$ small enough

(Quote did not write cty in time. We will prove that $B \hookrightarrow$ cts in time fns.)

w/ $0 \le s \le 1$ and $\frac{1}{2} < b < \frac{3}{4}$ fixed.

Lemma: For $b > \frac{1}{2}$, $X^{s,b}_{T_0} \hookrightarrow C_t H^s_x([-T_0,T_0] \times \mathbb{R})$

cty is only reason we need $b > \frac{1}{2}$

"whatever solution we get will be cts in time"

Pf: hard analysis: bound.
Soft analysis to prove cty.

First show: $\|u\|_{L^\infty_t H^s_x([-T_0,T_0]\times\mathbb{R})} \lesssim \|u\|_{X^{s,b}_{T_0}}$ (1)

By def of restricted $X^{s,b}_{T_0}$ norm,
Let $v \in X^{s,b}$ w/ $v \equiv u$ on $|t| \le T_0$ (we'll pass to inf at end)

For $|t| \le T_0$,

$$\|u(t)\|_{H^s_x} = \|v(t)\|_{H^s_x} = \|e^{t\partial_x^3} v(t)\|_{H^s_x} = \| \langle \xi \rangle^s e^{-it\xi^3} [\mathcal{F}_x v](t,\xi) \|_{L^2_\xi}$$

$= u$ on $|t| \le T_0$; inf; can pass to interaction variable for free b/c Airy propg is unitary

$$= \frac{1}{\sqrt{2\pi}} \| \langle \xi \rangle^s \int e^{it\xi^3} \hat{v}(\tau,\xi)\, d\tau \|_{L^2_\xi}$$

$\mathcal{F}_x$ FT wrt. both time & space

FIF $f(t) = \frac{1}{\sqrt{2\pi}} \int e^{it\tau} \hat{f}(\tau)\, d\tau$ apply to $f(t) := e^{-it\xi^3} [\mathcal{F}_x v](t,\xi)$

$$= \frac{1}{\sqrt{2\pi}} \left\| \langle \xi \rangle^s \int e^{it\xi^3} \hat{v}(\tau+\xi^3, \xi)\, d\tau \right\|_{L^2_\xi}$$

$\tau \to \tau + \xi^3$ — to simplify phase, no osc in ξ var

$$\lesssim \int \left\| \langle \xi \rangle^s \hat{v}(\tau+\xi^3, \xi) \right\|_{L^2_\xi} d\tau$$

"Δ-ineq"

$$\lesssim \left\| \langle \xi \rangle^s \langle \tau \rangle^b \hat{v}(\tau+\xi^3, \xi) \right\|_{L^2_{\tau,\xi}} \left\| \langle \tau \rangle^{-b} \right\|_{L^2_\tau}$$

$\lesssim 1$ for $b > \frac{1}{2}$.

bring up $X^{s,b}$ norm w/ CBS

also how we prove $H^{\frac{1}{2}+} \hookrightarrow C$ (cts)

$$\lesssim \|v\|_{X^{s,b}} \quad \text{if } b > \tfrac{1}{2}.$$

variable change

t disappeared when Δ-ineq

Taking infimum over such v we get $\|u(t)\|_{H^s_x} \lesssim \|u\|_{X^{s,b}_{T_0}}$ unif in $|t| \le T_0$.

If remains to prove cty in t. using our quantitative est.

(Idea 2: (Sahan) Schwartz $u_n \xrightarrow{X^{s,b}_{T_0}} u$ $\|u_n - u_m\|_{L^\infty_t H^s_x} \overset{(\star)}{\lesssim} \|u_n - u_m\|_{X^{s,b}}$)

For $t' < t$ in $[-T_0, T_0]$, $-T_0 \le t' < t \le T_0$

$$\|u(t) - u(t')\|_{H^s_x} = \|v(t) - v(t')\|_{H^s_x} \le \left\| e^{t\partial_x^3}[v(t) - v(t')] \right\|_{H^s_x}$$

again v agrees w/ u on $[-T_0, T_0]$

Δ-ineq

$$\le \left\| e^{t\partial_x^3} v(t) - e^{t'\partial_x^3} v(t') \right\|_{H^s_x} + \left\| \left(e^{t\partial_x^3} - e^{t'\partial_x^3}\right) v(t') \right\|_{H^s_x}$$

inf

move to interaction viewpoint as before, and Δ-ineq

(I) (II)

(I): on Fourier side, and inverse FT

$$\left\| e^{t\partial_x^3} v(t) - e^{t'\partial_x^3} v(t') \right\|_{H^s_x} = \frac{1}{\sqrt{2\pi}} \left\| \langle \xi \rangle^s \left[\int e^{it\tau - it\xi^3} \hat{v}(\tau,\xi)\, d\tau - \int e^{it'\tau - it'\xi^3} \hat{v}(\tau,\xi)\, d\tau \right] \right\|_{L^2_\xi}$$

same computations as before.

$$= \frac{1}{\sqrt{2\pi}} \left\| \langle \xi \rangle^s \int \left[e^{it\tau} - e^{it'\tau} \right] \hat{v}(\tau + \xi^3, \xi)\, d\tau \right\|_{L^2_\xi}$$

$\tau \to \tau + \xi^3$ var change again in order to kill phase(s)

Δ-ineq to bring L^2_ξ inside

$$\lesssim \left\| \langle \xi \rangle^s \langle \tau \rangle^b \left| e^{it\tau} - e^{it'\tau} \right| \hat{v}(\tau + \xi^3, \xi) \right\|_{L^2_{\tau,\xi}} \left\| \langle \tau \rangle^{-b} \right\|_{L^2_\tau}$$

again sharp $\langle \tau \rangle^b$ and CBS

bdd by $X^{s,b}$ norm, i.e. $\lesssim \|v\|_{X^{s,b}}$

$\lesssim 1$ if $b > \frac{1}{2}$

Go DCT applies and $|e^{it\tau} - e^{it'\tau}|$ sinks it to 0.

(II)

$$\left\| \left(e^{t\partial_x^3} - e^{t'\partial_x^3}\right) v(t') \right\|_{H^s_x} = \left\| \left[e^{(t-t')\partial_x^3} - 1\right] \underbrace{e^{t'\partial_x^3} v(t')}_{\text{in } H^s_x \lesssim \|v\|_{X^{s,b}}} \right\|_{H^s_x} \xrightarrow[t' \to t]{\text{DCT}} 0$$

on Fourier side

$$\lesssim \left\| \langle \xi \rangle^s \int \left[e^{it\tau - it\xi^3} - e^{it'\tau - it'\xi^3} \right] \hat{v}(\tau,\xi)\, d\tau \right\|_{L^2_\xi}$$

$\tau \to \tau + \xi^3$

only eliminate one term + usual CBS, etc.

$$\lesssim \left\| \langle \xi \rangle \langle \tau \rangle^b \left| e^{-i(t-t')\xi^3} - 1 \right| \hat{v}(\tau+\xi^3, \xi) \right\|_{L^2_{\tau,\xi}} \cdot \underbrace{\left\| \langle \tau \rangle^{-b} \right\|_{L^2_\tau}}_{\lesssim 1 \text{ for } b > 1} \longrightarrow 0.$$

again as before (DCT as $|t'-t| \to 0$)

Proving $\Phi : B \to B$ and is contraction.

$\|\Xi u\|_{X^{s,b}_{T_0}} \le \|\eta(t) e^{-t\partial_x^3} u_0\|_{X^{s,b}_{T_0}} + 3\|\eta(t)\int_0^t e^{-(t-t')\partial_x^3}\partial_x(u^2)(t')\,dt'\|_{X^{s,b}_{T_0}}$

The linear term in $\Phi(u)$:

b/c $\eta(t)e^{\cdots}u_0$ is extension of to all spacetime.

$$\|\eta(t)e^{-t\partial_x^3}u_0\|_{X^{s,b}_{T_0}} \le \inf \|\eta(t)e^{-t\partial_x^3}u_0\|_{X^{s,b}}$$

infimum over all exts, but it itself is such an ext.

move to interaction variable to go to std $H^b_t H^s_x$ norm

$$= \|e^{t\partial_x^3}\eta(t)e^{-t\partial_x^3}u_0\|_{H^b_t H^s_x}$$

fn of time only so commutes

product that has decoupled space & time!

$$= \|\eta\|_{H^b_t}\|u_0\|_{H^s_x} \lesssim \|u_0\|_{H^s_x}.$$

that's why we need smooth cutoff η, not sharp!

The nonlinear term in $\Phi(u)$:

Lemma. For $b > \frac{1}{2}$, $\left\|\eta(t)\int_0^t e^{-(t-\tau)\partial_x^2} F(\tau)\,d\tau\right\|_{X^{s,b}_{T_0}} \lesssim \|F\|_{X^{s,b-1}_{T_0}}$

"just like FTC where $\int_0^t$ gives one degree of smoothness, same is true even in $X^{s,b}_{T_0}$ norm"

"smoothing effect; if F any $b-1$ level of regular, then integrate, gain 1 level of regularity."

Pf. w/ usual starting w/ restr. norms, consider extension $\tilde F \equiv F$ on $|t| \le T_0$.

again is extension of

"doubling trick" - Ettas!

$$\left\|\eta(t)\int_0^t e^{-(t-\tau)\partial_x^3}F(\tau)\,d\tau\right\|_{X^{s,b}_{T_0}} \le \inf \left\|\eta(t)\int_0^t e^{-(t-\tau)\partial_x^3}\tilde F(\tau)\,d\tau\right\|_{X^{s,b}}$$

pass to interaction var

$$= \left\|e^{t\partial_x^3}\,\eta(t)\int_0^t e^{-(t-\tau)\partial_x^3}\tilde F(\tau)\,d\tau\right\|_{H^b_t H^s_x}$$

fn of time only, so commute

$$= \left\|\eta(t)\int_0^t e^{\tau\partial_x^3}\tilde F(\tau)\,d\tau\right\|_{H^b_t H^s_x}$$

doesn't matter order of $\int$, but correct order is

$$\lesssim \left\|\eta(t)\int_0^t \|e^{\tau\partial_x^3}\tilde F(\tau)\|_{H^s_x}\,d\tau\right\|_{H^b_t}$$

b/c we're going to use the FTC intuition "$\int_0^t$ is 1 higher degree of reg"

So if we can show $\left\|\eta(t)\int_0^t f(\tau)\,d\tau\right\|_{H^b_t} \lesssim \|f\|_{H^{b-1}_t}$ (2)

then would get $\left\|\eta(t)\int_0^t e^{-(t-\tau)\partial_x^2}F(\tau)\,d\tau\right\|_{X^{s,b}_{T_0}} \lesssim \|e^{\tau\partial_x^2}\tilde F(\tau)\|_{H^{b-1}_t H^s_x} = \|\tilde F\|_{X^{s,b-1}}$

take inf over exts $\tilde F$ of F

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Lec 17

"Must bring FT into game, how else work w/ H^{b-1}_t?"

So, boil down to proving (2).

$$= \frac{1}{\sqrt{2\pi}}\int_0^t\int e^{i\omega t'}\hat f(\omega)\,d\omega\,dt' = \frac{1}{\sqrt{2\pi}}\int \hat f(\omega)\int_0^t e^{i\omega t'}dt' = \frac{1}{\sqrt{2\pi}}\int \hat f(\omega)\cdot\frac{e^{i\omega t}-1}{i\omega}\,d\omega$$

$$\int_0^t f(\tau)\,d\tau = \int \chi_{[0,t]}(\tau)\,f(\tau)\,d\tau \sim \int \hat\chi_{[0,t]}(-\omega)\,\hat f(\omega)\,d\omega \sim \int \frac{e^{i\omega t}-1}{\omega}\hat f(\omega)\,d\omega$$

factors of 2π

$$\hat\chi_{[0,t]}(\omega) \sim \int e^{i\omega\tau}\chi_{[0,t]}\,d\tau \sim \frac{e^{i\omega t}-1}{\omega}$$

factors of 2π

so, $\| \eta(t) \int_0^t f(\tau)\,d\tau \|_{H^b_t} = \| \langle\tau\rangle^b \int e^{-it\tau} \eta(t) \left(\int \frac{e^{iwt}-1}{w} \hat f(w)\,dw \right) dt \|_{L^2_\tau}$ $= \frac{1}{(2\pi)}\| \int \eta(t) \hat f(w) \frac{e^{iwt}-1}{w} dw \|_{H^b_t} =: \|\cdot\|_{H^b_t}$ unwrap

"w caught up in too many places, but $\int_t$ is clean!"

$\overset{\text{Fubini}}{=} \| \langle\tau\rangle^b \int \frac{\hat\eta(\tau-w) - \hat\eta(\tau)}{w} \hat f(w)\,dw \|_{L^2_\tau}$

"ppl should be scared by naked w in denom" but 2 regimes 2 techniques

$|w|\le 1$: FTC expect cancellation

$|w|\ge 1$: replace w $\langle w\rangle$

as foreshadowed

$|w|\le 1$ cancellation

FTC, and pull abs val inside

$\| \langle\tau\rangle^b \int_{|w|\le 1} \frac{\hat\eta(\tau-w)-\hat\eta(\tau)}{w} \hat f(w)\,dw \|_{L^2_\tau} \lesssim \| \langle\tau\rangle^b \int_{|w|\le 1} \left(\int_0^1 |\hat\eta'(\tau-\theta w)|\,d\theta \right) |\hat f(w)|\,dw \|_{L^2_\tau}$

$\lesssim \left| \frac{1}{\langle\tau\rangle^b} \int_{|w|\le 1} |\hat f(w)|\,dw \right|_{L^2_\tau}$ (no τ's !!!)

$\lesssim \frac{1}{\langle\tau-\theta w\rangle^{10}}$ (Schwartz) $\sim \frac{1}{\langle\tau\rangle^{10}}$ (θ, w small)

end up "2b" is good enough, kill off the $\langle\tau\rangle^b$

$\lesssim \| \langle\tau\rangle^{b-10} \|_{L^2_\tau} \int_{|w|\le 1} |\hat f(w)|\,dw$ (finite for "10" small large)

$\overset{b>\frac12}{\lesssim} \int_{|w|\le 1} |\hat f(w)|\,dw \overset{\text{CBS}}{\lesssim} \| \langle w\rangle^{b-1} \hat f \|_{L^2_w} \| \langle w\rangle^{1-b} \|_{L^2_{|w|\le 1}}$

$\lesssim \| \langle w\rangle^{b-1} \hat f \|_2 \| \langle w\rangle^{1-b} \|_{L^2(|w|\le 1)}$ (finite no matter what b is.)

$\lesssim \| f \|_{H^{b-1}}$

as foreshadowed replace w in denom by $\langle w\rangle$

$|w|\ge 1$ no cancellation, Δ-ineq

$\| \langle\tau\rangle^b \int_{|w|\ge 1} \frac{|\hat\eta(\tau)| \cdot |\hat f(w)|}{\langle w\rangle} dw \|_{L^2_\tau} \lesssim \| \langle\tau\rangle^b \hat\eta \|_{L^2_\tau} \| \langle w\rangle^{-1} \hat f \|_1$

decoupled integrals

$\| \langle\tau\rangle^b \hat\eta \|_{L^2_\tau} \lesssim 1$ b/c $\hat\eta$ decay (Schwartz)

we learned $\hat\eta$ can absorb $\langle\tau\rangle^b$

$\lesssim_{\text{CBS}} \| \langle w\rangle^{b-1} \hat f \|_2 \| \langle w\rangle^{-b} \|_2 \lesssim \| f \|_{H^{b-1}}$ since $b > \frac12$.

can shove onto $\hat\eta$

if only $\lesssim \langle\tau-w\rangle^b$, since then get clean convolution. not true, but it is true that $\lesssim \langle\tau-w\rangle^b \langle w\rangle^b$ (consider $\tau \lesssim w$, $\tau \sim w$, $\tau \gtrsim w$)

Baby Lemma: $\langle x+y\rangle^b \sim \langle \max(x,y)\rangle^b \lesssim \langle x\rangle^b \langle y\rangle^b$

$\| \langle\tau\rangle^b \int_{|w|\ge 1} \frac{|\hat\eta(\tau-w)| \cdot |\hat f(w)|}{\langle w\rangle} dw \|_{L^2_\tau} \lesssim \| \int \langle\tau-w\rangle^b |\hat\eta(\tau-w)| \langle w\rangle^{-1+b} |\hat f(w)|\,dw \|_{L^2_\tau}$

convolution $= \| \langle w\rangle^b |\hat\eta| * \langle w\rangle^{b-1} |\hat f(w)| \|_{L^2_\tau}$

put in L^2 to give us H^{b-1}

means has to go in L^1, but $\hat\eta$ has fast decay. (Young's)

Young $\lesssim \| \langle w\rangle^{b-1} \hat f \|_2 \| \langle\tau-w\rangle^b \hat\eta(t-w) \|_1$ ($\lesssim 1$)

$\lesssim \| f \|_{H^{b-1}}$

QED gain of derivatives lemma/claim

So far we have proven

$\| \mathcal{I}(u) \|_{X^{s,b}_{T_0}} \lesssim \| u_0 \|_{H^s_x} + \| \partial_x (u^2) \|_{X^{s,b-1}_{T_0}}$ using Lemma.

we'll show $\| u \|_{X^{s,b}_{T_0}} \lesssim T_0^{\tilde b - b} \| u \|_{X^{s,\tilde b}_{T_0}}$ here, power of T_0!

source of smallness need, as long as $\tilde b > b$. But fortunately, we gained a whole "1"!

to do contraction mapping, always

and $\|\mathcal{I}u - \mathcal{I}v\|_{X^{s,b}_{T_0}} = 3\,\|\eta(t)\int_0^t e^{-(t-t')\partial_x^3}\partial_x(u^2-v^2)(t')\,dt'\|_{X^{s,b}_{T_0}} \lesssim_{\text{Lemma}} \|\partial_x(u^2-v^2)\|_{X^{s,b-1}_{T_0}}$

(linear term cancels)

need to learn how to bound $\partial(f\cdot g)$! product of fns

Thus far, we have showed that the mapping

$$\Phi(u)(t) = \eta(t)\,e^{-t\partial_x^3}u_0 + 3\eta(t)\int_0^t e^{-(t-\tau)\partial_x^3}\partial_x(u^2)(\tau)\,d\tau$$

where $\eta \in C_c^\infty((-2,2))$ with $\eta \equiv 1$ on $|t|\le 1$ satisfies

$$\begin{cases} \|\Phi(u)\|_{X^{s,b}_{T_0}} \lesssim \|u_0\|_{H^s_x} + \|\partial_x(u^2)\|_{X^{s,b-1}_{T_0}} \\ \|\Phi(u)-\Phi(v)\|_{X^{s,b}_{T_0}} \lesssim \|\partial_x[u^2-v^2]\|_{X^{s,b-1}_{T_0}} \end{cases}$$

for all $s\ge 0$, $b>1/2$.

It remains to estimate the nonlinear terms above. We start by exhibiting the small parameter necessary to prove Φ is a contraction on

$$B = \{u \in X^{s,b}_{T_0} : \|u\|_{X^{s,b}_{T_0}} \le C_0\|u_0\|_{H^s}\}$$

where C_0 is an absolute constant.

→ **Lemma** For any $-1/2 < b' < b < 1/2$ and $s\in\mathbb{R}$ we have (b/c we can't gain more than 1 derivative (in our H^{b-1} lemma))

$$\|u\|_{X^{s,b'}_{T_0}} \lesssim T_0^{b-b'}\|u\|_{X^{s,b}_{T_0}} \quad (\star)$$

Proof: Claim: It suffices to ~~We'll~~ prove the claim for $0\le b'<b<1/2$. Then, by duality, for

negative case: $-1/2 < b' < b \le 0 \Rightarrow 0 \le -b \le -b' \le \frac{1}{2}$ we have

$$\|u\|_{X^{s,b'}_{T_0}} = \sup_{\|g\|_{X^{-s,-b'}_{T_0}}\lesssim 1} \langle u, g\rangle \quad \text{(HW! (can ignore sup))} \lesssim \sup_g \|u\|_{X^{s,b}_{T_0}}\|g\|_{X^{-s,-b}_{T_0}}$$

(which I know is finite b/c always assume RHS of ineq like $(\star)$ are finite (o.w. trivial))

"Looks very sensible but trickier than you think!"

(same condition as g)

$$\lesssim \sup_g \|u\|_{X^{s,b}_{T_0}} T_0^{-b'+b}\|g\|_{X^{-s,-b'}_{T_0}} \quad \text{(using } (\star) \text{ positive case)} \quad 0\le -b < -b' < 1/2$$

$$\lesssim \|u\|_{X^{s,b}_{T_0}} \cdot T_0^{b-b'}$$

HW!

Straddle case: actually easy!

Finally, for $-1/2 < b' < 0 < b < 1/2$ we have

$$\|u\|_{X^{s,b'}_{T_0}} \le T_0^{-b'}\|u\|_{X^{s,0}_{T_0}} \le T_0^{b-b'}\|u\|_{X^{s,b}_{T_0}}$$

positive case:

So fix $0 \le b' < b < 1/2$. Let's show

$$\|u\|_{X^{s,b'}_{T_0}} \lesssim T_0^{b-b'} \|u\|_{X^{s,b}_{T_0}}$$

Let $v(t) = u(t)$ for $|t| \le T_0$. (extension) Using the fractional product rule and Sobolev embedding, we may bound.

$$\|u\|_{X^{s,b'}_{T_0}} \le \|\eta(\tfrac{t}{T_0}) v(t)\|_{X^{s,b'}} := \|\eta(\tfrac{t}{T_0}) e^{t\partial_x^3} v(t)\|_{H_t^{b'} H_x^s} = \| \| \eta(\tfrac{t}{T_0}) e^{t\partial_x^3} u \|_{H_t^{b'}} \|_{H_x^s}$$

($v = u$ on support of $\eta(\tfrac{t}{T_0})$)

$$\lesssim \Big\| \|\eta(\tfrac{t}{T_0})\|_{L_t^{\frac{1}{b}}} \|e^{t\partial_x^3} v(t)\|_{L_t^{\frac{2}{1-2b}}} \Big\|_{H_x^s}$$

$$+ \Big\| \| |\nabla_t|^{b'} \eta(\tfrac{t}{T_0})\|_{L^{1/b}} \|e^{t\partial_x^3} v(t)\|_{L_t^{\frac{2}{1-2b}}} \Big\|_{H_x^s}$$

$$+ \Big\| \|\eta(\tfrac{t}{T_0})\|_{L_t^{\frac{1}{b-b'}}} \| |\nabla_t|^{b'} e^{t\partial_x^3} v(t)\|_{L_t^{\frac{2}{1-2(b-b')}}} \Big\|_{H_x^s}$$

$$\lesssim \|e^{t\partial_x^3} v(t)\|_{H_t^b H_x^s} \cdot \{ T_0^b + T_0^{b-b'} \}$$

$$\lesssim T_0^{b-b'} \|v\|_{X^{s,b}}$$

Taking the infimum over such v, we derive the claim ∎

Holder: $\|\eta(\tfrac{t}{T_0}) e^{t\partial_x^3} u\|_{L_t^2} \lesssim \|e^{t\partial_x^3} u\|_{L_t^p} \cdot \|\eta(\tfrac{t}{T_0})\|_{L_t^q}$, $p = \frac{2}{1-2b}$ chosen to Sobolev embed, $q = \frac{1}{b}$

$\lesssim \| \partial_t^b e^{t\partial_x^3} u\|_{L_t^2} T_0^b$

$\lesssim T_0^{b-b'} \|e^{t\partial_x^3} u\|_{H_t^b}$

$\| \partial_t^{b'} (\eta(\tfrac{t}{T_0}) e^{t\partial_x^3} u)\|_{L_t^2} \le$ fractional chain rule

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PED "derivative for powers of T_0 lem" — trade

Lemma For $b > 1/2$, $s \ge 0$, and $b' \le -1/4$ we have

$$\|\partial_x (uv)\|_{X^{s,b'}_{T_0}} \lesssim \|u\|_{X^{s,b}_{T_0}} \|v\|_{X^{s,b}_{T_0}}$$

Remark: For $\frac{1}{2} < b < \frac{3}{4}$ (so $-\frac{1}{2} < b-1 < -\frac{1}{4}$) and $s \ge 0$, this gives

$$\|\partial_x (u^2)\|_{X^{s,b-1}_{T_0}} \lesssim T_0^{\frac{3}{4}-b} \|\partial_x (u^2)\|_{X^{s,-\frac{1}{4}}_{T_0}} \lesssim T_0^{\frac{3}{4}-b} \|u\|^2_{X^{s,b}_{T_0}}$$

and so $\|\Phi(u)\|_{X^{s,b}_{T_0}} \lesssim \|u_0\|_{H^s} + T_0^{\frac{3}{4}-b} \|u\|^2_{X^{s,b}_{T_0}}$ ($\frac{3}{4}-b = -\frac{1}{4}-(b-1)$)

this is why we restrict $b < \frac{3}{4}$.

Thus for $T_0 = T_0(\|u_0\|_{H^s})$, $\Phi : B \to B$.

Finally, to see that Φ is a contraction, for $u, v \in B$ we estimate

$$\|\phi(u)-\phi(v)\|_{X^{s,b}_{T_0}} \lesssim \|\partial_x(u^2-v^2)\|_{X^{s,b-1}_{T_0}}$$

$$\lesssim T_0^{\frac{3}{4}-b} \|\partial_x[(u-v)(u+v)]\|_{X^{s,-\frac{1}{4}}_{T_0}} \quad \text{(Lemma)}$$

$$\lesssim T_0^{\frac{3}{4}-b} \|u-v\|_{X^{s,b}_{T_0}} \left[\|u\|_{X^{s,b}_{T_0}} + \|v\|_{X^{s,b}_{T_0}}\right]$$

$$\lesssim T_0^{\frac{3}{4}-b} \|u-v\|_{X^{s,b}_{T_0}} \cdot \|u_0\|_{H^s}$$

$$< \tfrac{1}{2}\|u-v\|_{X^{s,b}_{T_0}} \quad \text{provided } T_0 = T_0(\|u_0\|_{H^s}) \text{ is small enough.}$$

To reiterate, $\forall u \in B := \{u \in X^{s,b}_{T_0} : \|u\|_{X^{s,b}_{T_0}} \le C_0\|u_0\|_{H^s}\}$, get $\|\Phi u\|_{X^{s,b}_{T_0}} \lesssim \|u_0\|_{H^s} + T_0^{3/4-b}\|u_0\|_{H^s}^2$

$\|\Phi u - \Phi v\|_{X^{s,b}_{T_0}} \lesssim T_0^{3/4-b}\|u_0\|_{H^s}\|u-v\|_{X^{s,b}_{T_0}}$. namely $T^{3/4-b}\|u_0\|_{H^s} \ll 1$

$\hookrightarrow \Phi$ is contraction on B!

It remains to prove the lemma.

Proof: We'll ignore the T_0 and argue by duality. We have to show

$$\left|\iint \partial_x(uv)\cdot w \, dx\,dt\right| \lesssim \|u\|_{X^{s,b}}\|v\|_{X^{s,b}}\|w\|_{X^{s,-b'}}$$

recall $\|\partial_x(uv)\|_{X^{s,-\frac{1}{4}}_{T_0}} \lesssim \sup_{\|w\|_{X^{-s,\frac{1}{4}}_{T_0}}} \langle \partial_x(uv), w\rangle_{t,x}$

Still calling extensions same name u, v, w, suffices to establish

Recall $\|u\|_{X^{s,b}} = \|\langle\xi-\tau^3\rangle^b\langle\xi\rangle^s \hat{u}(\tau,\xi)\|_{L^2_{\tau,\xi}}$

"pass to Fourier side, write out resulting orgy of integrals"

$$\left|\iint \partial_x(uv)\cdot w\,dx\,dt\right| \simeq \left|\iint \xi\cdot\widehat{uv}(\tau,\xi)\,\hat{w}(-\tau,-\xi)\,d\tau\,d\xi\right|$$

$$\simeq \left|\iiiint \xi\,\hat{u}(\tau_1,\xi_1)\,\hat{v}(\tau-\tau_1,\xi-\xi_1)\,\hat{w}(-\tau,-\xi)\,d\tau_1\,d\tau\,d\xi_1\,d\xi\right|$$

"can only use Young & Hölder" so isolate things I can put in L^p spaces

$$\lesssim \iiiint \frac{\langle\xi\rangle^s|\xi|\langle\tau-\xi^3\rangle^{b'}}{\langle\xi_1\rangle^s\langle\xi-\xi_1\rangle^s\langle\tau_1-\xi_1^3\rangle^b\langle\tau-\tau_1-(\xi-\xi_1)^3\rangle^b} f_1(\tau_1,\xi_1)\,f_2(\tau-\tau_1,\xi-\xi_1)\,f_3(\tau,\xi)\,d\tau_1\,d\tau\,d\xi_1\,d\xi$$

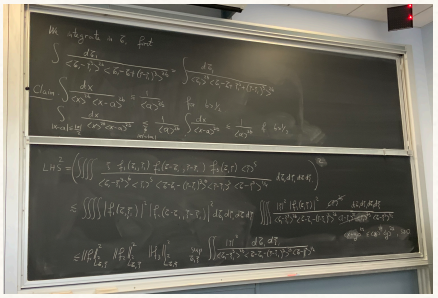
where $f_1(\tau,\xi) = \langle\tau-\xi^3\rangle^b\langle\xi\rangle^s|\hat{u}(\tau,\xi)|$ — we do this b/c $\|f_1\|_{L^2_{\tau,\xi}} = \|u\|_{X^{s,b}}$

$f_2(\tau,\xi) = \langle\tau-\xi^3\rangle^b\langle\xi\rangle^s|\hat{v}(\tau,\xi)|$

$f_3(\tau,\xi) = \langle\tau-\xi^3\rangle^{-b'}\langle\xi\rangle^{-s}|\hat{w}(-\tau,-\xi)|$

in 2026, focused on $\langle\tau-\xi^3\rangle^{1/4}\langle\xi\rangle^{-s}|\hat{w}(-\tau,-\xi)|$

Idea: CBS $\iiiint |f_1(\tau_1,\xi_1)|^2\,|f_2(\tau-\tau_1,\xi-\xi_1)|^2$ integrate in τ,ξ first, decouple ξ,τ & ξ',τ'!



By Cauchy-Schwarz,

$$\left|\iint 2_x(uv)\cdot w\,dxdt\right|^2 \lesssim \iiiint |f_1(\xi_1,\xi_1)|^2 |f_2(\xi-\xi_1,\xi-\xi_1)|^2 d\xi_1 d\xi d\xi_1' d\xi'$$

$\lesssim 1$ by cute identity last lec $\langle x+y\rangle^{2s} \lesssim \langle x\rangle^{2s}\langle y\rangle^{2s}$ $\forall s\geq 0$

$$\cdot \iiiint \frac{\langle \xi\rangle^{2s} |\xi|^2 \langle \xi-\xi_1^3\rangle^{2b'}}{\langle \xi_1\rangle^{2s}\langle \xi-\xi_1'\rangle^{2s}\langle \xi_1,-\xi_1^3\rangle^{2b}\langle \xi-\xi_1,-(\xi-\xi_1')^3\rangle^{2b}} |f_3(\xi,\xi)|^2 d\xi_1 d\xi d\xi_1' d\xi'.$$

$$\lesssim \|f_1\|_{L^2_{\xi,\xi}}^2 \|f_2\|_{L^2_{\xi,\xi}}^2 \cdot \iiiint \frac{|\xi|^2\langle \xi-\xi_1^3\rangle^{2b'}|f_3(\xi,\xi)|^2}{\langle \xi_1,-\xi_1^3\rangle^{2b}\langle \xi-\xi_1,-(\xi-\xi_1')^3\rangle^{2b}} d\xi_1 d\xi_1' d\xi d\xi'.$$

$$\leq \|f_3\|_{L^2_{\tau,\xi}}^2 \sup_{\tau,\xi} \iint \frac{|\xi|^2}{\langle \tau_1-\xi_1^3\rangle^{2b}\langle \tau-\tau_1-|\xi-\xi_1|^3\rangle^{2b}\langle \tau-\xi^3\rangle^{2b'}} d\tau_1 d\xi_1,$$

Babylem

We use the fact that

$$\int_{\mathbb{R}} \frac{1}{\langle x\rangle^{2b}\langle x-\alpha\rangle^{2b}} dx \lesssim \langle\alpha\rangle^{-2b} \quad \text{for } b>1/2$$

Indeed $\int_{|x-\alpha|\leq \frac{|\alpha|}{2}} \frac{1}{\langle x\rangle^{2b}\langle x-\alpha\rangle^{2b}} dx \lesssim \langle\alpha\rangle^{-2b} \int_{\mathbb{R}} \frac{1}{\langle x-\alpha\rangle^{2b}} dx \lesssim \langle\alpha\rangle^{-2b}$

(used up fast: $|x|\sim|\alpha|$, can $\int_{\mathbb{R}}$ now)

and $\int_{|x-\alpha|>\frac{|\alpha|}{2}} \frac{1}{\langle x\rangle^{2b}\langle x-\alpha\rangle^{2b}} dx \lesssim \langle\alpha\rangle^{-2b} \int_{\mathbb{R}} \frac{1}{\langle x\rangle^{2b}} dx \lesssim \langle\alpha\rangle^{-2b}$

yes we can do better but we assume $b>\frac{1}{2}$ for cly in time anyways.

ith core "shape" is:

integrate in τ_1 first b/c ξ_1 has cube ξ_1^3, very entangled

$$\int \frac{d\tau_1}{\langle \tau_1-\xi_1^3\rangle^{2b}\langle \tau_1-\tau+(\xi_1-\xi_1')^3\rangle^{2b}} \overset{\text{CoV}}{=} \int \frac{d\tau_1}{\langle \tau_1\rangle^{2b}\langle \tau_1-\ldots\rangle^{2b}}$$

$a := \tau+\xi_1'^3+(\xi-\xi_1')^3$

Thus $$\int \frac{d\xi_1}{\langle \xi_1,-\xi_1^3\rangle^{2b}\langle \xi-\xi_1,-(\xi-\xi_1')^3\rangle^{2b}} = \int \frac{d\xi_1}{\langle \xi\rangle^{2b}\langle \xi_1-\underbrace{\xi+\xi_1'^3+(\xi-\xi_1')^3}_{=:a}\rangle^{2b}} \overset{\text{babylem}}{\lesssim} \langle \xi-\underbrace{\xi_1'^3-(\xi-\xi_1')^3}_{=:a}\rangle^{-2b}$$

$$\lesssim \langle \xi-3\xi_1'^2\xi+3\xi_1'\xi^2-\xi^3\rangle^{-2b}$$

Now $$\int_{\mathbb{R}} \langle \xi-3\xi_1'^2\xi+3\xi_1'\xi^2-\xi^3\rangle^{-2b} d\xi_1' = \int \langle \xi_1^2-\xi_1\cdot\sqrt{3}|\xi|^{3/2}+\xi^3-\xi\rangle^{-2b} |3\xi|^{-1/2} d\xi_1$$

$\xi_1 \mapsto \xi_1+\frac{\sqrt{3}}{2}|\xi|^{3/2}$

$$= \int \langle \xi_1^2+\xi^3-\xi-\tfrac{3|\xi|^3}{4}\rangle^{-2b} |3\xi|^{-1/2} d\xi_1$$

$\xi_1^2 \mapsto \eta$

$$\lesssim |\xi|^{-1/2} \int \langle \eta+\xi^3-\xi-\tfrac{3|\xi|^3}{4}\rangle^{-2b} |\eta|^{-1/2} d\eta$$

$$\lesssim |\xi|^{-1/2}\cdot \langle \xi^3-\xi-\tfrac{3|\xi|^3}{4}\rangle^{-1/2}$$

$$\lesssim |\xi|^{-1/2} \langle 4\xi^3-3|\xi|^3-4\xi\rangle^{-1/2}$$

Here we use the easy bound

Babylem 2: $$\text{LHS} = \int_{\mathbb{R}} \frac{dx}{|x|^{1/2}\langle x-\alpha\rangle^{2b}} \lesssim \langle\alpha\rangle^{-1/2} \quad \text{for } b>1/2$$

Hint: This is immediate if $|\alpha|\leq 1$. ($\text{LHS} \sim \int_{\mathbb{R}} \frac{dx}{|x|^{1/2}\langle x\rangle^{2b}} \lesssim 1$)

If $|\alpha|>1$ treat separately the cases $|x|\geq 2|\alpha|$, $|x|\leq \frac{|\alpha|}{2}$, and $\frac{|\alpha|}{2}<|x|<2|\alpha|$.

↳ $\frac{1}{|\alpha|^{1/2}}\int \frac{dx}{\langle x-\alpha\rangle^{2b}} \lesssim \frac{1}{|\alpha|^{1/2}}$ ↳ $\int_{|x|<|\alpha|/2} \frac{dx}{|x|^{1/2}\langle x-\alpha\rangle^{2b}} \lesssim \frac{1}{\langle\alpha\rangle^{2b}}\int_{|x|<|\alpha|/2} \frac{1}{|x|^{1/2}} \lesssim \frac{|\alpha|^{1/2}}{\langle\alpha\rangle^{2b}} \lesssim \frac{1}{\langle\alpha\rangle^{1/2}}$ for $b>\frac{1}{2}$.

integral $\lesssim \frac{1}{|\alpha|^{1/2}}\int_{\mathbb{R}} \frac{1}{\langle x-\alpha\rangle^{2b}} \lesssim \frac{1}{\langle\alpha\rangle^{1/2}}$ ($|\alpha|\geq 1$)

splitting "3 cases always works don't waste time finding 2 cases"

Remains to do ξ_1 integral

$$\sup_{\tau,\xi} \int \frac{|\xi|^2\, d\xi_1}{\langle a\rangle^{2b}\langle \tau-\xi^3\rangle^{1/2}} \overset{\text{algebra complete square}}{=} \sup_{\tau,\xi} \int \frac{|\xi|^2\, d\xi_1'}{\langle 3\xi(\xi_1'-\frac{\xi}{2})^2+\frac{\xi^3}{4}-\tau\rangle^{2b}\langle \xi^3-\tau\rangle^{1/2}}$$

$$\lesssim \overset{\text{CoV}}{\underset{\eta=3\xi\xi_1'^2 \leftrightarrow \xi_1'\approx\sqrt{\frac{|\eta|}{|\xi|}} \text{ Jacobian}}{}} \sup_{\tau,\xi} \int \frac{|\xi|^2}{|\eta|^{1/2}|\xi|^{1/2}} \frac{d\eta}{\langle \eta+\frac{\xi^3}{4}-\tau\rangle^{2b}\langle \xi^3-\tau\rangle^{1/2}} \lesssim \sup_{\tau,\xi} \frac{|\xi|^{3/2}}{\langle \tau-\xi^3\rangle^{1/2}\langle \tau-\frac{\xi^3}{4}\rangle^{1/2}}$$

cov by shift invariance

just more basic math

Square it to rid $\sqrt{\ }$, cov $\xi^3 \leftrightarrow \xi$

$\sim \sup_{\tau,\xi} \frac{|\xi|}{\langle \tau-\xi\rangle \langle 4\tau-\xi\rangle}$

3 cases: $|\xi| < \frac{|\tau|}{2}$, $\lesssim \frac{|\tau|}{\langle\tau\rangle^2} \lesssim 1$. ✓

$|\xi| > 2|\tau|$, $\lesssim \frac{|\xi|}{\langle\xi\rangle} \lesssim 1$. ✓ b/c $\langle\tau-\xi\rangle \sim \langle\xi\rangle$

$|\xi| \sim |\tau|$, $\langle 4\tau-\xi\rangle \sim \langle\tau\rangle$, $\frac{|\tau|}{\langle\tau\rangle} \lesssim 1$ ✓

"1 = min, big-time"

Get $$\left| \iint \partial_x(uv)\cdot w\,dx\,dt \right|^2 \lesssim \|f_1\|_{L^2_{\tau,\xi}}^2 \|f_2\|_{L^2_{\tau,\xi}}^2 \cdot \underbrace{\iint \frac{|\xi|^{3/2}\,|f_3(\tau,\xi)|^2}{\langle\tau-\xi^3\rangle^{-25}\langle 4\xi^3-3|\xi|^3-4\tau\rangle^{1/2}}\,d\tau\,d\xi}_{\lesssim \|f_3\|_{L^2_{\tau,\xi}}^2}$$

provided $$\left\| \frac{|\xi|^{3/2}}{\langle\tau-\xi^3\rangle^{-25}\langle 4\xi^3-3|\xi|^3-4\tau\rangle^{1/2}} \right\|_{L^\infty_{\tau,\xi}} = \left\| \frac{\xi}{\langle\tau-\xi\rangle^{-45'}\langle 4\xi-3|\xi|-4\tau\rangle} \right\|_{L^\infty_{\tau,\xi}} \lesssim 1$$

If $|\xi| \geq 2|\tau| \Rightarrow \frac{|\xi|}{\langle\tau-\xi\rangle^{-45'}} \sim \frac{|\xi|}{\langle\xi\rangle^{-45'}} \lesssim 1$ for $b' \leq -\frac{1}{4}$

$|\xi| \leq \frac{|\tau|}{2} \Rightarrow \frac{|\xi|}{\langle\tau-\xi\rangle^{-45'}} \lesssim \frac{|\tau|}{\langle\tau\rangle^{-45'}} \lesssim 1$ for $b' \leq -\frac{1}{4}$

$\frac{|\tau|}{2} \leq |\xi| \leq 2|\tau| \Rightarrow \frac{|\xi|}{\langle 4\xi-3|\xi|-4\tau\rangle} = \begin{cases} \frac{|\xi|}{\langle\xi-4\tau\rangle} \lesssim 1 & \text{if } \xi \geq 0 \\ \frac{|\xi|}{\langle 4\tau+7|\xi|\rangle} \lesssim 1 & \text{if } \xi < 0 \end{cases}$

3 cases

$\blacksquare$

→ global well posed b/c local well posed$^{\text{time}}$ & evolve by L^2 norm of init data and L^2 norm conserved.

Chapter V

Conservation laws

Lec 19 Miura, Gardner → Hamiltonian flow
mKdV mmKdV

5/6/24 **Conservation laws** ("with caveats")

(historical approach, will look like "voodoo" b/c physicists derived w/o even knowing solutions ∃)

5/11/26 Lec 19 "discovery of Miura first led to realization that KdV special"

Rmk: If v is solution to **modified KdV** (mKdV)

$$v_t + v_{xxx} = 6v^2 v_x$$

cubic nonlinearity not quadratic like KdV play better w/ local well-posedness results: higher power ⇒ higher nonlinearity smaller.

OTOH, GWP ↑ higher power are dangerous outside perturbative regime but we don't by LWP & conservation laws anyways so LWP is the regime to focus on

8 11:34 why ↑ nicer.

Miura transform $u := v^2 + v_x$ then solves KdV!

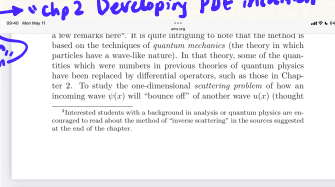
$$u_t + u_{xxx} - 6uu_x = (2v + \partial_x)(v_t + v_{xxx} - 6v^2 v_x) = 0.$$

$= 2v v_t + v_{tx} + 2v v_{xxx} + \binom{3}{1} 2v_x v_{xx} + v_{xxxx} - 6(v^2 + v_x)(2vv_x + v_{xxx})$

$= 2v(v_t + v_{xxx}) + \partial_x(v_t + v_{xxx}) + 6v v_{xxx} - 12v^3 v_x - 12v v_{xxx} - 6v^2 v_x^2 - 6v_x v_{xx}$

"miura transform intuition"

→ chp 2 Developing PDE intuition"

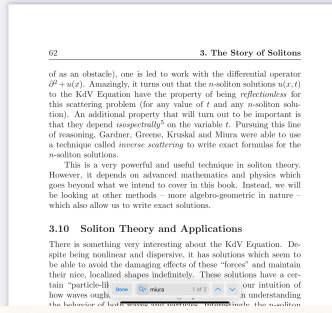


a few remarks here". It is quite intriguing to note that the method is based on the techniques of *quantum mechanics* (the theory in which particles have a wave-like nature). In that theory, some of the quantities which were numbers in previous theories of quantum physics have been replaced by differential operators, such as those in Chapter 2. To study the one-dimensional *scattering problem* of how an incoming wave $\psi(x)$ will "bounce off" of another wave $u(x)$ (thought

[1]Interested students with a background in analysis or quantum physics are encouraged to read about the method of "inverse scattering" in the sources suggested at the end of the chapter.

Amazingly, $\tilde{u} = v^2 - v_x$ also solves KdV!

$$\tilde{u}_t + \tilde{u}_{xxx} - 6\tilde{u}\tilde{u}_x = (2v - \partial_x)(v_t + v_{xxx} - 6v^2 v_x) = 0$$



62 3. The Story of Solitons

of as an obstacle), one is led to work with the differential operator $\partial^2 + u(x)$. Amazingly, it turns out that the n-soliton solutions $u(x,t)$ to the KdV Equation have the property of being *reflectionless* for this scattering problem (for any value of t and any n-soliton solution). An additional property that will turn out to be important is that they depend *isospectrally* on the variable t. Pursuing this line of reasoning, Gardner, Greene, Kruskal and Miura were able to use a technique called *inverse scattering* to write exact formulas for the n-soliton solutions.

This is a very powerful and useful technique in soliton theory. However, it depends on advanced mathematics and physics which goes beyond what we intend to cover in this book. Instead, we will be looking at other methods – more algebro-geometric in nature – which also allow us to write exact solutions.

3.10 Soliton Theory and Applications

There is something very interesting about the KdV Equation. Despite being nonlinear and dispersive, it has solutions which seem to be able to avoid the damaging effects of these "forces" and maintain their nice, localized shapes indefinitely. These solutions have a certain "particle-li

→ Darboux transformation, "create solitons out of thin air!"

The mapping $v \longmapsto u := v^2 + v_x$ is called the **Miura transform**

sol to mKdV → sol to KdV

reduce study of KdV to that of mKdV (which is friendlier for LWP)

If could invert, would conclude that well-posedness theory for mKdV and KdV are the same.

But to invert Miura transform, want to find v s.t. $v^2 + v_x = u$. ↔ $v_x = -v^2 + u$ ODE

(given u)

Looks like Riccati eq, so inspired by theory of Riccati equations,

seek solutions of the form $v = \frac{\psi'}{\psi}$ for nowhere vanishing fn. ψ. ("log derivative", to get cancellation)

Then $u = v^2 + v_x = \left(\frac{\psi'}{\psi}\right)^2 + \frac{\psi''}{\psi} - \frac{(\psi')^2}{\psi^2} = \frac{\psi''}{\psi}$ i.e. $-\psi'' + u\psi = 0$.

$\iff (-\partial^2 + u)\psi = 0$

ψ is solution to Schrödinger eq, w/ potential u

i.e. formal eigenfn for operator w/ eigenvalue 0.

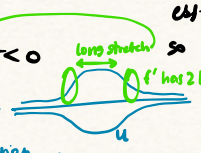
i.e. we want nowhere vanishing solution ψ to $(-\partial^2 + u)\psi = 0$.

Sadly, this doesn't exist even if u is really nice.

If u is of form  (negative Gaussian or whatever)

then $-\partial^2+u$ has negative eigenvalue. ↺ 11:41 (pf: perturbation...)

$\langle(-\partial^2+u)f, f\rangle = \int |f'|^2 + u|f|^2 < 0$



essential spectrum $[0,\infty)$

so any negative... must automatically be eigenvalue.

so $-\partial^2+u$ has negative spectrum, and Weyl's criterion, this consists of (discrete?) negative eigenvalues

By Sturm-Liouville oscillation theory, ψ (which is formal eigenfunction of eigenvalue 0) must vanish at least once.

↑ first EVal has Efn don't cross x axis
but next one crosses once,
next one twice (and zeroes interlace)

(u positive this isn't problem, but we want robust well posedness theory, not require silly restriction like init data >0 or whatever)

OTOH, Gardner:

If instead we considered $(-\partial^2+u)\psi = -k^2\psi$. (1)

Look "deep into neg spectrum find first Evec, nonvanishing"

then we could find a nowhere vanishing sol. ψ provided $k = k(u) \gg 1$.
(k suff large depending on u).

if u vanishes $\pm\infty$, then at large values, ODE theory says $\psi(x) = ae^{kx} + be^{-kx}$ as $x \to \infty$
$\in C_c^\infty(\mathbb{R})$ fundamental sol

e.g. if $k^2 > \|u\|_\infty$, then

$$\langle(-\partial^2+u+k^2)f, f\rangle = \int |f'|^2 + \underbrace{(u+k^2)}_{>0}|f|^2 > 0$$

so ↗ is positive definite operator.

Why does pos. def. operator $\longrightarrow$ $\exists$ nowhere vanishing ψ satisfying (1)?

If $u \in C_c^\infty$, then (1) becomes $-\psi'' = -k^2\psi$ as $x \to \pm\infty$.

So $\exists$ a solution ψ s.t. $\psi(x) = e^{kx}$ as $x \to -\infty$.

Also by ODE theory, at ∞, $\psi(x) = ae^{kx} + be^{-kx}$ as $x \to +\infty$.

let's pretend we know $a \neq 0$ ↺ 11:50 ✓

But Eigenfn better in L^2 (finite momentum)

Note $\dfrac{\psi'}{\psi} = \dfrac{kae^{kx} - kbe^{-kx}}{ae^{kx} + be^{-kx}} \longrightarrow k$ as $x \to \infty$

b/c $\dfrac{\psi'}{\psi} - k = \dfrac{-2bk}{ae^{2kx}+b} \xrightarrow{x\to\infty} 0$



In middle, such a solution can't vanish. Indeed if $\exists x_0 \in \mathbb{R}$ s.t. $\psi(x_0) = 0$, (↺ 11:50 and x_0 is first pt ψ cross axis "leftmost")

then $0 < \int_{-\infty}^{x_0} |\psi'|^2 + (u+k^2)|\psi|^2 = \int_{-\infty}^{x_0} (\psi\psi')' - \psi\psi'' + (u+k^2)|\psi|^2\,dx$

b/c k large $\Leftarrow k(u)$ or more fancy: for u in L^2 not just L^∞

$$= \psi\psi'\Big|_{-\infty}^{x_0} + \int_{-\infty}^{x_0} \psi\cdot(-\partial^2+u+k^2)\psi\,dx$$

$-\partial^2+u+k^2 > 0$ if $k = k(u)$ large enough
We $\langle f, f\rangle = \int (f')^2 + uf^2 + k^2f^2\,dx$

$$= 0 + 0 \quad \not\downarrow$$

$|\int uf^2| \le \|u\|_2\|f\|_4^2$
$\lesssim \|u\|_2 \|f'^{1/4}f\|_2^2 \lesssim \|u\|_2\|f'\|_2^{1/2}\|f\|_2^{3/2} \le \frac{1}{2}\|f'\|_2^2 + C(\|u\|_2)\cdot\|f\|_2^2$

modified Miura transform

So can invert this, but so what?

Eqn (1) correspond to seeking a solution of the form

$$w = \frac{\psi'}{\psi} - k.$$

to the eqn $u = w^2 + w_x + (\dots)$ (+2kw)

$$= \left(\frac{\psi'}{\psi}\right)^2 - 2k\frac{\psi'}{\psi} + k^2 + \frac{\psi''}{\psi} - \left(\frac{\psi'}{\psi}\right)^2 + 2k\frac{\psi'}{\psi} - 2k^2$$

choose so that these things in circle cancel.

$$= \frac{\psi''}{\psi} - k^2$$

$$\Leftrightarrow -\psi'' + u\psi = -k^2\psi.$$

The mapping $w \longmapsto u = w^2 + w_x + 2kw$ is called the **Gardner transform.** beauty is that it's invertible

we saw that it is invertible if $k = k(w) \gg 1$.

so we've reduced KdV to Gardner eq but not a clear win b/c Gardner eq still has KdV term

What eq does w satisfy? well of course, the **Gardner eq!**

$$w_t + w_{xxx} = \underbrace{6w^2 w_x}_{\text{"mKdV" term}} + \underbrace{12k\, w w_x}_{\text{"KdV term"}}$$

i.e. $$\partial_t w + \partial_x(w_{xx} - 2w^3 - 6kw^2) = 0 \quad (2)$$

microscopic conservation form,

Rewrite Gardner eq as

$\partial_t w + \partial_x(w_{xx} - 2w^3 - 6kw^2) = 0$

↑ write everything not ∂_t of spatial derivative of something

Why Gardner interested in this? He wanted conservation law for KdV.

$u = w^2 + w_x + 2kw.$

expand w as power series in $\frac{1}{k}$:

"true for every k so coeffs of k^0 must match"

plug it in to itself!

$$w = \frac{1}{2k}(u - w^2 - w')$$

$$= \frac{1}{2k}u - \left(\frac{1}{2k}\right)^3(u - w^2 - w')^2 - \left(\frac{1}{2k}\right)^2(u - w^2 - w')' + \frac{1}{8k^3}(u - w^2 - w')''$$

$$= \frac{1}{2k}u - \frac{1}{4k^2}u' - \frac{1}{8k^3}(u^2 - u'') + O\left(\frac{1}{k^4}\right)$$

Plug this expansion into eqn (2):

we get infinitely many conservation laws! of the type: $\partial_t \underbrace{\rho}_{\text{density}} + \partial_x \underbrace{j}_{\text{diverseme of "currents"}} = 0$

"just for fun I calculated the first 3"

"I'm not crazy I didn't memorize them... well I know the first 2 but the 3rd one I could never learn"

the first one is:

only keep the terms that give $\frac{1}{k}$

$$\frac{1}{2k}\partial_t u + \partial_x\left(\frac{1}{2k}u'' - 6k\frac{1}{(2k)^2}u^2\right) = 0$$

$$\Rightarrow \partial_t u + \partial_x(u'' - 3u^2) = 0, \quad \text{the KdV eqn!}$$

Integrating these microscopic conservation laws, like (holds pointwise) $\Rightarrow \int u(t,x)\,dx = \int u(0,x)\,dx \quad \forall t \in \mathbb{R}$ (conservation of mass)

get macroscopic conservation laws immediately

But microscopic conservation laws more powerful than macroscopic, b/c can integrate micro laws w/ all sorts of fns!

↳ cf. previous 2 SIR how we derived virial & Morawetz.

The next two are:

$$\partial_t(u^2) + \partial_x(-4u^3 + 2uu_{xx} - u_x^2) = 0$$

$$\partial_t\left(u^3 + \tfrac{1}{2}u_x^2\right) + \partial_x\left(-\tfrac{9}{2}u^4 + 3u^2u_{xx} - 6uu_x^2 + u_xu_{xxx} - \tfrac{1}{2}u_{xx}^2\right) = 0$$

(linearly independent, b/c each next density or current involves higher and higher derivatives)

known to KdV (just $\int$ KdV against u)

$$\Rightarrow \int u(t,x)^2\,dx = \int u_0(x)^2\,dx$$

integrate in space; cons. of momentum

Whitham 1965 (not by this $\frac{1}{k}$ method ... I did what he was doing")

$$\int u^3(t,x) + \tfrac{1}{2}(u'(t,x))^2\,dx = \int u_0^3 + \tfrac{1}{2}(u')^2\,dx$$

"energy", or Hamiltonian. KdV is Hamiltonian flow!

2 more conservation laws were discovered by Kruskal–Zabusky 1967

4 . — — —— —— in quick succession. Princeton Plasma physics

Rumor only 9 cons laws.

Miura spent a week discovered the tenth.

Computer program for the 11th. "Miura says huge achievement b/c prev. computer prog ran out of memory by 5th"

The above $\frac{1}{k}$ expansion is method of Gardner that produces all these, and infinitely many more.

12:15 on different "definitions" of complete integrability

> "many completely integrable eqs related to each other by tranformations. e.g. Hashimoto trans. Landau–Lifshitz ⟷ cubic NLS ?

"KdV not actually related to Schrödinger in this way; the Schröd connection we saw in today's lecture ⟷ Lax pair"

Rmk: KdV is Galilean invt (mKdV is NOT)

meaning if u is sol to KdV, then $\tilde{u}(t,x) = u(t, x+6ht) + h$

is also sol to KdV.

check: $\tilde{u}_t + \tilde{u}_{xxx} - 6\tilde{u}\tilde{u}_x = u_t + 6hu_x - 6(u+h)u_x = u_t + u_{xxx} - 6uu_x = 0.$

Taking $h = k^2$, $u = 0$, we get a solution to KdV $\tilde{u} = k^2$.

Let $\psi = \cosh(kx) \Rightarrow v = \frac{\psi'}{\psi} = k\tanh(kx)$.

then $v^2 + v_x = \frac{\psi''}{\psi} = k^2 = u$

Compute $v^2 - v_x = 2\left(\frac{\psi'}{\psi}\right)^2 - \frac{\psi''}{\psi} = 2k^2\tanh(kx)^2 - k^2$

$= 2k^2(\tanh^2(kx) - 1) + k^2$

$= k^2 - \frac{2k^2}{\cosh^2(kx)}$ (to cancel)

Undo the Galilean transform, $h = -k^2$,

we get a solution to KdV $\tilde{u}(t,x) = \frac{-2k^2}{\cosh(k(x - 12k^2t))}$

Start w/ 0 solution → boosted water level $+k^2$, $\cosh(kx)$ is crazy that when makes $v+v_x$ is constant.

↓

Darboux transform $v+v_x$ to $v-v_x$

← miraculously end w/ soliton solution!

5/8/24

Poisson structure and Hamiltonian flows

Def: Poisson str is a mapping $\{\cdot,\cdot\}: A \times A \to A$ (A an algebra) that satisfies the algebraic properties

Example: A = set of smooth functions on manifold

1) antisymmetry $\{F,G\} = -\{G,F\}$.

2) Linearity in both entries $\{aF+bG, H\} = a\{F,H\} + b\{G,H\}$.
(2nd entry follows by (1)).

3) Jacobi identity $\{\{F,G\},H\} + \{\{H,F\},G\} + \{\{G,H\},F\} = 0$.
"cycle last position to 1st position"

4) Leibniz rule $\{FG,H\} = \{F,H\}G + F\{G,H\}$
"differentiate FG in direction H"

Examples: 1) $A = M_{n\times n}$ matrices

$\{F,G\} := [F,G]$ commutator $FG-GF$.

2) $A = C^\infty(\underbrace{\mathbb{R}^n_x \times \mathbb{R}^n_p}_{\mathbb{R}^{2n}})$ closer to finite dim DOE

$$\{F,G\} := \sum_{j=1}^{n} \frac{\partial F}{\partial x_j}\frac{\partial G}{\partial p_j} - \frac{\partial F}{\partial p_j}\frac{\partial G}{\partial x_j}$$

split into 2 groups, position & momentum

Note that $(x_j, p_j)_{1\le j\le n}$ define canonical coordinates, i.e. ← def of "canonical coord"

$\{x_j, x_k\} = 0$ (b/c $\frac{\partial x}{\partial p} = 0$), and similarly $\{p_j, p_k\} = 0$

But $\{x_j, p_k\} = \delta_{jk}$.

$u \in \mathcal{H} \longrightarrow [v \mapsto \lim_{\varepsilon\to 0} \frac{F(u+\varepsilon v)-F(u)}{\varepsilon}] \in \mathcal{H}^*$ ↔ representative given by Riesz repr. $\frac{\delta F}{\delta u} \in \mathcal{H}$

$\mathcal{H}^* \ni$ denoted as $\frac{\delta F}{\delta u}$ i.e. $\frac{\delta F}{\delta u} = \langle \frac{\delta F}{\delta u}, \cdot \rangle_{\mathcal{H}}$ $: \mathcal{H} \to \mathbb{R}$

so the map $[u \mapsto \frac{\delta F}{\delta u}]$ $\mathcal{H} \to \mathcal{H}$

3) A is set of smooth real valued fns on complex Hilbert space $\mathcal{H}$

and $\{F,G\} := \operatorname{Im} \langle \frac{\delta F}{\delta u}, \frac{\delta G}{\delta u} \rangle_{\mathcal{H}}$

($\frac{\delta F}{\delta u}$: conjugate linear in 1st entry; $\frac{\delta G}{\delta u}$: linear in 2nd entry)

$\frac{\delta F}{\delta u}$ functional derivative $\mathcal{H} \to \mathcal{H}$ (so better notation $\frac{\delta F}{\delta \bar{u}}$?)

meaning

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} F(u+\varepsilon v) =: \operatorname{Re}\langle \frac{\delta F}{\delta u}, v \rangle$$

"directional derivative in direction v"

"actually easy though notationally dense"

Example: $\mathcal{H} = L^2(\mathbb{R})$, $F[u] := \frac{1}{2}\int |u(x)|^2 dx$.

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} F(u+\varepsilon v) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \frac{1}{2}\int |u(x)|^2 + 2\varepsilon \operatorname{Re}(\bar{u}v)(x) + \varepsilon^2 |v(x)|^2 dx$$

$$= \operatorname{Re}\int (\bar{u}v)(x)\,dx \Rightarrow \boxed{\frac{\delta F}{\delta u} := u.}$$

inspired by (3) but generalizing (2) to infinite dim

3') complex Hilbert space $\mathcal{H} := \ell^2(\mathbb{Z}, \mathbb{C})$

$\mathcal{A} = C^\infty(\mathcal{H}(\mathbb{Z},\mathbb{C}), \mathbb{R})$ ← smooth fcn on Banach space $F: E \to \mathbb{R}$: $\lim_{t\to 0} \frac{F(v+tw)-F(v)}{t}$

For $u \in \mathcal{H}$, write $u =: (u_j)_{j\in\mathbb{Z}} =: (x_j + ip_j)_{j\in\mathbb{Z}}$. Then for $F: \mathcal{H} \to \mathbb{R}$ (smooth),

By chain rule $\frac{\delta F}{\delta u} : \mathcal{H} \to \mathcal{H}$ has jth entry $\in \mathbb{C} = \mathcal{H} = \ell^2(\mathbb{Z})$, $\left(\frac{\delta F}{\delta u}\right)_j := \frac{\partial F}{\partial x_j} + i\frac{\partial F}{\partial p_j} : \mathcal{H} \to \mathbb{Z}$

so $\{F,G\} := \operatorname{Im} \left\langle \frac{\delta F}{\delta u}, \frac{\delta G}{\delta u} \right\rangle_{\ell^2(\mathbb{Z},\mathbb{C})}$ (conjugate in!)

$= \sum_{j\in\mathbb{Z}} \frac{\partial F}{\partial x_j}\frac{\partial G}{\partial p_j} - \frac{\partial F}{\partial p_j}\frac{\partial G}{\partial x_j}$

} Poisson bracket responsible for Schrödinger eqs.

$u: \mathbb{R}^d \to \mathbb{C}$

$\{\operatorname{Re} u(x), \operatorname{Im} u(x)\}_{x\in\mathbb{R}^d}$

canonical coords indexed over uncountable

Phase space is $(x_j, p_j)_{j\in\mathbb{Z}} = (\operatorname{Re} u_j, \operatorname{Im} u_j)_{j\in\mathbb{Z}}$

↑ canonical coordinates

---

4) Poisson bracket for kdV.

$\mathcal{H} = L^2(\mathbb{R}, \mathbb{R})$ $\quad \mathcal{A} = C^\infty(L^2(\mathbb{R},\mathbb{R}), \mathbb{R}) \Rightarrow F, G$

$\{F,G\}(u) := \int \frac{\delta F}{\delta u}(x) \left(\frac{\delta G}{\delta u}\right)'(x)\, dx$ ← pretend can take derivative of $\frac{\delta G}{\delta u} \in L^2(\mathbb{R},\mathbb{R})$. (Since $\frac{\delta G}{\delta u}$ is probably pretty nice, hard to prove it is smooth or diff.)

$u \in \mathcal{H}$; also pretend the result is smooth $\mathcal{H} \to \mathbb{R}$.

where again functional derivative $\frac{\delta F}{\delta u} := \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} F(u+\varepsilon v) = \int \frac{\delta F}{\delta u}(x)\cdot v(x)\, dx$ (Riesz representation?)

$\left[v \to \lim_{\varepsilon\to 0} \frac{F(u+\varepsilon v)-F(u)}{\varepsilon}\right]$

$\frac{\delta F}{\delta u} \in \mathcal{H}$.

To find canonical coords, we compute

$\{F,G\} = \int \frac{\delta F}{\delta u}(x) \left(\frac{\delta G}{\delta u}\right)'(x)\, dx = \int \widehat{\frac{\delta F}{\delta x}}(\xi) \cdot \widehat{\left(\frac{\delta G}{\delta u}\right)'}(-\xi)\, d\xi$ (Plancherel to Fourier side)

$= \int \widehat{\frac{\delta F}{\delta x}}(\xi) \cdot (-i\xi)\widehat{\left(\frac{\delta G}{\delta u}\right)}(\xi)\, d\xi$

$= \int_0^\infty \widehat{\frac{\delta F}{\delta u}}(\xi) \cdot (-i\xi)\widehat{\frac{\delta G}{\delta u}}(\xi)\, d\xi + \int_0^\infty \widehat{\frac{\delta F}{\delta u}}(-\xi)\cdot(i\xi)\overline{\widehat{\frac{\delta G}{\delta u}}(-\xi)}\, d\xi$ (variable change $\xi \to -\xi$)

$= \int_0^\infty \widehat{\frac{\delta F}{\delta u}}(\xi)\cdot(i\xi)\overline{\widehat{\frac{\delta G}{\delta u}}(\xi)}\, d\xi$

$= 2\operatorname{Re}\int_0^\infty \widehat{\frac{\delta F}{\delta u}}(\xi)\cdot(-i\xi)\overline{\widehat{\frac{\delta G}{\delta u}}}(\xi)\, d\xi$

$= 2\operatorname{Im}\int_0^\infty \xi\, \widehat{\frac{\delta F}{\delta u}}(\xi)\cdot\overline{\widehat{\frac{\delta G}{\delta u}}}(\xi)\, d\xi$

$$= 2 \int_0^\infty \xi \cdot \left[-\mathrm{Re}\, \widehat{\frac{\delta F}{\delta u}}(\xi) \cdot \mathrm{Im}\, \widehat{\frac{\delta G}{\delta u}}(\xi)\, d\xi + \mathrm{Im}\, \widehat{\frac{\delta F}{\delta u}}(\xi) \cdot \mathrm{Re}\, \widehat{\frac{\delta G}{\delta u}}(\xi)\, d\xi \right] d\xi$$

canonical coord given by pieces of Fourier transform.

So the canonical coordinates are

$$C_\xi(u) = \frac{1}{\sqrt{2\pi}} \int \cos(x\xi)\, u(x)\, dx \qquad \xi > 0.$$

$$S_\xi(u) = \frac{1}{\sqrt{2\pi}} \int \frac{\sin(x\xi)}{\xi}\, u(x)\, dx$$

need to divide by 1 ξ, decide to put it w/ the sin.

So e.g. Poisson bracket of position (indexed by $\xi > 0$) & momentum (indexed by $\eta > 0$)

$$\{C_\xi(u), S_\eta(u)\} = \int \frac{\delta C_\xi}{\delta u} \left(\frac{\delta S_\eta}{\delta u}\right) = \frac{1}{2\pi} \int \cos(x\xi) \left(\frac{\sin(x\eta)}{\eta}\right)' dx$$

differentiate with x

$$= \frac{1}{2\pi} \int \cos(x\xi) \cos(x\eta)\, dx$$

$$= \delta_0(\xi - \eta)$$

interpreted in sense of distributions

write as $e^{ix\xi}$ $e^{ix\eta}$, integrate against Schwarz, get Fourier transform

Exc: $\{C_\xi(u), C_\eta(u)\} = 0 \quad \forall \xi, \eta > 0.$

same w/ S.

12:35pm.

---

e.g. one for Schrödinger, or one for KdV.

Def: given a Poisson bracket, and real-valued Hamiltonian H (abstract element of algebra)

we define a vector field X_H w/ $X_H(f) := \{f, H\}$.

$: \mathcal{H} \to \mathcal{H}$

Lemma: $\{F, G\} = -[X_F, X_G]$ (commutator)

i.e. 2 Hamiltonians that Poisson-commute $\Rightarrow$ vector fields commute!

pf: symbol pushing.

$$-[X_F, X_G](f) = -X_F X_G(f) + X_G X_F(f)$$

$$= -\{X_G(f), F\} + \{X_F(f), G\}$$

$$= -\{\{f, G\}, F\} + \{\{f, F\}, G\}$$

$= -\{\{F, f\}, G\}$ (antisym)

$= + \{\{G, F\}, f\}$ (Jacobi id)

$= \{f, \{F, G\}\}$ (antisym)

cycle!

G-cycle to beginning

by definition, $X_{\{F,G\}}(f)$.

Def: given Poisson bracket and real-valued Hamiltonian H. (abstract element of algebra)

the associated Hamiltonian flow (flow generated by H)

$$\frac{d}{dt} u = X_H(u) := \{u, H\}.$$

Example: $A = C^\infty(\mathbb{R}^n_x \times \mathbb{R}^n_p)$, recall $\{F,G\} := \sum_{j=1}^n \frac{\partial F}{\partial x_j}\frac{\partial G}{\partial p_j} - \frac{\partial F}{\partial p_j}\frac{\partial G}{\partial x_j}$

Let $H(\vec{x},\vec{p}) := \frac{1}{2m}|\vec{p}|^2 + V(\vec{x})$

Kinetic energy

Then evolution under Hamiltonian?

only term that contribute

$$\begin{cases} \frac{d}{dt} x_j := \{x_j, H\} = \frac{\partial H}{\partial p_j} = \frac{1}{m} p_j \\ \frac{d}{dt} p_j := \{p_j, H\} = -\frac{\partial H}{\partial x_j} = -\frac{\partial}{\partial x_j} V(\vec{x}). \end{cases}$$

(Newtons equations).

is the flow associated to H

Example KdV: use Poisson bracket from Ex 4 — 4) Poisson bracket for KdV.

$\mathcal{H} = L^2(\mathbb{R}, \mathbb{R})$ $\quad F, G : \mathcal{H} \to \mathbb{R}$ smooth for $\mathcal{A}$ algebra

$$\{F,G\} := \int \frac{\delta F}{\delta u}(x) \left(\frac{\delta G}{\delta u}\right)'(x)\, dx.$$

$\{F,G\}$: smooth to $\mathcal{H} \to \mathbb{R}$, $\mathcal{H}$

$\left(\frac{\delta G}{\delta u}\right)'$: this is derivative

Claim: Hamiltonian flow associated to H is

$$\frac{d}{dt} u = \left(\frac{\delta H}{\delta u}\right)'$$

Indeed, if $G(u) := u(x)$ (functional; "Dirac delta", eg) (pretend we can evaluate L^2 fns pointwise) then $\frac{\delta G}{\delta u} := \delta_x$

$$\Rightarrow \frac{d}{dt} u(t,x) := \{u(x), H\} = \int \frac{\delta u}{\delta u}(m) \left(\frac{\delta H}{\delta u}\right)'(m)\, dx = \left(\frac{\delta H}{\delta u}\right)'(x).$$

The Hamiltonian flow corresponding to $P(u) = \frac{1}{2}\int |u(x)|^2 dx$

is $\frac{d}{dt} u = \left(\frac{\delta P}{\delta u}\right)' = u' \Rightarrow u(t,x) = f(t+x)$

This is why we call P the momentum (it generates translations)!

"mass" in Schrödinger but "momentum" here!

The Hamiltonian flow corresponding to $H(u) = \int \frac{1}{2}|u'(x)|^2 + u^3(x)\, dx$

is $\frac{d}{dt} u = \left(\frac{\delta H}{\delta u}\right)' = \partial_x(-u'' + 3u^2)$

$\frac{1}{2}(u'+\varepsilon v')^2 = \frac{1}{2}u'^2 + \varepsilon u'v' + \frac{\varepsilon^2}{2}v'^2$; $(u+\varepsilon v)^3 = u^3 + 3\varepsilon u^2 v + \dots$

$= -u_{xxx} + 6uu_x$ KdV! (KdV realized as Hamiltonian flow wrt. Poisson str

(the same Hamiltonian will give rise to NLS! wrt. Poisson str in Ex 3

8/10/24 **Recall** 4 Poisson bracket: $\{F,G\} := \int \frac{\delta F}{\delta u}(x) \left(\frac{\delta G}{\delta u}\right)'(x)\,dx$. ← always space derivative

Hamiltonian flow associated to Hamiltonian H is

$$\frac{d}{dt}u = \left(\frac{\delta H}{\delta u}\right)'$$

The Hamiltonian flow corresponding to $P(u) = \frac{1}{2}\int |u(x)|^2\,dx$

is $\frac{d}{dt}u = \left(\frac{\delta P}{\delta u}\right)' = u' \Rightarrow u(t,x) = f(t+x)$

This is why we call P the momentum (it generates translations)!

b/c generates KdV, call it the Hamiltonian for KdV.

The Hamiltonian flow corresponding to $H(u) = \int \frac{1}{2}|u'(x)|^2 + u^3(x)\,dx$

is $\frac{d}{dt}u = \left(\frac{\delta H}{\delta u}\right)' = \partial_x(-u'' + 3u^2)$ ($\frac{1}{2}(u'+\varepsilon v')^2$ ⇝ $\int u'v'$; $\int 3u^2 v$)

$= -u_{xxx} + 6uu_x$ KdV! (KdV realized as Hamiltonian flow wrt Poisson str)

---

do a computation:

$$\{P,H\} := \int \frac{\delta P}{\delta u}(x)\cdot\left(\frac{\delta H}{\delta u}\right)'(x)\,dx$$

$$= \int u\cdot(-u''' + 6u')\,dx \overset{\text{plug in results above}}{=} \int u'u'' + 6u^2u'\,dx \overset{\text{using IBP}}{=} \int \partial_x\left[\tfrac{1}{2}(u'(x))^2 + 2(u(x))^3\right]dx$$

$$= 0$$

for nice enough fns u, P & H commute!

This has physical interpretation, but first a

Remark: Let H be Hamiltonian and u flow by this Hamiltonian $\frac{d}{dt}u = \left(\frac{\delta H}{\delta u}\right)'$ (⋆) ← Hamiltonian flow corresponding to H.

Given another Hamiltonian F (means element of algebra $\mathcal{A}$), how does $F(u(t))$ evolve in time?

$$\frac{d}{dt}F(u(t)) \overset{\text{chain rule}}{=} \left\langle \frac{\delta F}{\delta u}(u(t)), \frac{d}{dt}u \right\rangle = \left\langle \frac{\delta F}{\delta u}(u(t)), \left(\frac{\delta H}{\delta u}\right)'(u(t)) \right\rangle = \{F,H\}(u(t))$$

Poisson bracket evaluated at?

$$= (X_H F)(u(t))$$

vector field at the pt $u(t)$ on manifold.

Similarly, w/ ϕ_t denoting flow corresponding to (⋆),

i.e. $u_0 \longmapsto u(t)$ where $u(t)$ solution to $\begin{cases} \frac{d}{dt}u = \left(\frac{\delta H}{\delta u}\right)' \\ u(0) = u_0 \end{cases}$

we can also compute (u initial data u_0)

$$\frac{d}{dt}\widetilde{F\circ\phi_t} = \lim_{s\to 0}\frac{F\circ\phi_t\circ\phi_s - F\circ\phi_t}{s} = \frac{d}{ds}\Big|_{s=0}(\underbrace{F\circ\phi_t}_{\text{"F"}})\circ\phi_s = \{F\circ\phi_t, H\}$$

$\widetilde{F\circ\phi_t} = F(u(t))$

def of derivative

Thus we have written in 2 ways:

$$\{F,H\}\circ\phi_t = \frac{d}{dt}F\circ\phi_t = \{F\circ\phi_t, H\}.$$

In ptcle, solutions to Hamiltonian flow (Φ) conserve the Hamiltonian b/c

$$\frac{d}{dt}H(u(t)) = \{H,H\}(u(t)) = 0$$

anti-commute

bringing this back to thing above $\{P,H\}=0$,

we have also found that solutions to KdV also conserve the momentum!

(again, $\frac{d}{dt}P(u(t)) = \{P,H\}(u(t)) = 0$)

or equivalently!, the class of solutions to KdV is left-invariant under spatial translations.

The following theorem explains this best, and in general (not just for Poisson bracket for KdV as above)

Thm: Poisson commuting Hamiltonians yield commuting flows.

Pf: start w/ 2 Hamiltonians F, G that Poisson commute $\{F,G\}=0$.

Let X_F, X_G be vector fields corresponding to F, G.

Then, recalling final last time, $-[X_F, X_G] = X_{\{F,G\}} = 0$.

We want to prove: Let ϕ_t be flow corresponding to F,

ψ_s - - - - - - - G.

WTS $\phi_t\circ\psi_s = \psi_s\circ\phi_t$ $\forall s,t$ (i.e. $\forall$ pts on manifold u_0, $\phi_t\psi_s u_0 = \psi_s\phi_t u_0$)

e.g. ϕ = KdV flow, ψ = translations, then it is the statement

We do one simple 2nd order Taylor series computation:

Let f be a smooth functional on the manifold

$$\frac{\partial^2}{\partial s\partial t}\Big|_{(s,t)=(0,0)} f(\phi_t\psi_s u) = \frac{\partial}{\partial s}\Big|_{s=0}\underbrace{(X_F f)}_{\text{functional on manifold, the new type}}(\psi_s u) = (X_G(X_F f))(u)$$

taking derivative wrt. t using above fact

taking derivative wrt. s.

OTOH, $\frac{\partial^2}{\partial s \partial t}\Big|_{(s,t)=(0,0)} f(\psi_s \phi_t u) = \frac{\partial}{\partial t}\Big|_{t=0} (X_G f)(\phi_t u) = (X_F(X_G f))(u)$

first der. wrt. s ; again

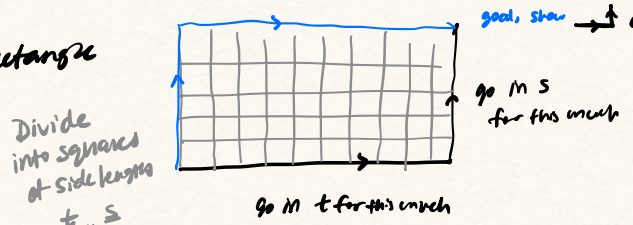
Thus, $\frac{\partial^2}{\partial s \partial t}\Big|_{(s,t)=(0,0)} f(\phi_t \psi_s u) - f(\psi_s \phi_t u) = ([X_G, X_F] f)(u) = 0$.

By Taylor Prod, for general smooth f,

$$f(\phi_t \psi_s u) - f(\psi_s \phi_t u) = \underbrace{0}_{} + \underbrace{0}_{\text{1st derivatives}} + \underbrace{0}_{\text{2nd derivative}} + O(s^3+t^3)$$

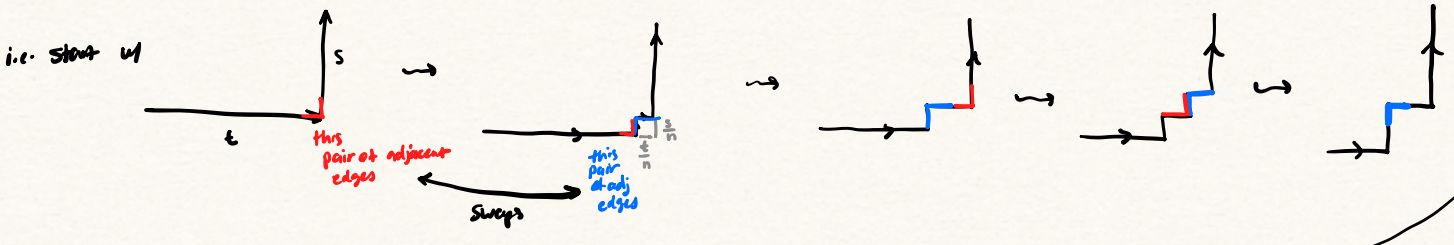
for $s,t \in$ nbhd of $(0,0)$

Divide rectangle



Divide into squares of side length $\frac{t}{n} \times \frac{s}{n}$ (so n^2 many squares)

we express $\phi_t \psi_s - \psi_s \phi_t$ as n^2 many "telescopic" differences where each difference swaps one pair of adjacent edges of a small square by the other pair:



Note that each difference is of the order of $O(\frac{1}{n^3})$

But we only have n^2 many pieces (i.e. terms in sum!)

Letting $n \to \infty$, we get that $f(\phi_t \psi_s) = f(\psi_s \phi_t)$ $\forall$ smooth f.

$\Rightarrow \phi_t \psi_s = \psi_s \phi_t$.

more naive idea/pairs:
$\frac{d}{dt} \phi_t \psi_s(u) = X_F(\phi_t \psi_s u)$
$\frac{d}{dt} u = X_F(u) = \{u, F\}$

Prop: Hamiltonian flows preserve Poisson structure.

i.e. Let H be Hamiltonian, let ϕ_t denote its flow.
Say have 2 other Hamiltonians F, G
WTS $\{F \circ \phi_t, G \circ \phi_t\} = \{F, G\} \circ \phi_t$

pf: Recall already proved this if $G = H$: $\frac{d}{dt} F \circ \phi_t = \{F \circ \phi_t, H\}$ and we know $H = H \circ \phi_t$ b/c H preserved by flow.

$= \{F, H\} \circ \phi_t$

well, let's just calculate $\frac{d}{dt}\left[\{F\circ\phi_t, G\circ\phi_t\} - \{F,G\}\circ\phi_t\right]$

$= \{\{F\circ\phi_t, H\}, G\circ\phi_t\} + \{\{F\circ\phi_t, \{G\circ\phi_t, H\}\}$ (Leibniz rule) $- \{\{F,G\}\circ\phi_t, H\}$.

$\{\{F\circ\phi_t, \{G\circ\phi_t, H\}\}$ // antisym $-\{\{G\circ\phi_t, H\}, F\circ\phi_t\}$ // Jacobi cycle.

$= \{\{F\circ\phi_t, G\circ\phi_t\}, H\} + \{\{H, F\circ\phi_t\}, G\circ\phi_t\}$.

antisym

$= \{\{F\circ\phi_t, G\circ\phi_t\} - \{F,G\}\circ\phi_t, H\}$

$= X_H(\{F\circ\phi_t, G\circ\phi_t\} - \{F,G\}\circ\phi_t)$

So $\{F\circ\phi_t, G\circ\phi_t\} - \{F,G\}\circ\phi_t$ solves init value problem $\begin{cases} \frac{d}{dt} f = X_H(f) \\ f(0) = 0 \end{cases} \Rightarrow f(t) = e^{tX_H} f(0) = 0$

X_F vector field nice as can be

(i.e. uniqueness of sol of ODE, 0 is solution, hence is only solution)

We are now ready for **Lax Pair Formulation of KdV**

Main objects of study for remainder of the quarter:

Lax operator $L(t,x)f = (-\partial_x^2 + q(t,x))f$ (operator acting on f) self adjoint ($q: \mathbb{R}\times\mathbb{R} \to \mathbb{R}$)

"Peter" operator $P(t)f = (-4\partial_x^3 + 3(q\cdot\partial_x + \partial_x q\cdot))f$ antiself adjoint.

and Lax pair formulation to KdV is

q solves KdV $\iff \frac{d}{dt} L(t) = [P(t), L(t)]$.

5/13/24

Lax pair formulation of KdV

Introduced last time,

$$L(t) = -\partial_x^2 + q(t,x)$$ ← only dependence on time

$$P(t) = -4\partial_x^3 + 3(q\partial + \partial q) = -4\partial_x^3 + 6q\partial + 3q'.$$

∂ w/o subscript is ∂_x; multiply by q operator

as we are only interested in real-valued q, L is self adjoint, P is antiself adjoint

we have $\frac{d}{dt}L(t) = [P(t), L(t)] \iff q$ solves KdV.

pf: $\frac{d}{dt}L(t) - [P(t), L(t)] = q_t - [-4\partial^3 + 6q\partial + 3q', -\partial^2 + q]$

(exc. in expanding commutator)

$$= q_t - \{[-4\partial^3, -\partial^2+q] + 6[q\partial, -\partial^2+q] + 3[q', -\partial^2+q]\}$$

b/c ∂^3 and ∂^2 commute; Leibniz rule $[AB,C] = [A,C]B + A[B,C]$; b/c multiplying by ∂ and multiplying by q' commute

$$= q_t - \{-4(q''' + 3q''\partial + 3q'\partial^2 + 0) + 6[q, -\partial^2+q]\partial + 6q[\partial, -\partial^2+q] + 3(q''' + 2q''\partial + 0)\}$$

all ∂^3 hit f, commutator doesn't see it; multiplying by q commute; ∂ and ∂^3 commute

$$= q_t - \{-4q''' - 12q''\partial - 12q'\partial^2 + 6(q'' + 2q'\partial)\partial + 6qq' + 3q''' + 6q''\partial\}$$

$$= q_t - \{-q''' + 6qq'\}$$

$$= q_t + q''' - 6qq'.$$

---

restrict for now to finite dim setting to explain why Lax pair formulation useful.

Rmk: If $L(t), P(t)$ were finite dimensional matrices s.t. $P(t)$ is antiself adjoint and $\frac{d}{dt}L(t) = [P(t), L(t)]$, then the solutions to the initial value problem $\begin{cases} \frac{d}{dt}U(t) = P(t)U(t) \\ U(0) = Id \end{cases}$ would be unitary.

Indeed: $\frac{d}{dt}(U^*(t)U(t)) = [P(t)U(t)]^* U(t) + U^*(t)P(t)U(t) = U^*(t)\underbrace{[P^*(t) + P(t)]}_{=0 \text{ by antiself adj}}U(t) = 0$

$\Rightarrow U^*(t)U(t) = U^*(0)U(0) = Id \Rightarrow U$ is an isometry

$\Rightarrow U$ is unitary

Antiself adjoint matrices: Lie algebra for unitary matrices...?

In infinite dim, the computation above still $\rightsquigarrow$ U is isometry, though can not immediately conclude unitarity.

Unitarity of $U(t,s)$ follows from the well-posedness theory (explained by Fredholm theory) of $\frac{d}{dt}\psi = P(t)\psi$

Then $U(t,s)$ takes $\psi(s)$ and outputs $\psi(t)$ "fundamental solution for this problem"

Cond uniqueness of ——————

gives $U(t,s)\,U(s,t) = Id$

can read out adjoint of $U(t,s)$.

Going back to finite dim. $\frac{d}{dt}(U^* L\, U)(t) = U^* P^* L U + U^*[P,L]U + U^* L P U$

(time derivative hits U^*; time der. hits L; time der. hits U)

$= U^*[-PL + PL - LP + LP]U = 0.$

So $U^*(t)\,L(t)\,U(t) = U^*(0)\,L(0)\,U(0) = L(0)$.

$\Rightarrow \det((U^*LU)(t) - \lambda) = \det(L(0) - \lambda)$

characteristic poly

$\Rightarrow \det(L(t) - \lambda) = \det(L(0) - \lambda)$ $\Rightarrow$ eigenvalues of $L(0)$ preserved as time t

U unitary

Spectrum of L preserved by flow.

14:43 for some back and forth about these topics.

Back in infinite dim, would like to build the "characteristic polynomial" of $L(t) = -\partial^2 + q(t)$

"Fredholm $\det(-\partial^2 + q + k^2)$"

Sadly this doesn't make sense even if $q \equiv 0$.

E.g. on 1 dim torus $\mathbb{T}$, can compute eigenvalues of $-\partial^2 + k^2$ explicitly: $\{n^2 + k^2\}_{n \in \mathbb{Z}}$

b/c $(-\partial^2 + k^2)\, e^{inx} = (n^2 + k^2)\, e^{inx}$

So $\det(-\partial^2 + k^2) = \prod_{n \in \mathbb{Z}} (n^2 + k^2) = \infty$.

However, if we <u>normalize</u> correctly, do get something that converges

By comparison $\prod_{n \in \mathbb{Z} \setminus \{0\}} \frac{n^2+k^2}{n^2} = \frac{1}{\pi k^2} \sinh(\pi k^2)$

Inspired by this, ppl lead to <u>perturbation determinants</u>

written poetically as: $\det\left(\frac{-\partial^2+q+k^2}{-\partial^2+k^2}\right)$

Here $\frac{1}{-\partial^2+k^2}$ stands for the resolvent $R_0(k) = (-\partial^2+k^2)^{-1}$.

↑ correspond to $q=0$

On Fourier side, $\widehat{R_0(k) f}(\xi) = \frac{1}{\xi^2+k^2} \hat{f}(\xi)$ → inverse FT.

$[R_0(k) f](x) = \frac{1}{2k} \int e^{-k|x-y|} f(y)\, dy$

Now ambiguity of if $R_0(k)$ acting on left or right?

i.e. Can interpret $\det\left(\frac{-\partial^2+q+k^2}{-\partial^2+k^2}\right)$ as $\det(1+R_0 q)$ or as $\det(1+qR_0)$.

Note $qR_0(k)$, $R_0(k)q$ is bdd map $L^2 \to L^2$ if $q \in L^2_{loc}$.

adjoints b/c q real-valued

14:56 long story on history of KdV problem pushing down to H^{-1}. (H^{-1} first done on Torus, using Birkhoff coordinates)

Ultimately want potential q in H^{-1}. The above interpretations of $\det(\ldots)$ do not work.

Instead of R_0 on left <u>or</u> right, we do R_0 on <u>left and</u> right (by splitting it in half):

→ interpret as $\det\left(1+\sqrt{R_0(k)}\, q\, \sqrt{R_0(k)}\right)$

We'll show $\sqrt{R_0}\, q\, \sqrt{R_0}$ is <u>Hilbert Schmidt</u> iff $q \in H^{-1}$.

Def: Let A is an operator of L^2 w/ integral kernel $K(x,y)$

$$(Af)(x) = \int K(x,y) f(y)\, dy$$

We say that A is **Hilbert-Schmidt** if $\iint |K(x,y)|^2 dx\, dy < \infty$ $\quad = \|K\|_{L^2_{x,y}} =: \|A\|_{HS}$

(Lin finite-dim, $K(x,y)$ is matrix, A is mult by matrix, and $\|\cdot\|_{HS}$ is sum of squares of all entries)

Note $\|A\|_{op} = \sup_{\|f\|=\|g\|=1} |\langle Af, g\rangle| \geq \sup_{f,g} \left| \iint K(x,y) f(y) dy\ \bar{g}(x) dx \right|$

$$\leq \sup_{\substack{g \\ \text{CBS in} \\ \text{variable } x}} \underbrace{\|g\|_2}_{=1} \left\| \int K(x,y) f(y) dy \right\|_{L^2_x}$$

$$\leq \sup_{\substack{f \\ \text{CBS} \\ \text{in var } y}} \underbrace{\|f\|_2}_{=1} \|K(x,y)\|_{L^2_{x,y}} =: \|A\|_{HS}.$$

Rmk: A is Hilbert Schmidt $\Rightarrow$ A is cpct operator, i.e. $A(B_{0,1}(L^2))$ is precompact in L^2.

How to show set in L^2 is precompact?

Recall in 247, Riesz thm (for all L^p!) says this equiv to $A(B_{0,1})$ being bdd, tight, equicts

Bdd: Let $\|f\|_2 < 1$. Then $\|Af\|_2 \leq \|A\|_{op} \|f\|_2 \leq \|A\|_{HS}$.

Tight: (very little L^2 norm in spatial infinity) unif in f.

$$\|Af\|_{L^2(|x|>R)} \leq \|K\|_{L^2_y L^2_x (|x|\geq R)} \|f\|_2 < \varepsilon \quad \text{for } R = R(\varepsilon) \gg 1 \text{ indep of } f.$$

Equicts: $\|(Af)(x+\delta) - (Af)(x)\|_{L^2_x} \leq \|K(x+\delta, y) - K(x,y)\|_{L^2_y L^2_x} \|f\|_2 < \varepsilon$ for $\delta = \delta(\varepsilon) \ll 1$ b/c translations cts in L^2.

Def: Let A be operator on L^2 w/ cts integral kernel.

The **trace** of A is $\operatorname{tr}(A) := \int K(x,x) dx$

"integrate along diagonal".

Rmk: if A is HS w/ integral kernel $K(x,y)$,

the kernel of A^* (adjoint) is $\overline{K(y,x)}$: $(A^*f)(x) = \int \overline{K(y,x)}\, f(y)\, dy$.

The kernel of A^*A is: ($x \xleftarrow{*}$ "takes z", $z \leftarrow y$ "takes y")

$$\int \overline{K(z,x)}\, K(z,y)\, dz.$$

So $\operatorname{tr}(A^*A) = \iint |K(z,x)|^2\, dz\, dx =: \|A\|_{HS}^2$

In practice this is how we compute HS norm (as trace of A^*A).

5/15/24 Recall: an operator $A: L^2 \to L^2$ is Hilbert-Schmidt (HS) (w integral kernel $K(x,y)$)

if $\|A\|_{HS}^2 := \|K\|_{L^2_{x,y}}^2 < \infty$.

we saw such A are cpct, and had trace finite $\text{tr}(A^*A) = \|A\|_{HS}^2$.

Thm: if A cpct operator on Hilbert space $\mathcal{H}$ (for us, L^2)

then A^*A is nonneg operator (meaning quadratic form $\langle A^*Af, f\rangle \geq 0 \ \forall f \in \mathcal{H}$)

selfadjoint

and cpct

Thm by spectral thm, $\exists$ ON-basis $\{\varphi_n\}$ of $\mathcal{H}$ consisting of eigenvectors of A^*A.

Let $\sigma_n := \sigma_n(A)$ denote singular values of A

$=$ eigenvalues of A^*A, sqrt

i.e. $A^*A\varphi_n = \sigma_n^2 \varphi_n$

$\Rightarrow AA^*(A\varphi_n) = \sigma_n^2 (A\varphi_n)$

i.e. σ_n also singular values for A^* w eigenvector $A\varphi_n$. for $\sigma_n \neq 0$.

Observe $\langle A\varphi_n, A\varphi_m\rangle = \langle A^*A\varphi_n, \varphi_m\rangle = \langle \sigma_n^2 \varphi_n, \varphi_m\rangle = \sigma_n^2 \delta_{n=m}$.

(note $\sigma_n = 0 \Rightarrow A\varphi_n = 0$)

Discarding the $\sigma_n = 0$, we define $\psi_n := \frac{A\varphi_n}{\sigma_n}$.

These ψ_n form ON-set and $A\varphi_n = \sigma_n \psi_n$

$A^*\varphi_n = \sigma_n \varphi_n$

and $AA^*\psi_n = \sigma_n^2 \psi_n$

Thus the nonzero singular values of A are precisely the nonzero singular values of A^* (reversing argument to get reverse direction)

Using that $\{\varphi_n\}$ is a basis for $\mathcal{H}$,

$$Af = A\left(\sum \langle \varphi_n, f\rangle \varphi_n\right) = \sum \langle \varphi_n, f\rangle A\varphi_n = \sum \sigma_n \langle \varphi_n, f\rangle \psi_n.$$

i.e. can write our cpct operator A as $\sum \sigma_n \langle \varphi_n, \bullet\rangle \psi_n$ and $A^* = \sum \sigma_n \langle \psi_n, \bullet\rangle \varphi_n$.

and $\mathrm{tr}(A) := \sum \langle A e_n, e_n \rangle = \sum \sigma_n \langle \psi_n, e_n \rangle$

on basis independent property of trace!

Def: For $1 \le p < \infty$, the **Schatten class** $\mathcal{I}_p$ ("ideal") is defined as

$$\mathcal{I}_p := \{ A \text{ cpct} : \|A\|_{\mathcal{I}_p}^p := \sum |\sigma_n(A)|^p < \infty \}$$

(only nonzero σ_n contribute).

Examples: If A cpct, $\|A\|_{\mathcal{I}_2}^2 := \sum |\sigma_n(A)|^2 = \sum \langle A^* A e_n, e_n \rangle = \mathrm{tr}(A^*A) = \|A\|_{HS}^2$

recall $A^*A e_n = \sigma_n^2 e_n$

Example: Fix $1 \le p < \infty$. If $A \in \mathcal{I}_p$, B bdd operator,

$AB, BA \in \mathcal{I}_p$. (i.e. ideal in {bdd operators}).

In fact even stronger, $\sigma_n(BA) \le \|B\|_{op} \sigma_n(A)$ (*)

In ptcl, $\|BA\|_{\mathcal{I}_p} \le \|B\|_{op} \|A\|_{\mathcal{I}_p}$.

"$\ell^p \le \ell^\infty \cdot \ell^p$"

↳ there is actually a Hölder inequality, but we won't talk about it.

Note that $\sigma_n(AB) = \sigma_n(B^*A^*) \overset{(*)}{\le} \|B^*\|_{op} \sigma_n(A^*) = \|B\|_{op} \sigma_n(A)$.

How to prove (*)?

min max characterization of singular values:

$$\sigma_n(BA) = \min_{f_1 \cdots f_{n-1} \text{ LI}} \max_{g \perp \mathrm{span}\{f_1 \cdots f_{n-1}\},\ \|g\|=1} \|BAg\|.$$

$$\le \|B\|_{op} \|Ag\|,$$

then use min max characterization to recover $\sigma_n(A)$

Remark: Last time we saw $\|A\|_{HS} \ge \|A\|_{op}$.

Here, $\forall 1 \le p < \infty$, $\|A\|_{op} \le \|A\|_{\mathcal{I}_p}$. ($\uparrow$ is $p=2$).

Pf: $\|A\|_{op} = \sup_{\substack{\|f\|=1 \\ \|g\|=1}} |\langle Af, g\rangle|$. But recall $A = \sum \sigma_n \langle \varphi_n, \cdot \rangle \cdot \psi_n$.

$$= \sup_{f,g} \left| \sum \sigma_n \langle \varphi_n, f\rangle \langle \psi_n, g\rangle \right|$$

$$\leq \underset{\text{Hölder}}{\sup_{f,g}} \|\sigma_n\|_{\ell^p_n} \left\| \langle \varphi_n, f\rangle \langle \psi_n, g\rangle \right\|_{\ell^{p'}}$$

$$\leq \|\langle \varphi_n, f\rangle\|_{\ell^{2p}_{n=1}} \|\langle \psi_n, g\rangle\|_{\ell^{2p}_{n=1}}.$$

$$\leq \|\langle \varphi_n, f\rangle\|_{\ell^2} \|\langle \psi_n, g\rangle\|_{\ell^2} \leq \|f\| \cdot \|g\| = 1.$$

$$\leq \|\sigma_n\|_{\ell^p_n} \cdot 1 = \|A\|_{\mathcal{I}_p}.$$

Lemma: Let A_j be HS operators on $L^2(\mathbb{R}^d)$ w/ integral kernels K_j.

Then 1) trace cyclic $\operatorname{tr}(A_1 A_2) = \operatorname{tr}(A_2 A_1)$. (caution, $[A_1, A_2] \in \mathcal{I}_1 \not\Rightarrow \operatorname{tr}(A_1 A_2) = \operatorname{tr}(A_2 A_1)$; need $A_1 A_2$ and $A_2 A_1$ separately in trace class $\mathcal{I}_1$)

2) $|\operatorname{tr}(A_1 \cdots A_\ell)| \leq \prod_{j=1}^{\ell} \|A_j\|_{\mathcal{I}_2}$ $\forall \ell \geq 2$.

Pf: (1) follow from Fubini:

$$\operatorname{tr}(A_1 A_2) = \int \underbrace{\int K_1(x,y)}_{\text{1 end at } x} \underbrace{K_2(y,x)\, dy}_{\text{start at } x} \, dx \underset{\substack{K_1, K_2 \\ \text{in } L^2, \\ \text{Fubini}}}{=} \int\int K_2(y,x) K_1(x,y)\, dx\, dy = \operatorname{tr}(A_2 A_1).$$

(2) follows from CBS

$$|\operatorname{tr}(A_1 \cdots A_\ell)| = \left| \int_{\mathbb{R}^d} K_1(x_1, x_2) K_2(x_2, x_3) \cdots \underbrace{K_\ell(x_\ell, x_1)}\, dx_1 \cdots dx_\ell \right|$$

[applies $A_2 \cdots A_{\ell-1}$ to the function $\underbrace{K_\ell(\cdot, x_1)}_{\text{parameter}}$] evaluated at x_2

$$\underset{\Delta\text{-ineq}}{\leq} \int \left| \langle K_1(x_1, \cdot), A_2 \cdots A_{\ell-1} K_\ell(\cdot, x_1) \rangle_{x_2} \right| dx_1$$

$$\underset{\text{CBS}}{\leq} \int \|K_1(x,y)\|_{L^2_{x,y}} \underbrace{\|A_2 \cdots A_{\ell-1} K_\ell(\cdot, x_1)\|_{L^2_y}}\, dx_1$$

$$\leq \|A_1 \cdots A_\ell\|_{op} \|K_\ell(y,x)\|_{L^2_y}$$

$$\underset{\text{CBS}}{\leq} \|K_1(x,y)\|_{L^2_{x,y}} \|K_\ell(y,x)\|_{L^2_{x,y}} \prod_{j=2}^{\ell-1} \|A_j\|_{op}$$

$\leq \|A_j\|_{HS}$

$\|A_1\|_{HS}$ $\quad$ $\|A_\ell\|_{HS}$

CBS requires 2 guys, everyone else can be in operator norm. ✓

Recall last time perturbation determinants

$$\det\left(\frac{-\partial^2+q+k^2}{-\partial^2+k^2}\right) = \det\left(1+\sqrt{R_0(k)}\, q\, \sqrt{R_0(k)}\right).$$

remember split $R_0(k)$ to deal w/ low regularity issue

Another issue, in order to define $\det(I+A)$ one needs eigenvalues of A summable $\Rightarrow$ $q \in L^1$.

We will address that issue later

where $R_0(k) = (-\partial^2+k)^{-1}$

$$(R_0(k) f)(x) := \frac{1}{2k}\int e^{-k|x-y|} f(y)\, dy.$$

and promised that $\|V\|_{HS} < \infty \iff q \in H^{-1}$.

Calculate:

$$\|\sqrt{R_0(k)}\, q\, \sqrt{R_0(k)}\|_{HS}^2 = \operatorname{tr}\left\{ \left(\sqrt{R_0(k)}\, q\, \sqrt{R_0(k)}\right)^* \left(\sqrt{R_0(k)}\, q\, \sqrt{R_0(k)}\right) \right\}$$

$$= \operatorname{tr}\left(\sqrt{R_0(k)}\, q\, \sqrt{R_0(k)}\, \sqrt{R_0(k)}\, q\, \sqrt{R_0(k)} \right)$$

self adjoint...

cycle

$$= \operatorname{tr}\left(R_0(k)\, q\, R_0(k)\, q \right)$$

$$= \iint \frac{e^{-k|x-y|}}{2k} q(y) \frac{1}{2k} e^{-k|y-x|} q(x)$$

bring back to x — end at y — start at x

so new starting pt is y.

$$= \frac{1}{k}\iint q(x) \frac{e^{-2k|x-y|}}{4k} q(y)\, dy\, dx$$

$R_0(2k)$ kernel!

$$= \frac{1}{k}\langle q, R_0(2k)\, q\rangle$$

$$= \frac{1}{k}\langle \hat{q}, \frac{1}{\xi^2+4k^2}\hat{q}\rangle \quad \text{Plancherel}$$

$$= \frac{1}{k}\int \frac{|\hat{q}(\xi)|^2}{\xi^2+4k^2}\, d\xi =: \frac{1}{k}\|q\|_{H_k^{-1}}^2$$

"vaguely rescaled H^{-1} norm"

Conclusion, $\sqrt{R_0(k)}\, q\, \sqrt{R_0(k)} \in \mathcal{I}_2 \iff \|q\|_{H_k^{-1}} < \infty \iff \|q\|_{H^{-1}} < \infty$.

Rmk. taking $q \geq 0$, $\sqrt{R_0(k)}\, q\, \sqrt{R_0(k)} \geq 0$ (in sense of quadratic forms).

($\sqrt{R_0(k)}$: operator)

$\Rightarrow$ trace of $\uparrow$ = trace class norm of $\uparrow$.

$$\operatorname{tr}\left(\sqrt{R_0(k)}\, q\, \sqrt{R_0(k)}\right) = \operatorname{tr}\left(R_0(k) q\right) = \int \frac{1}{2k} e^{\overbrace{-k|x-x|}^{\text{end at } x \text{, start at } x}} q(x)\, dx = \frac{1}{2}\int q(x)\, dx$$

(cycle)

So $\det\left(1+\sqrt{R_0(k)}\, q\, \sqrt{R_0(k)}\right) := \prod(1+\lambda) < \infty \iff \sum \lambda < \infty \iff q \in L^1$.

We want to treat slower decaying potentials. Lead us to "found object" $\det_2$.

$$\det_2(1+A) := \det(1+A) \cdot e^{-\operatorname{tr}(A)}$$

- $\det$: recall we build up perturbation det by things left invt by KdV flow.
- Adding in factor: should still preserve b/c $\operatorname{tr}(A) \sim \frac{1}{2k}\|q\|_{L^1}$, preserved by KdV flow.

This makes sense for $A \in \mathcal{I}_2$ (less stringent than $A \in \mathcal{I}_1$).

$$\det_2(1+A) = \prod_\lambda (1+\lambda) e^{-\lambda}$$

$$\Rightarrow \log \det_2(1+A) = \sum_\lambda \lambda - \log(1+\lambda) \underset{\text{expand powers}}{=} \sum_\lambda \sum_{\ell \geq 2} \frac{(-1)^\ell}{\ell} \lambda^\ell = \sum_{\ell \geq 2} \frac{(-1)^\ell}{\ell} \operatorname{tr}(A^\ell).$$

This converges if $\|A\|_{HS} < 1$ (by geom series) b/c $|\operatorname{tr}(A^\ell)| \leq \|A\|_{\mathcal{I}_2}^\ell$ if $\ell \geq 2$.

Lemma: $\|A\|_{\mathcal{I}_2}$ (indeed $\leq \frac{1}{k}\|q\|_{H^{-1}}^2$, so taking k large enough (depending on q) is < 1)

5/20/24

By analogy w/

$$\det_2(1+A) = \det(1+A)e^{-\mathrm{tr}(A)}$$

extend Fredholm beyond trace class operators to HS operators

$$-\log \det_2(1+A) = -\log \det(1+A)e^{-\mathrm{tr}A} = \sum_{\ell\geq 2} \frac{(-1)^\ell}{\ell}\mathrm{tr}(A^\ell) < \infty$$

convergence of series

$\|A\|_{I_2} < 1.$

$\|A\|_{HS}$

we define our central object

$$\alpha(k,q) = \sum_{\ell\geq 2} \frac{(-1)^\ell}{\ell}\mathrm{tr}\left\{\left(\sqrt{R_0(ik)}\, q\, \sqrt{R_0(ik)}\right)^\ell\right\}.$$

"power series in potential q"

we computed $\left\|\sqrt{R_0(ik)}\, q\, \sqrt{R_0(ik)}\right\|_{I_2}^2 = \frac{1}{k}\int \frac{|\hat{q}(\xi)|^2}{\xi^2+4k^2} < \frac{1}{k}\|q\|_{H^{-1}}^2 \quad \forall k \geq 1.$

So taking k suff large, $k > 1 + \|q\|_{H^{-1}}^2$, we have < 1, so

the power series defining α converged (is well defined).

---

convergent geom series proportional to dominant term; here, $\ell = 2$.

If we prove α conserved by kdV flow, gives conservation law at H^{-1} level.

Prop: Let q be Schwartz solution to kdV. If $k \geq 1 + \sup_{t\in I}\|q(t)\|_{H^{-1}}^2$

then $\frac{d}{dt}\alpha(k, q(t)) = 0 \quad \forall t \in \mathring{I}.$

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correct, but if H^{-1} preserved, same k keeps working, so

...

pf: $k \geq 1 + \sup_{t\in I}\|q(t)\|_{H^{-1}}^2 \rightarrow \left\|\sqrt{R_0(ik)}\, q\, \sqrt{R_0(ik)}\right\|_{I_2}^2 < \frac{1}{k}\|q\|_{H^{-1}}^2 < 1 \quad \forall t \in I.$

so series of α converge uniformly, so to take $\frac{d}{dt}$, termwise.

$$\frac{d}{dt}\alpha(k;q(t)) = \sum_{l\geq 2} \frac{(-1)^l}{l} \frac{d}{dt} \operatorname{tr}\left\{ \left(\sqrt{R_0}\, q\, \sqrt{R_0}\right)^l \right\}$$

↙ cyclicity of tr.

$$= \sum_{l\geq 2} \frac{(-1)^l}{l} \frac{d}{dt} \operatorname{tr}\left\{ [R_0 q(t)]^l \right\}$$

$$= \sum_{l\geq 2} \frac{(-1)^l}{l} \sum_{k=0}^{l-1} \operatorname{tr}\left\{ (R_0 q(t))^k R_0 \frac{d}{dt} q \underline{(R_0 q)^{l-k-1}} \right\}$$

cycle term $\frac{d}{dt}q$ in last position

$$= \sum_{l\geq 2} (-1)^l \operatorname{tr}\left\{ (R_0 q)^{l-1} R_0 (-q'' + bqq') \right\}$$

To show this is 0, look at term which give same "#s of q's"

(1) quadratic term, only $-q''$ (gives 1 "q"), so only $l=2$ possibly give "2 q's":

i.e. WTS $\operatorname{tr}\{R_0 q\, R_0 q''\} = 0$.

(2) WTS: ($l+1$ many q's")

$$(-1)^l \operatorname{tr}\left\{ (R_0 q)^{l-1} R_0 bqq' \right\} + (-1)^{l+1} \operatorname{tr}\left\{ (R_0 q)^{(l+1)-1} R_0 (-q'') \right\} = 0 \quad \forall l \geq 2$$

$$\Leftrightarrow \operatorname{tr}\left\{ (R_0 q)^l R_0 (-q'') \right\} = \operatorname{tr}\left[(R_0 q)^{l-1} R_0 bqq' \right]$$

notice minus sign!

Pf (1): $\operatorname{tr}\{R_0 q\, R_0 q''\} = \iint \frac{1}{2k} e^{-k|y-x|} q(y) \frac{1}{2k} e^{-k|x-y|} q''(x)\, dy\, dx$

$e^{-k|y-x|}$: take from y back to x; $q(y)$: y; $\frac{1}{2k} e^{-k|x-y|}$: taken from x to y

$$= \frac{1}{k} \iint q'''(x) \underbrace{\frac{e^{-2k|x-y|}}{4k}}_{R_0(2k)\text{ related!}} q(y)\, dy\, dx$$

$$= \frac{1}{k} \langle q''', R_0(2k)\, q \rangle$$

Plancherel $= \frac{1}{k} \int \frac{-i\xi^3 |\hat{q}(\xi)|^2}{\xi^2 + 4k^2}\, d\xi$

$$= \frac{1}{K}\int_0^\infty \frac{-i\xi^3 |\hat{q}(\xi)|^2}{\xi^2+4k^2}\,d\xi + \frac{1}{K}\int_0^\infty \frac{+i\xi^3 |\hat{q}(-\xi)|^2}{\xi^2+4k^2}\,d\xi$$

$\int_{-\infty}^{0}$ change of variable $\xi \leftrightarrow -\xi$.

$$= 0 \quad \text{b/c } q \text{ is real valued} \Rightarrow \hat{q}(-\xi) = \overline{\hat{q}(\xi)}$$

PR (2): start w/ observation that $[\partial, q] = q'$

$$(\partial q - q\partial)f = \partial(qf) - q\cdot(\partial f) = q'\cdot f + q\cdot(\partial f) - q\cdot(\partial f)$$

So $-q''' = -[\partial, [\partial, [\partial, q]]]$.

$$= -[\partial, [\partial, \partial q - q\partial]]$$

$$= -[\partial, \partial^2 q - \partial q\partial - \partial q\partial + q\partial^2] = -[\partial, \partial^2 q - 2\partial q\partial + q\partial^2]$$

$$= -\{\partial^2 q' - 2\partial^2 q\partial + 2\partial q\partial^2 + q'\partial^2\}$$

We now "build" inverse of $R_0(z)$ in each of these terms.

$(-\partial^2+k^2)$

$$= (-\partial^2+k^2)q' - k^2 q' - 2(-\partial^2+k^2)q\partial + 2k^2 q\partial + 2\partial q(-\partial^2+k^2) - 2k^2\partial q + q'(-\partial^2+k^2) - k^2 q'$$

$$= -4k^2[\partial, q] + (-\partial^2+k^2)q' + q'(-\partial^2+k^2) - 2(-\partial^2+k)q\partial + 2\partial q(-\partial^2+k^2)$$

So, plugging this in,

$$\mathrm{tr}\{(R_0 q)^\ell R_0(-q'')\} = -4k^2\,\mathrm{tr}\{(R_0 q)^\ell R_0[\partial, q]\} + \mathrm{tr}\{(R_0 q)^\ell R_0 (-\partial^2+k^2)q'\} + \mathrm{tr}\{(R_0 q)^\ell R_0 q'(-\partial^2+k^2)\}$$

inverses; 1 power; $(R_0 q)^{\ell-1} R_0 q' (-\partial^2+k^2) R_0 q$ inverses

$$-2\,\mathrm{tr}\{(R_0 q)^\ell R_0(-\partial^2+k^2)q\partial\} + 2\,\mathrm{tr}\{(R_0 q)^\ell R_0 \partial q(-\partial^2+k^2)\}$$

inverses; 1 power

$$= -4k^2\,\mathrm{tr}\{(R_0 q)^\ell R_0 \partial q\} + 4k^2\{(R_0 q)^\ell R_0 q\partial\} + \mathrm{tr}\{(R_0 q)^{\ell-1} R_0 q' q'\} + \mathrm{tr}\{(R_0 q)^{\ell-1} R_0 q' q\}$$

commute, look on Fourier side; must commute; same term

$$-2\,\mathrm{tr}\{(R_0 q)^{\ell-1} R_0 q^2\partial\} + 2\,\mathrm{tr}\{(R_0 q)^{\ell-1} R_0 \partial q^2\}$$

$$= -4k^2 \operatorname{tr}\{(R_0 q)^\ell \partial R_0 q\} + 4k^2 \{(R_0 q)^\ell R_0 q\} \quad + \quad 2 \operatorname{tr}\{(R_0 q)^{\ell-1} R_0 q' \}$$

cycle, match up; so this whole underlined bit $= 0$.

$$+ 2 \operatorname{tr}\{(R_0 q)^{\ell-1} R_0 \underbrace{[\partial, q^2]}_{= (q^2)' = 2qq'}\}$$

$$= 6 \operatorname{tr}\{(R_0 q)^{\ell-1} R_0 q q'\}.$$

"α is macroscopic conservation law"

Thm: Let q be Schwarz sol to KdV. Fix $-1 \le s < 0$ and let

$$k \ge 1 + 10 \|q(0)\|_{H^{-1}}^2 .$$ Then

$$\|q(t)\|_{H^s} \lesssim_s \left(1 + \|q(0)\|_{H^{-1}}^2\right)^{|s|} \|q(0)\|_{H^s} \quad \forall t \in \mathbb{R}.$$

"bootstrap, control H^{-1} at $t=0$, control it forever more"

Pf: by cty of the flow, $\exists$ small interval $I \ni 0$ s.t.

$$k \ge 1 + 4 \|q(t)\|_{H^{-1}}^2 \quad \forall t \in I.$$

Then, $\left\| \sqrt{R_0(k)}\, q(t) \sqrt{R_0(k)} \right\|_{I_2}^2$ (Monica missed the 2 oops!) $< \frac{1}{k} \|q(t)\|_{H^{-1}}^2 < \frac{1}{4}$ $\quad \forall t \in I.$

So α is well-defined and conserved on I.

Moreover, α comparable to dominant term:

$$\left| \alpha(k; q(t)) - \frac{1}{2} \left\| \sqrt{R_0(k)}\, q(t) \sqrt{R_0(k)} \right\|_{I_2} \right| \le \sum_{\ell \ge 3} \frac{1}{\ell} \left\| \sqrt{R_0(k)}\, q(t) \sqrt{R_0(k)} \right\|_{I_2}^\ell$$

$$\le \frac{1}{3} \left\| \sqrt{R_0(k)}\, q(t) \sqrt{R_0(k)} \right\|_{I_2}^2 \left(\frac{\left\| \sqrt{R_0(k)}\, q(t) \sqrt{R_0(k)} \right\|_{I_2}}{1 - \|\cdots\|_{I_2}} \right)$$ ($\downarrow$ geom series)

$$\le \frac{1}{3} \left\| \sqrt{R_0(k)}\, q(t) \sqrt{R_0(k)} \right\|_{I_2}^2 \cdot \frac{\frac{1}{2}\|\cdots\|}{1 - \frac{1}{2}\|\cdots\|}$$ (b/c $\|\cdots\|^2$ is small by $\frac{1}{4}$)

$= \frac{1}{10}$

$$\Rightarrow \quad \frac{1}{6} \| \sqrt{R_0(t)}\, q(t)\, \sqrt{R_0(t)} \|_{I_2}^2 \le \alpha(K, q(t)) \le \frac{5}{6} \| \sqrt{R_0(t)}\, q(t)\, \sqrt{R_0(t)} \|_{I_2}^2 \quad \forall t \in I.$$

$\frac{7}{15}$ $\qquad$ $\frac{8}{15}$

α conserved! $\Longrightarrow$

$$\| \sqrt{R_0(t)}\, q(t)\, \sqrt{R_0(t)} \|_{I_2}^2 \le \frac{15}{7} \alpha(K, q(t)) = \frac{15}{7} \alpha(K, q(0)) \le \frac{8}{7} \| \sqrt{R_0(t)}\, q(0)\, \sqrt{R_0(t)} \|_{I_2}^2$$

$$\le \frac{8}{7} \frac{1}{K} \| q(0) \|_{H^1}^2 < \frac{1}{2} \frac{8}{7} \frac{1}{20} = \frac{2}{35}$$

b/c $K \ge 1 + \frac{\Phi}{20} \| q(0) \|_{H^1}^2$

and indeed $\frac{2}{35} < \frac{1}{11}$.

5/22/24

Thm: Let q be a Schwartz solution to KdV. Fix $-1 \le s < 0$, $K \ge 1 + 20\|q(0)\|_{H^{-1}}^2$

Then $\sup_{t\in\mathbb{R}} \|q(t)\|_{H^s} \lesssim (1+\|q(0)\|_{H^{-1}}^2)\,\|q(0)\|_{H^s}$.

Pf: By ctty of flow

$\exists I \ni 0$ s.t. $K \ge 1 + 9\|q(0)\|_{H^{-1}}^2 \quad \forall t\in I$.

Then

$$\|\sqrt{R_0(K)}\, q(t) \sqrt{R_0(K)}\|_{I_2}^2 \le \frac{1}{K}\|q(t)\|_{H^{-1}}^2 < \frac{1}{9} \quad \forall t\in I \quad (\star)$$

Since α given by geom series in q, proportional to leading order term

$$\left|\alpha(K;q(t)) - \frac{1}{2}\|\sqrt{R_0(K)}\, q(t) \sqrt{R_0(K)}\|_{I_2}^2\right| \le \frac{1}{3}\|\sqrt{R_0(K)}\, q(t) \sqrt{R_0(K)}\|_{I_2}^2 \cdot \frac{\|\sqrt{R_0(K)}\, q(t) \sqrt{R_0(K)}\|_{I_2}}{1-\|\sqrt{R_0(K)}\, q(t) \sqrt{R_0(K)}\|_{I_2}}$$

$$< \underbrace{\frac{1}{3}\cdot\frac{\frac{1}{3}}{1-\frac{1}{3}}}_{=\frac{1}{6}} \|\ldots\|$$

so $\frac{1}{3}\|\sqrt{R_0(K)}\, q(t) \sqrt{R_0(K)}\|_{I_2}^2 \le \alpha(K;q(t)) \le \frac{2}{3}\|\sqrt{R_0(K)}\, q(t) \sqrt{R_0(K)}\|_{I_2}^2 \quad \forall t\in I$

Since α is conserved, $\|\sqrt{R_0(K)}\, q(t) \sqrt{R_0(K)}\|_{I_2}^2 \le 3\alpha(K;q(t)) = 3\alpha(K;q(0)) \le 2\|\sqrt{R_0(K)}\, q(0) \sqrt{R_0(K)}\|_{I_2}^2 < 2\cdot\frac{1}{10} < \frac{1}{9}$

bootstrap then yields $I = \mathbb{R}$.

---

above ineq

$$\frac{1}{4K^3}\int \frac{|\hat q(t,\xi)|^2}{\xi^2+1}\,d\xi \le \frac{1}{K}\int \frac{|\hat q(t,\xi)|^2}{\xi^2+4K^2}\,d\xi = \|\sqrt{R_0(K)}\, q(t) \sqrt{R_0(K)}\|_{I_2}^2 \le 2\|\sqrt{R_0(K)}\, q(t) \sqrt{R_0(K)}\|_{I_2}^2 = \frac{2}{K}\int \frac{|\hat q(0,\xi)|^2}{\xi^2+4K^2}\,d\xi \le \frac{2}{K}\|q(0)\|_{H^{-1}}^2$$

$(4K^2 \ge 1)$

$$\frac{1}{4K^3}\int \frac{|\hat q(t,\xi)|^2}{\xi^2+1}\,d\xi = \frac{1}{4K^3}\|q(t)\|_{H^{-1}}^2$$

$$\Rightarrow \|q(t)\|_{H^{-1}}^2 \le 8K^2\|q(0)\|_{H^{-1}}^2$$

Taking $K = K_0 := 1 + 20\|q(0)\|_{H^{-1}}^2$ gives the claim when $s=-1$.

"a priori bounds at level of H^{-1}"

for remaining s,

Note $\int_{k_0}^{\infty} \frac{1}{\xi^2+4k^2} k^{1+2s}\, dk \sim (\xi^2+k_0^2)^s$

"Build less fast decreasing fcn by adding fast decreasing fcn"

Pf: if $k_0 \geq |\xi|$ $\int_{k_0}^{\infty} \frac{1}{\xi^2+4k^2} k^{1+2s}\, dk \sim \int_{k_0}^{\infty} k^{-1+2s}\, dk \sim k_0^{2s} \sim (k_0+|\xi|^2)^s$

$k \geq k_0 \geq |\xi| \Rightarrow \xi^2+4k^2 \sim k^2$

if $k_0 < |\xi|$, $\int_{k_0}^{\infty} \frac{k^{1+2s}}{\xi^2+4k^2}\, dk = \int_{k_0}^{|\xi|} \dots + \int_{|\xi|}^{\infty} \dots$

$\lesssim \int_{k_0}^{|\xi|} \frac{k^{1+2s}}{|\xi|^2}\, dk + \int_{|\xi|}^{\infty} k^{-1+2s}\, dk$

$2+2s>0$ ← if integrable at 0 so extend to 0 so FTC lower endpt is 0.

$\leq \int_0^{|\xi|} \frac{k^{1+2s}}{|\xi|^2}\, dk$

$\frac{|\xi|^{2+2s}}{|\xi|^2}$

$\sim |\xi|^{2s} \sim (k_0^2+\xi^2)^s$.

integrate both $\int_{k_0}^{\infty} k^{1+2s}\, dk$.

$$\int_{k_0}^{\infty} \int \frac{|\hat{q}(t,\xi)|^2}{\xi^2+4k^2}\, d\xi\, k^{1+2s}\, dk \leq 2 \int_{k_0}^{\infty} \int \frac{|\hat{q}(0,\xi)|^2}{\xi^2+4k^2}\, d\xi\, k^{1+2s}\, dk$$

above Note ⇒

$$\frac{1}{k_0^{2s}} \int \frac{|\hat{q}(t,\xi)|^2}{(\xi^2+1)^{|s|}}\, d\xi \lesssim \int \frac{|\hat{q}(t,\xi)|^2}{(\xi^2+k_0^2)^{|s|}}\, d\xi \lesssim \int \frac{|\hat{q}(0,\xi)|^2}{(\xi^2+k_0^2)^{|s|}}\, d\xi \lesssim \int \frac{|\hat{q}(0,\xi)|^2}{(\xi^2+1)^{|s|}}\, d\xi = \|q(0)\|_{H^s}^2$$

$$\Rightarrow \boxed{\|q(t)\|_{H^s} \lesssim k_0^{|s|} \|q(0)\|_{H^s}}$$ a priori bds in H^s.

Cor: Fix $-1 \le s < 0$. The orbit of a Schwartz sol to KdV is equicts in H^s

Pf: by Plancherel, tightness on Fourier side $\Longleftrightarrow$ equicts physical side

$\Downarrow$

So suffices to show $\forall \varepsilon > 0$, $\exists K = K(\varepsilon)$ s.t.

$$\sup_{t \in \mathbb{R}} \int_{|\xi| > K} \frac{|\hat{q}(t,\xi)|^2}{(\xi^2+1)^{|s|}} \, d\xi < \varepsilon.$$

In previous results we showed that

$$\int_{|\xi| \ge K} \frac{|\hat{q}(t,\xi)|^2}{(\xi^2+1)^{|s|}} d\xi \le \int \frac{|\hat{q}(t,\xi)|^2}{(\xi^2+k^2)^{|s|}} d\xi \lesssim \int \frac{|\hat{q}(0,\xi)|^2}{(\xi^2+k^2)^{|s|}} d\xi$$

(in place of $\int_{k_0}^{\infty}$ above of $\int_{K}^{\infty}$)

$$\lesssim \int_{|\xi| \le h} \left(\frac{h^2}{k^2}\right)^{|s|} \frac{|\hat{q}(0,\xi)|^2}{(\xi^2+1)^{|s|}} d\xi + \int_{|\xi| \ge h} \frac{|\hat{q}(0,\xi)|^2}{(\xi^2+1)^{|s|}} d\xi.$$

$$\lesssim \left(\frac{h}{k}\right)^{2|s|} \|q(0)\|_{H^s} + \text{higher freq pieces of } H^s \text{ norm of init data.}$$

If have many init data all bdd unif. in H^s can choose h just based on this unif bd, i.e. choose h uniform in choice of init data in bdd class:

Then, choose $h = h(\varepsilon)$ large to make 2nd term small (MCT and bddness of H^s norm of $q(0)$),

and then choose K large enough that 1st term small

Cor: If $Q \in S(\mathbb{R})$ is bdd and equicts in H^s for $-1 \le s < 0$,

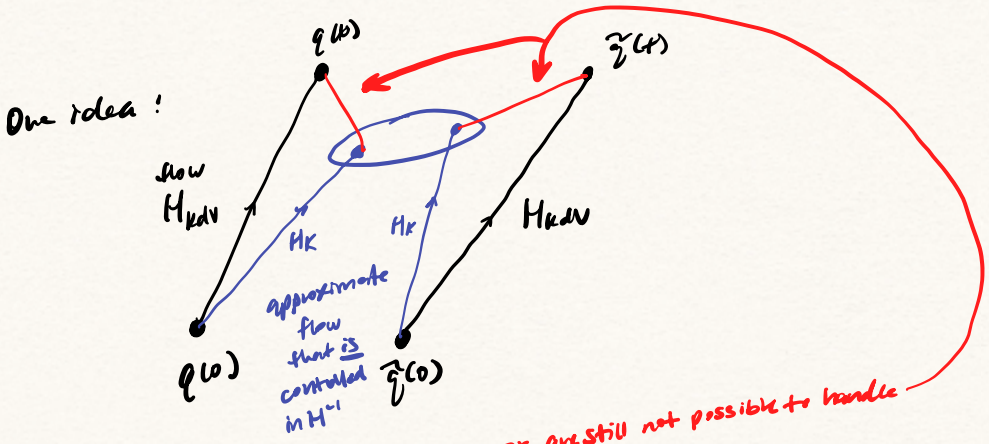
then orbits emanating from Q form a bdd and equicts in H^s.

17:29

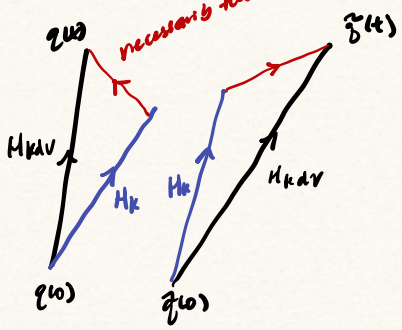
Ill posedness of KdV in H^{-1}.

contraction mapping fails that $\Rightarrow$ crossing control, but we know $s < -\frac{3}{4}$

data $\to$ sol not unif cts.



The idea is to find H_K flow commuting:



17:34

start computation.

$$\alpha(k;q) = \sum_{l\geq 2} \frac{(-1)^l}{l} \mathrm{tr}\{(R_0 q)^l\}$$

$$= \frac{1}{2k}\int \frac{|\hat{q}|^2}{\xi^2+4k^2}\,d\xi - \frac{1}{3(2\pi)^{3/2}}\iiint \frac{\hat{\xi}(\xi-\mu)}{\xi^2+k^2}\frac{1}{\mu^2+k^2}\hat{q}(\mu-\eta)\frac{1}{\eta^2+k^2}\hat{q}(\eta-\xi)\,d\xi\,d\eta\,d\mu + O(k^{-7})$$

(ξ_2: back to η; η: back to ξ; start at ξ)

lower order in k

at least 4 R_0, each k^2, but integral gives ak.

expand in $\frac{1}{k}$ powerseries

$$= \underbrace{\frac{1}{2k\cdot 4k^2}\int |\hat{q}|^2 \sum_{l\geq 0}\left(\frac{-\xi^2}{4k^2}\right)^l}_{\text{Plancherel}} - \underbrace{\frac{1}{3(2\pi)^{3/2}}\iiint \frac{\hat{q}(\xi_1)\,\hat{q}(\xi_2)\,\hat{q}(-\xi_1-\xi_2)}{(\xi_1^2+k^2)\left((\xi_1+\xi_1)^2+k^2\right)\left((\xi_1+\xi_1+\xi_2)^2+k^2\right)}\,d\xi_1\,d\xi_2}_{\text{integrate wrt. } \xi_1 \text{ (easy for computer)}} + O(k^{-7})$$

$$= \frac{1}{8k^3}\int |q|^2 - \frac{1}{32k^5}\int |q'|^2 + O(k^{-7})$$

$$-\frac{1}{3(2\pi)^{1/2}K}\iint \frac{\hat{q}(\xi_1)\,\hat{q}(\xi_2)\,\hat{q}(-\xi_1-\xi_2)\,(\xi_1^2+\xi_1\xi_2+\xi_2^2+12k^2)}{(\xi_1^2+4k^2)(\xi_2^2+4k^2)((\xi_1+\xi_2)^2+4k^2)}\,d\xi_2\,d\xi_1$$

(appears when we integrate; K appears after integral)

$$= \frac{12k^2}{(4k^2)^3}\iint \hat{q}(\xi_1)\,\underbrace{\hat{q}(\xi_2)\hat{q}(-\xi_1-\xi_2)}_{\text{convolution}}\,d\xi_2\,d\xi_1$$

+ next term has $(k^2)^4$ in denom, and will contain $\frac{1}{K}$, indeed $O(\frac{1}{k^7})$.

$$\sqrt{2\pi}\langle q, \underbrace{\hat{q}*\hat{q}}_{q^2}\rangle = \sqrt{2\pi}\int q(x)^3\,dx$$

$$= \frac{1}{4k^3}\underbrace{P(q)}_{\text{momentum}} - \frac{1}{16k^5}\int \tfrac{1}{2}|q'|^2 + q(x)^3\,dx + O(k^{-7})$$

$= \frac{1}{4k^3} P(q) - \frac{1}{16k^5} H_{KdV} + O(k^{-7})$

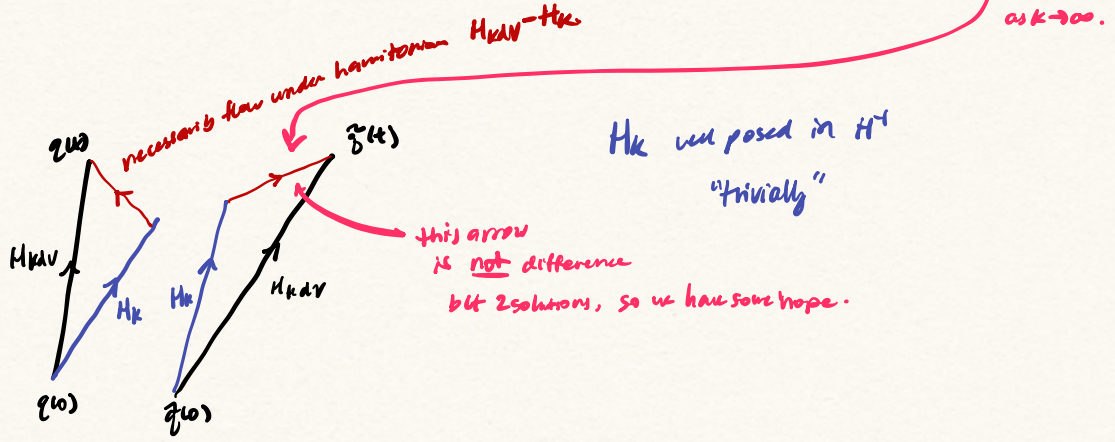
conserved quantities of Gardner... Miura!!!

(α was built from spectral data ... 17:48)

commute w/ H_K b/c $P(q)$, H_{KdV}, ... conserved by H_{KdV}. "everybody commute w everybody" 17:57.

Choose $H_K = -16k^5 \alpha(k; \beta) + 4k^2 P(q) = H_{KdV} + O(k^{-2}) \xrightarrow[\text{Kamis Hamilt converge}]{k\to\infty} H_{KdV}$.

the hope is that these errors go to 0 as $k \to \infty$.



H_K well posed in H^1 "trivially"

5/24/24

Recall $H_k(q) := -16k^5 \alpha(k;q) + 4k^2 p(q) = H_{kdV} + O(k^{-2})$.

(Hamiltonians)

Claim: flow corresponding to H_k is globally well posed in H^{-1}.

The flow corresponding to $H_k(q)$ is (recalling Poisson str discussed previously for KdV)

$$\frac{d}{dt} q = \left(\frac{\delta H_k}{\delta q}\right)' = -16k^5 \left(\frac{\delta \alpha(k)}{\delta q}\right)' + 4k^2 q'$$

- $\frac{\delta H_k}{\delta q}$: functional derivative
- $\frac{\delta \alpha(k)}{\delta q}$: "forcing term"
- q': functional derivative of momentum is q.

Recall functional derivative defined to satisfy $\int \frac{\delta \alpha(k)}{\delta q}(x) f(x)\,dx = \frac{d}{ds}\Big|_{s=0} \alpha(q+sf)$.

(k) omit from notation b/c it's always fixed k.

$$\alpha(q+sf) = \sum_{l\geq 2} \frac{(-1)^l}{l} \operatorname{tr}\left\{ (R_0(q+sf))^l \right\}$$

$$\frac{d}{ds}\Big|_{s=0} \alpha(q+sf) = \sum_{l\geq 2} (-1)^l \operatorname{tr}\left\{ (R_0 q)^{l-1} R_0 f \right\}$$

derivative attaches one of l copies of $(R_0 \cdot)^l$, by cyclicity can always assume it's last copy, so l total copies of something, cancel $\frac{1}{l}$.

$$= -\sum_{l\geq 1} \operatorname{tr}\left\{ (-1)^l (R_0 q)^l R_0 f \right\}$$

Let $R(k,q)$ denote the resolvent of $-\partial^2 + q + k^2$.

Resolvent identity says $R(k,q) - \underbrace{R_0(k)}_{\| \; R(k,0) \text{ take potential } = 0} = -R(k,q)\,\underbrace{(q-0)}_{\text{difference of potentials}}\,R_0(k)$

(i.e. hint look at $\frac{1}{-\partial^2 + q + k^2} - \frac{1}{-\partial^2 + k^2}$ ..)

$$= \dots$$

iterating ...

$$= \sum_{l\geq 1} (-1)^l (R_0 q)^l R_0$$ "Born series"

So $\frac{d}{ds}\Big|_{s=0} d(q+sf) = -\operatorname{tr}\{(R(k,q) - R_0(k))f\} = +\frac{1}{2k}\int f(x)\,dx \; - \int G(x,x;k,q) f(x)\,dx$

(integral kernel associated to resolvent evaluated at diagonal.)

We know $\operatorname{tr}\{R_0(k)f\}$, exact computation ↺ 18:01

where $G(x,y;k,q)$ denotes the integral kernel of the resolvent $R(k,q)$

(well defined, and this so it makes sense to restrict to zero once set diagonal)

and we denote $g(x;k,q) = G(x,x;k,q)$ restrict to diagonal

we get $\frac{\delta d(k)}{\delta q} = \frac{1}{2k} - g(x)$ ("diagonal green fn associated to q")

$$\Rightarrow \boxed{\frac{d}{dt}q = 16k^5 g' + 4k^2 g'} \quad \text{this is the Hk flow.}$$

Thm: Fix $q \in H^1(\mathbb{R})$. Then $\exists!$ self adj operator L associated to the quadratic form

Lax operator $-\partial^2 + q$ ⟶ quadratic form $\int |\psi'(x)|^2 + q(x)|\psi(x)|^2\,dx$

(perturbation)

w/ domain H^1. It is semibounded.

Moreover, if $k \geq 1 + 2\|q\|_{H^1}^2$, then resolvent $R(k,q)$ given by norm convergent series

$$R(k,q) = \sum_{l\geq 0} (-1)^l (R_0(k)q)^l R_0(k)$$ exactly Born series above

$$= \sum_{l\geq 0} (-1)^l \sqrt{R_0(k)} \left(\sqrt{R_0(k)}\, q\, \sqrt{R_0(k)}\right)^l \sqrt{R_0(k)}$$

(recall operator norm bdd by HS norm.)

Pf: To construct L we use **KLMN Thm** — in Reed & Simon.

Thm (KLMN): Let A positive self adj operator (in our case, the Laplacian) w/ quadratic form domain $Q(A)$ (in our case H^1).

Let $\beta(f,g)$ be symm bilinear form (in our case $\beta(f,g) = \overline{\beta(g,f)}$ like Hermpod)

"perturbation of A" — quadratic form assoc to A.

s.t. $|\beta(f,f)| \le a \underbrace{|\langle f, Af\rangle|}_{= \langle f, Af\rangle \text{ bc } A \text{ pos.}} + b\|f\|^2 \quad \forall f \in Q(A)$ for some $0 \le a < 1$, $b \in \mathbb{R}$

Then $\exists!$ self adj operator C w/ $Q(C) = Q(A)$

s.t. $\langle f, Cg\rangle = \langle f, Ag\rangle + \beta(f,g) \quad \forall f,g \in Q(C)$

($\langle f, Cg\rangle$ = quadratic form associated to C)

(And C bdd below by $-b$) ← easy, just Δ-ineq.

So by this thm, to construct L, suffices to show

$$\int q(x)\,|h(x)|^2\,dx \le a\int |\psi'(x)|^2\,dx + b\int |\psi|^2(x)\,dx. \quad \text{for some } 0 \le a < 1,\ b \in \mathbb{R}.$$

We ultimately want $q \in H^{-1}$, so ask this we can do is

$$\int q(x)\,|h(x)|^2\,dx \le \|q\|_{H^{-1}}\,\||\psi|^2\|_{H^1} \lesssim \|q\|_{H^{-1}}\,\|\psi\|_{H^1}\,\|\psi\|_{L^\infty}$$

How to show $\|\psi\|_{L^\infty}$ controlled by H^1?

$$\|\psi\|_{L^\infty} \le \|\hat\psi\|_{L^1} \le \|\underbrace{(\xi^2+\varkappa^2)^{1/2}\hat\psi}_{\text{Parseval side}}\|_{L^2_\xi}\,\|\underbrace{\frac{1}{(\xi^2+\varkappa^2)^{1/2}}}_{\text{Mikapp}}\|_{L^2_\xi} \quad \text{for } \varkappa \ge 1.$$

$$\lesssim \varkappa^{1/2}\|q\|_{H^{-1}} \int |\psi'(x)|^2 + \varkappa^2|\psi(x)|^2\,dx$$

$\varkappa^{1/2}\|q\|_{H^{-1}} < 1$ if $\varkappa \gg \|q\|^2_{H^{-1}}$

Also wanted to say... $\|q\|_{H^{-1}}\,\|\psi\|^2_{H^1} \le \varepsilon\|\psi'\|_2 + \frac{1}{\varepsilon}\|q\|^2_{H^{-1}}$ (doesn't depend on ψ)

18:24

Finally, $\left\|\sum_{\ell\ge 0}(-1)^\ell \sqrt{R_0}\,(\sqrt{R_0}\,q\,\sqrt{R_0})^\ell \sqrt{R_0}\right\|_{op} \lesssim \sum_{\ell\ge 0}\|\sqrt{R_0}\|^2_{op}\,\underbrace{\|\sqrt{R_0}\,q\sqrt{R_0}\|^\ell_{op}}_{\text{bdd by HS norm}} \le \sum_{\ell\ge 0}\frac{1}{k^2}\left(\frac{1}{\sqrt{k}}\|q\|_{H^{-1}}\right)^\ell \lesssim \frac{1}{k^2}$

18:25 "as expected"

Previously, we fixed q arbitrary and looked at k large ($\geq 1 + 20\|q\|_{H^1}^2$)

By scaling, equivalent to look at $k \geq 1$, and restrict to all q small

This is how we phrase the next thm.

equivalent to take q in ball B_δ in H^{-1}, and look only at $k \geq 1 + 20 \sup_{q \in B_\delta} \|q\|_{H^{-1}}$

Thm: $\exists \delta > 0$ s.t. $\forall k \geq 1$ we have the following:

1) $\forall q \in B_\delta := \{ q \in H^{-1}, \|q\|_{H^{-1}} < \delta \}$ the integral kernel $G(x,y;k,q)$ is cts in x,y.

↳ Thm we may define $g(x,k,q) = G(x,x,k,q)$.

2) This means that $q \longmapsto g - \frac{1}{2k}$ and $q \mapsto k - \frac{1}{2g}$

are real analytic diffeos from B_δ to H^1!

18:29.

3) if $q \in S(\mathbb{R})$, then both $g - \frac{1}{2k}, k - \frac{1}{2g} \in S(\mathbb{R})$ — subtract $\frac{1}{2k}$ to go to 0 at ∞.

"g gains 2 degrees of regularity compared to q"

In fact Schwartz in quantitative way

equivalent $\longleftrightarrow$ $\|\partial_x^n g'\|_2 \lesssim \|q\|_{H^{n-1}}$

$\|g - \frac{1}{2k}\|_{H^{n+1}} \lesssim \|q\|_{H^{n-1}}$ and $\|\langle x \rangle^n g'\|_2 \lesssim \|\langle x \rangle^n q\|_{H^{-1}}$

Pf: (1) we'll show $G - G_0 \in H^1_x H^1_y$ (where G_0 = green fct for free resolvent ($q=0$))

then, in $d=1$, $H^1 \hookrightarrow C$, and G_0 cts (we already know), so G cts in x,y.

So we define $g(x) = \sum_{\ell \geq 0} (-1)^\ell \langle \delta_x, \sqrt{R_0} (\sqrt{R_0} q \sqrt{R_0})^\ell \sqrt{R_0} \delta_x \rangle$

↑ integral kernel restricted to diag.

$$= \frac{1}{2k} + \sum_{\ell \geq 1} (-1)^\ell \langle \delta_x, \sqrt{R_0} (\sqrt{R_0} q \sqrt{R_0})^\ell \sqrt{R_0} \delta_x \rangle$$

↑ "kernel of R_0 restr. to diag"

How to show)?

Test against H^{-1} fns: let $f, h \in H^{-1}$.

$$\left| \iint h(x) (G-G_0)(x,y) f(y) \, dx dy \right| \quad \text{WTS} \quad \leq \|f\|_{H^{-1}} \|h\|_{H^{-1}}.$$

$$= \left| \mathrm{tr}\{ h (R-R_0) f \} \right|$$

$$\leq \underset{\text{Born series}}{\sum_{\ell \geq 1}} \left| \mathrm{tr}\{ h (R_0 g)^\ell R_0 f \} \right|$$

$$= |\langle h, (R-R_0) f \rangle|$$

if CBS now, put h in L^2, $(R-R_0) f$ in L^2.

instead insert operators so left guy in L^2 gets $\|h\|_{H^{-1}}$.

$$= \left| \left\langle \frac{1}{\sqrt{-\partial^2 + k^2}} h, \; \sqrt{-\partial^2+k^2} (R-R_0) f \right\rangle \right|$$

$\frac{1}{\sqrt{-\partial^2+k^2}} h$: L^2

$\sqrt{-\partial^2+k^2} (R-R_0) \sqrt{-\partial^2+k^2}$: op norm

$\frac{1}{\sqrt{-\partial^2+k^2}} f$: L^2

$$\leq \|f\|_{H^{-1}} \|h\|_{H^{-1}} \sum_{\ell \geq 1} \| \sqrt{R_0} g \sqrt{R_0} \|_{HS}^{\ell} \leq \|f\|_{H^{-1}} \|h\|_{H^{-1}} \left(\frac{\#}{\sqrt{k}} \|g\|_{H^{-1}} \right)$$

since k large enough that op norm ≤ 1, Born series

leading order term

$\|g\|_{H^{-1}} < \delta$, $k \geq 1$

$$\leq \|f\|_{H^{-1}} \|h\|_{H^{-1}}$$

Remember forcing term is g' in H_k slow

PDE theory g' Lipschitz $\Rightarrow$ ☺.

But g' is real analytic by (2) above

Last time we left off with

Prop: $\exists \delta > 0$ s.t. $\forall k \geq 1$,

1) $q \in B_\delta = \{q \in H^{-1} : \|q\|_{H^{-1}} < \delta\}$, the resolvent $R(k,q) = (-\partial^2 + q + k^2)^{-1}$ admits a (continuous) kernel $G(x,y;k,q)$. Furthermore, we may define diagonal

(we proved it's $\subset H^1$ in each component, $H^1 \hookrightarrow$ cts)

$$g(x;k,q) = G(x,x;k,q)$$

$$= \frac{1}{2k} + \sum_{\ell \geq 1} (-1)^\ell \left\langle \delta_x, \sqrt{R_0}\,(\sqrt{R_0}\, q \sqrt{R_0})^\ell \sqrt{R_0}\, \delta_x \right\rangle \quad (1)$$

"Born series"

2) mappings $q \mapsto g - \frac{1}{2k}$, $q \mapsto K - \frac{1}{2g}$ are real analytic diffeomorphisms from B_δ into H^1.

3) If $g \in S(\mathbb{R})$, then $g - \frac{1}{2k}, K - \frac{1}{2g} \in S(\mathbb{R})$

Moreover $\forall n \geq 0$, g gains 2 degrees of regularity over q (in the sense of derivatives)

$$\|\partial_x^n g'\|_2 \lesssim \|q\|_{H^{n-1}}$$

$$\|\langle x \rangle^n g'\|_2 \lesssim \|\langle x \rangle^n q\|_{H^{-1}}.$$

Pf. (1) last time.

(2) we prove that result actually lives in H^1.

By duality.

$f \in H^{-1}$, want to bound $\left| \int f (g - \frac{1}{2k}) dx \right|$ only in terms of $\|f\|_{H^{-1}}$.

$$\leq \sum_{\ell \geq 1} \left| \mathrm{tr}\{ f \sqrt{R_0} (\sqrt{R_0}\, q \sqrt{R_0})^\ell \sqrt{R_0} \} \right| \leq \sum_{\ell \geq 1} \|\sqrt{R_0} f \sqrt{R_0}\|_{HS} \|\sqrt{R_0}\, q \sqrt{R_0}\|_{HS}^\ell$$

$\leq$ Δ-ineq using Born series (1) above for $g - \frac{1}{2k}$

cycle

$$\leq \sum_{\ell \geq 1} \frac{1}{k} \|f\|_{H^{-1}} \left(\frac{1}{k} \|q\|_{H^{-1}} \right)^\ell$$

$$\leq \frac{2}{k} \|f\|_{H^{-1}} \cdot \delta.$$

$\|q\|_{H^{-1}} < \delta$, $k \geq 1$

$$\Rightarrow \|g - \tfrac{1}{2k}\|_{H^1} \leq \frac{2}{k} \delta.$$

H^1 controls L^∞

So taking $\delta \leq \frac{1}{8}$ (Suff small to absorb implicit constant $\|\hat{f}\|_{L^\infty} \leq \|f\|_1 \leq \|f\|_{H^1} \left\| \frac{1}{\langle \xi \rangle} \right\|_2$) $\Rightarrow \frac{1}{4k} \leq g \leq \frac{3}{4k}$ (2) so g is never zero (so $k - \frac{1}{2g}$ makes sense).

Look at this argument, we have actually proven that Born series (1) converges in H^1.

$\Rightarrow g$ is real analytic function of q.

Note that we may write $K - \frac{1}{2g} = K\,\frac{2K(g-\frac{1}{2K})}{1+2K(g-\frac{1}{2K})}$ and $f \mapsto \frac{f}{1+f}$ is a diffeomorphism from small ball in H^1 into H^1

So $g \mapsto K - \frac{1}{2g}$ is a diffeo from B_S into H^1

if $g \mapsto g - \frac{1}{2K}$ is a diffeo from B_S into H^1.

Together w/ (2) this would also show that $g \mapsto K - \frac{1}{2g}$ is real analytic (reciprocal of real analytic fn never 0).

So, to prove 3), remains to show . (using inverse fn theorem)

By resolvent identity $R(K, q+sf) - R(K,q) = -R(K;q+sf)\,sf\,R(K;q)$

$q+sf$: perturbation

$R(K,q+sf)$: numerator of Newton difference quotient

dividing by s and letting $s \to 0$, get

$$dg|_q(f) = -\int G(x,y)\, f(y)\, G(y,x)\, dy$$

$dg|_q$: derivative of g at potential q; f: "direction f"

$G(x,y)$: $x \leftarrow$ variable, $y \leftarrow$ fixed

$G(y,x)$: symmetric in x, $y \leftarrow$ variable

specializing $\Rightarrow$ $dg|_{q=0}(f) = -\int \frac{1}{2K} e^{-K|x-y|} f(y) \frac{1}{2K} e^{-K|y-x|}\, dy = -\frac{1}{K}\int \frac{1}{4K} e^{-2K|x-y|} f(y)\, dy$

trick from before, $R_0(2K)$ kernel!

$= -\frac{1}{K} R_0(2K) f$.

$R_0(2K)$: smooth 2 degrees; f: rough fn in H^{-1}

$\Rightarrow$ $dg|_{q=0}$ is an isom from H^{-1}_K into $H^1_K = \{f : (\xi^2+4K^2)^{\frac{1}{2}} \hat{f} \in L^2\}$

$\{f : (\xi^2+4K^2)^{-1/2} \hat{f} \in L^2\}$

Thm for δ small, $g - \frac{1}{2K}$ is diffeo from small ball $\{g : \|g\|_{H^{-1}_K} < \delta\}$ into H^1_K.

δ: depending on K

As $\|g\|_{H^{-1}_K} \le \|g\|_{H^{-1}}$ and $K \ge 1 \Rightarrow \|f\|_{H^1} \le \|f\|_{H^1_K}$.

This shows the mapping $q \mapsto g - \frac{1}{2k}$ is a diffeo from B_δ into H^1.

Almost fine ...

To see that δ may be chosen independently of k, we use contraction mapping pf of inverse fn thm.

To this end, suffices to show

$$\underbrace{\| dg|_q - dg|_{q=0} \|_{H^{-1}}}_{\text{map } H^{-1} \text{ to } H^1} \leq \delta \, \| (dg|_{q=0})^{-1} \|^{-1}_{H^1_k \to H^{-1}_k}$$

(recall in IFT, then the mapping $\Phi(q) := q + (dg|_{q=0})^{-1}(f - g(q))$ is a contraction $H^{-1}_k \to H^1_k$

"given f in image, find q s.t. $g(q) = f$" — want to

We $\| d\Phi|_q \|_{H^1_k \to H^1_k} = \| 1 - (dg|_{q=0})^{-1} dg|_q \|_{H^{-1}_k \to H^{-1}_k} \leq \| (dg|_{q=0})^{-1} \|_{H^1_k \to H^1_k} \| dg|_{q=0} - dg|_q \|_{H^{-1}_k \to H^1_k} \leq \delta$

pf of: by duality $f \in H^1_k$, $h \in H^{-1}_k$

$$\left| \langle (dg|_q - dg|_{q=0})(f), h \rangle \right| = \left| \operatorname{tr} \{ Rf R h - R_0 f R_0 h \} \right|$$

↓ expand R in Born series in both locations, q and $q=0$.

$$\leq \sum_{k+j \geq 1} \left| \operatorname{tr} f (R_0 q)^k R_0 f (R_0 q)^j R_0 h \right|$$

(at least one q appears.)

↓ usual trick, split $R_0 = \sqrt{R_0}\sqrt{R_0}$ and cycle $\sqrt{R_0}$ around.

$$\leq \sum_{k+j \geq 1} \| \sqrt{R_0} f \sqrt{R_0} \|_{HS} \| \sqrt{R_0} h \sqrt{R_0} \|_{HS} \| \sqrt{R_0} q \sqrt{R_0} \|^{k+j}_{HS}$$

$$\leq \frac{1}{k} \| f \|_{H^1_k} \frac{1}{k} \| h \|_{H^{-1}_k} \cdot 2\delta \quad \text{(sum)}$$

$$\leq 2\delta \, \| f \|_{H^{-1}_k} \| h \|_{H^{-1}_k} \left(\frac{1}{k} = \| (dg|_{q=0})^{-1} \|^{-1}_{H^1_k \to H^{-1}_k} \right)$$

---

3) if $g \in S(\mathbb{R})$, then $g - \frac{1}{2k}, \; k - \frac{1}{2g} \in S(\mathbb{R})$

Moreover $\forall n \geq 0$, g (in the self-adjoint setting) gains 2 degrees of regularity over g $\quad \| \partial_x^n g' \|_2 \lesssim \| g \|_{H^{n-1}}$

$\| \langle x \rangle^n g' \|_2 \lesssim \| \langle x \rangle^n g \|_{H^{-1}}$.

Assume $g \in S(\mathbb{R})$. How to translate derivative of g to derivative of g?

KdV spatial translation $g(x+h; g) = g(x; g(\cdot + h))$.

$$\left| \int \partial_x^n \left(g - \tfrac{1}{2k} \right) f \right| \leq \sum_{l \geq 1} \| \sqrt{R_0} f \sqrt{R_0} \|_{HS} \sum_{\sigma_1 + \cdots + \sigma_l = n} C_{\sigma_1 \cdots \sigma_l} \prod_{i=1}^{l} \| \sqrt{R_0}\, g^{(\sigma_j)} \sqrt{R_0} \|_{HS}.$$

Born series (1) for g.

derivatives on l-many copies.

at most l^n ("n choose l sort of thing")

$$\leq \frac{1}{\sqrt{k}} \| f \|_{H^{-1}} \sum_{l \geq 1} \sum_{\sigma_1 + \cdots + \sigma_l = n} l^n \prod_{j=1}^{l} \left(\frac{1}{\sqrt{k}} \| g^{(\sigma_j)} \|_{H^{-1}} \right)$$

Hölder on Fourier side / "interpolation": $\| g^{(\sigma_j)} \|_{H^{-1}} \lesssim \| g^{(n)} \|_{H^{-1}}^{\sigma_j/n} \| g \|_{H^{-1}}^{1 - \sigma_j/n}$

$$\lesssim \frac{1}{\sqrt{k}} \| f \|_{H^{-1}} \sum_{l \geq 1} l^n \| g^{(n)} \|_{H^{-1}}^{\sum \sigma_j / n = n} \left(\frac{1}{\sqrt{k}} \| g \|_{H^{-1}} \right)^{l-1} \cdot \left(\frac{1}{\sqrt{k}} \right)^{1}$$

$$\lesssim \frac{1}{k} \| g \|_{H^{n-1}} \| f \|_{H^{-1}}$$

$$\Rightarrow \| g - \tfrac{1}{2k} \|_{H^{n+1}} \lesssim \| g \|_{H^{n-1}}$$

(even a $\frac{1}{k}$ holding here.)

Rmk:

convert weights attaching g = powers sensing in g into weights attaching g, need to commute weight part $\sqrt{R_0}$. $\langle x \rangle$

commutator of inverse operator can be rewritten using this identity to move inverse outside!

$$[\langle x \rangle, R_0(k)] = R_0(k) \left[-\partial^2 + k^2, \langle x \rangle \right] R_0(k)$$

$$= -R_0(k) \left[\frac{x}{\langle x \rangle} \partial + \partial \frac{x}{\langle x \rangle} \right] R_0(k).$$

So commute another power of $\langle x \rangle$ means commute w/ this operator

$$\left[\langle x\rangle,\ \frac{x}{\langle x\rangle}\partial+\partial\frac{x}{\langle x\rangle}\right] \underset{\text{Leibniz rule.}}{=} \frac{x}{\langle x\rangle}[\langle x\rangle,\partial]+[\langle x\rangle,\partial]\frac{x}{\langle x\rangle} = -\frac{2x^2}{\langle x\rangle^2}$$

$\langle x\rangle$ w/ $\frac{x}{\langle x\rangle}\partial$ — both multiplication operator....

Inductively, similar computations show

$$\langle x\rangle^n R_0 = \sum_{j=0}^{n} \sqrt{R_0}\, \underbrace{A_{j,n}}_{\text{operator}}\, \sqrt{R_0}\,\langle x\rangle^j.$$

bdd on L^2 uniformly in j: $\|A_{j,n}\|_{op} \lesssim_n 1$.

$f \in H^{-1}$ for duality

so then $\left|\int \langle x\rangle^n (g-\frac{1}{2k})f\right| \le \sum_{l\ge 1} \left|\operatorname{tr} f\, \langle x\rangle^n (R_0 q)^l R_0\right|$

commute, using above fact. ONCE, i.e. $\langle x\rangle^n R_0^2$, and then set ∂

$$\le \sum_{l\ge 1}\sum_{j=0}^{n} \|\sqrt{R_0} f \sqrt{R_0}\|_{HS}\, \|A_{j,n}\|_{op}\, \|\sqrt{R_0}\langle x\rangle^j q \sqrt{R_0}\|_{HS}\, \|\sqrt{R_0} q \sqrt{R_0}\|_{HS}^{l-1}$$

$$\lesssim \frac{1}{k}\, \|\langle x\rangle^n g\|_{H^{-1}}.$$

(using that $\|\langle x\rangle^j g\|_{H^1} \le \|\langle x\rangle^n g\|_{H^{-1}}$)

Smooth scaling from here.

Green fn g' trace term. Now that we understand it well quantitatively, smooth scaling in H_k flow

05/31/19

HK flow

warmup

Prop: Let q be Schwartz sol to KdV w/ $q(0) \in B_\delta := \{\|q\|_{H^{-1}} < \delta\}$ and fix $K \geq 1$

Let $g(t,x) := g(x; K, q(t))$. Then q evolve under KdV $\Rightarrow$ g evolve under equation:
(diagonal reconstruction)

$$\frac{d}{dt} g = -2q' g + 2g g' - 4k^2 g'$$

$$\frac{d}{dt} \frac{1}{2g} = \left(\frac{g}{g} - \frac{2k^2}{g}\right)'$$

(2 on purpose b/c we had mapping $g \mapsto g - \frac{1}{2k}$, $g \mapsto k - \frac{1}{2g}$.)

Pf: chain rule - how to diff g wrt potential, which we do using resolvent identity.

$$\frac{d}{dt} g = -\int G(x,y) \left(\frac{d}{dt} q\right)(y)\, G(y,x)\, dy = -\int G(x,y)\left(-q''' + 6qq'\right)(y)\, G(y,x)\, dy$$

(kernel of resolvent; x to y, y to x)

The defining property of G is kernel of resolvent of Lax operator, so applying Lax operator to left or right, better get δ function!! I.e.,

$$(-\partial_y^2 + q(y) + k^2)\, G(y,x) = G(x,y)(-\partial_y^2 + q(y) + k^2) = \delta(y-x)$$

How to take advantage of this? We did something similar when proving $\frac{d}{dt} d = 0$.

But unlike then when we had free resolvent, $q=0$, we have $q \neq 0$.

Need to introduce a g.

We showed

$$-f''' + (-\partial^2 + k^2) f' + f'(-\partial^2 + k^2) - \lambda(-\partial^2 + k^2) f\partial + 2\partial f(-\partial^2 + k^2) - 4k^2 f'$$

$$(-\partial^2 + g + k^2) f' + f'(-\partial^2 + g + k^2) - \lambda(-\partial^2 + g + k^2) f\partial + 2\partial f(-\partial^2 + g + k^2) - 4k^2 f'$$

$$= -f''' + 2gf' - 2gf\partial + 2\partial fg.$$

$$= 2[\partial, fg] = 2(fg)'$$

$$= -f''' + 2gf' + 2(fg)'$$

$$\Rightarrow \int G(x,y)\left(-f''' + 2gf' + 2(fg)' + 4k^2 f'\right)(y)\, G(y,x)\, dy$$

$=$

$$\int G(x,y)\Big[\underbrace{(-\partial^2+q+k^2)f'}_{\text{annihilate to } \delta x} + f'\underbrace{(-\partial^2+q+k^2)}_{\text{annihilate to } \delta x} - 2\underbrace{(-\partial^2+q+k^2)}_{\text{annihilate to } \delta x} f\delta + 2\underbrace{f(-\partial^2+q+k^2)}_{\text{annihilate to } \delta x}\Big] G(y,x)$$

$=$ $\quad f'g \qquad f'g \qquad -2fg' \qquad 2g'f$

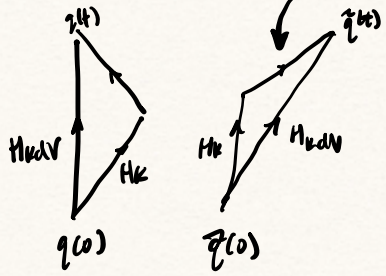
$$= 2f'g - 2fg'$$

$$g'(x) = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) G(x,y)\Big|_{x=y}$$

Taking $f = q \Rightarrow$ $\quad \int G(x,y)\left(-\partial^3 + 6q\partial + 4k^2\partial\right)(y)\, G(y,x) = 2gg - 2gg'$

So $\quad \frac{d}{dt} g = -(2g'g - 2gg') + 4k^2 \underbrace{\int G(x,y)\, g'(y)\, G(y,x)\, dy}_{= -g'(x)}$

So $\quad H_k(q) - 16k^5 a(k,q) + 4k^2 p(q) = H_{KdV} + O(k^{-2})$.



flow commute $\rightsquigarrow$ flow according to Hamiltonian $H_{KdV} - H_k$

1) show H_k globally well posed.
2) show $H_{KdV} - H_k$ "very shrinks" as $k \to \infty$.

$\Rightarrow$ win.

Today, Prop (H_k flows): Fix $k \geq 1$. Then $\exists \delta > 0$ small s.t. H_k flow is globally well posed on B_δ (small in H^{-1}).

- The H_k-flow conserves $a(x)$ $\forall x \geq 1$ and commutes w/ KdV flow.
- If init data Schwartz, sol remains Schwartz $\forall$ time.
- Finally $\quad \frac{1}{dt}\frac{1}{2g(x)} = \frac{-4k^5}{k^2 - x^2}\left(\frac{g(k)}{g(x)}\right)^2 + 4k^2\left(\frac{1}{2g(x)}\right)' \quad \forall x \neq k.$

KdV^2 scaling symmetry in $H^{-3/2}$, so can rescale to get H^{-1} norm smaller than δ. We are above critical regularity $-\frac{3}{2}$!

Pf: the H_k flow is

$$\frac{d}{dt} q = \left[\frac{\delta H}{\delta q}\right]' = -16k^5\left(\frac{\delta a(k)}{\delta q}\right)' + 4k^2 q' \overset{\text{recall power series in } \frac{1}{k}}{=} -16k^5\left(\frac{1}{2x} - q\right)' + 4k^2 q' = \underbrace{16k^5 q'}_{\text{forcing term is } g'} + 4k^2 q'$$

So letting $\tilde{q}(t,x) := q(t, x - 4k^2 t)$

thm $\frac{d}{dt} q(t,x) = 16k^5 g'(x-4k^2t, q(t))$

$\Rightarrow$ FTC in t $\quad q(t,x) = \underbrace{q(0,x)}_{\| \; q(0,x)} + \int_0^t 16k^5 g'(x-4k^2s, q(s))\,ds.$

$\Rightarrow$ undoing translation on LHS $\quad \boxed{q(t,x) = q(0, x+4k^2t) + 16k^5 \int_0^t g'(x+4k^2(t-s), q(s))\,ds}$

Easy ODE, Picard Iteration, as long as g' is Lipschitz on H^{-1}.

But we've shown $q \mapsto g\frac{1}{2k}$ is real analytic map from B_δ to H^1, g' is certainly Lipschitz:

$$\| g'(q_1) - g'(q_2) \|_{H^{-1}} \lesssim \| g(q_1) - g(q_2) \|_{H^1} \lesssim \| q_1 - q_2 \|_{H^{-1}}$$

g Lipschitz $H^1 \to H^{-1}$.

So Picard thm gives local-in-time wellposedness on B_δ.

Now, must extend to global-in-time

i.e. need to show HK flow has a priori bounds on H^{-1}.

Enough to prove $a(x)$ is conserved b/c $a(x)$ encodes a priori bds in H^{-1}:

$$a(x,q) \sim \frac{1}{\pi} \int \frac{|\hat{q}(\xi)|^2}{\xi^2 + x^2}\,d\xi.$$

leading order term in expansion

As the HK flow <u>locally wellposed</u> (LWP) in B_δ, suffices to prove conservation of $a(x)$ for Schwartz init data.

Now if I start w/ Schwartz, obviously solution also Schwartz:

Last time proved Schwartz potential $\Rightarrow$ diagonal Green fn $g(q) - \frac{1}{2k}$ also Schwartz

So Rookie ball cap, indeed Schwartz data $\to$ Schwartz sol.

So now, $\frac{d}{dt} a(x) = \{a(x), H_K\} = \int \frac{\delta a(x)}{\delta q} \left(\frac{\delta H_K}{\delta q}\right)' dx = \int \left(\frac{1}{2x} - g(x)\right)\left[-16k^5\left(\frac{1}{2x} - g(x)\right) + 4k^2 \xi_0\right]' dx$

Poisson bracket

shouldn't matter b/c g is Schwz

"Superstition" so IBP to move derivative away from g''

$$= 16k^5 \int g'(x)\, g(k)\, dx + 4k^2 \int g'(x) g\, dx.$$

Claim: $=0$ $\quad$ $=0$

by no means obvious.

A miracle of complete integrability.

$\rightsquigarrow$ diagonal Green fn satisfies elliptic eq! (first known in 1880).

$\forall$ spectral parameter x.

$$g'''(x) = 2g'g(x) + 4g\, g'(x) + 4x^2 g'(x).$$

rewrite as hoping know what's coming $= 2g\, g'(x) + 2(g\, g(x))' + 4x^2 g'(x)$

Begin w/ $\quad 2\int g'(x) g\, dx = \int g'''(x) - 2(g\, g(x))' - 4x^2 g'(x)\, dx$

all derivatives of Schwartz fns $\Rightarrow$ integral $=0$.

Next, $\quad 4x^2 \int g'(x) g(k) = \int g(k) g'''(x) - 2\int g' g(x) g(k) - 4\int g\, g'(x)\, g(k)$

$$= -2\int g' g(k) g(x) - 4\int g\, g'(k)\, g(x) + 4k^2 \int g(k) g'(x) - 2\int g' g(x) g(k) - 4\int g\, g'(x)\, g(k)$$

$$\Rightarrow \quad 0 = -4\int (g\, g(k) g(x))' = 4(x^2 - k^2) \int g'(x) g(k).$$

(g Schwartz $\Rightarrow$; $g \in C^\infty$)

$$\Rightarrow \int g'(x) g(k) = 0 \quad \forall\, x, k \geq 1 \quad \left(\text{if } x = k, = \int (g(x)^2)' \right)$$

Appendix: how to prove elliptic eq for g.

$$g'''(x) = (\partial_x + \partial_y)^3 G(x,y)\big|_{x=y}$$

Use fundamental nature of G: $(-\partial_x^2 + q(x))\, G(x,y) = (-\partial_y^2 + q(y))\, G(x,y) = -k^2 G(x,y) + \delta(x-y)$.

$$\Rightarrow (\partial_x^2 - \partial_y^2)\, G(x,y) = (q(x) - q(y))\, G(x,y)$$

$$\underbrace{(\partial_x - \partial_y)(\partial_x^2 - \partial_y^2)}_{\partial_x^3 + \partial_y^3 - \partial_y \partial_x^2 - \partial_x \partial_y^2}\, G(x,y) = (q'(x) + q'(y))\, G(x,y) + (q(x) - q(y))(\partial_x - \partial_y) G(x,y)$$

$\bullet\bullet\bullet$ etc.

6/12/24 Last day: finish pf of wellposedness of KdV in H^{-1}. (Guest lecture)

Recall apriori bound:

$\exists \delta > 0$ s.t.

$\forall$ Schwartz sol $q(t)$ to KdV w/ init data $q(0) \in B_\delta := \{f : \|f\|_{H^{-1}} < \delta\}$

have a priori bound (i) $\sup_{t\in\mathbb{R}} \|q(t)\|_{H^{-1}} \lesssim \|q(0)\|_{H^{-1}}$ (proved by perturbation determinant presented $\alpha(k; q(t)) = \alpha(k; q(0))$)

orbit cpct in H^{-1} (ii) $\{q(t) : t\in\mathbb{R}\}$ is cpct in H^{-1} ($\Leftrightarrow \|q(t)\|_{H^{-1}_K} \xrightarrow{K\to\infty} 0$ SS $K \cdot \alpha(K; q(t))$)

Can replace KdV by other flow that have α preserved

like $\underbrace{tH_{KdV} + sH_K}_{\text{Hamiltonian}}$ ($\{\alpha(k), tH_{KdV} + sH_K\} = 0$ Poisson commute)

Thm: Let q_n be seq of Schwartz solution to KdV w/ $q_n(0)$ converging in H^{-1}.

Then, $\forall T > 0$, $q_n(t)$ converges in H^{-1} uniformly for $|t| \le T$

Cor (global wellposedness of KdV): the data $\to$ sol extends (and uniquely so) from Schwartz space to a jointly cts map $\Phi : \mathbb{R}_t \times H^{-1}(\mathbb{R}) \to H^{-1}(\mathbb{R})$.

Moreover, $\forall q \in H^{-1}$, the orbit $\{\Phi(t;q) : t\in\mathbb{R}\}$ is bdd and cpct

Pf of Cor: $S(\mathbb{R})$ dense in $H^{-1}(\mathbb{R})$, so extension $\exists$ and is unique.

Why joint cts? Suffices to show for each $T>0$, $\Phi : [-T,T] \times H^{-1}(\mathbb{R}) \to H^{-1}(\mathbb{R})$ is joint cts.

We show:
(A) $\Phi(t;q)$ is cts in time t.
(B) $\Phi(t;q)$ is uniformly cts in q (cpct set time interval)

→ Std argument ε-ineq → joint cty.

To show (A), first **Lemma:** Schwartz sol to KdV are H^{-1}-cts in t

(Pf: recall energy method from earlier classes show H^2-cts in time $\Rightarrow$ H^{-1}-cts in time. (indeed Bourgain method show H^0-cts in time))

+ unif limit of cts fn is cts.

To show (B), std arguments "1st year analysis"

Why orbit bdd & cpct? Thm says true for Schwartz sol, so "Std arguments" says preserved in extension.

Pf of Thm: first rescale to small initial data: KdV enjoys the scaling symmetry

$$q(t,x) \longmapsto q_\lambda(t,x) := \lambda^2 q(\lambda^3 t, \lambda x)$$

(is $H^{-3/2}$ critical, but we are concerned about H^{-1} regularity, so

$$\|q_\lambda(t)\|_{H^{-1}}^2 = \lambda^4 \int \frac{\lambda^{-2} |\hat{q}(\lambda^3 t, \frac{\xi}{\lambda})|^2}{\xi^2+1}\, d\xi \overset{\text{change of var.}}{=} \int \frac{\lambda^3}{(\lambda\xi)^2+1} |\hat{q}(\lambda^3 t, \xi)|^2 d\xi$$

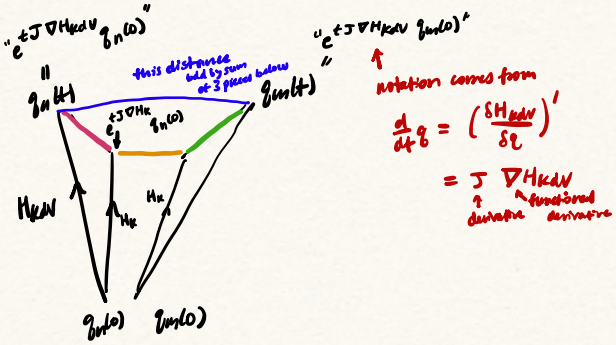
$$\lesssim \lambda \int \frac{1}{\xi^2+1} |\hat{q}(\lambda^3 t, \xi)|^2 d\xi \quad \text{provided } 0<\lambda<1$$

$$\lesssim \lambda \, \|q(\lambda^3 t)\|_{H^{-1}}$$

So for $t=0$, λ small enough, may assume

$$\|q(0)\|_{H^{-1}} < \delta$$

Now, the commuting flow argument.



notation comes from

$$\frac{d}{dt} q = \left(\frac{\delta H_{KdV}}{\delta q}\right)' = J \nabla H_{KdV}$$

(' = functional derivative, ∇ = gradient)

Lemma: $e^{tJ\nabla H_{KdV}} = e^{tJ\nabla(H_{KdV} - H_K)} \circ e^{tJ\nabla H_K}$

Pf: comes from Poisson commute $\{H_{KdV} - H_K, H_K\} = 0$.

So,

$$\|q_n(t) - q_m(t)\|_{H^{-1}} \le \|e^{tJ\nabla H_{KdV}} q_n(0) - e^{tJ\nabla H_{KdV}} q_m(0)\|_{H^{-1}} + \|e^{tJ\nabla(H_{KdV}-H_K)} e^{tJ\nabla H_K} q_n(0) - e^{tJ\nabla H_K} q_m(0)\|_{H^{-1}}$$

$$+ \quad \text{same but } m \text{ in place of } n.$$

As H_K flow globally wellposed in H^{-1},

$$\sup_{t\in S} \|e^{tJ\nabla H_{KdV}} q_n(0) - e^{tJ\nabla H_{KdV}} q_m(0)\|_{H^{-1}} \to 0 \text{ as } n,m \to \infty.$$

Denote $Q := \{q_n(0)\}_{n=1}^\infty \subseteq B_\delta$

$$Q^* := \{ e^{J\nabla(tH_{KdV} + sH_K)} q : q \in Q,\ t,s \in \mathbb{R},\ k \ge 1 \}$$

(i.e. results of any combination of H_{KdV}, H_K flow acting on Q.)

It sufficed to prove

$$\sup_{q\in Q^*} \sup_{t\in S} \| \underbrace{e^{tJ\nabla(H_{KdV}-H_K)} q}_{=: q(t) \text{ lies in } Q^*} - q \|_{H^{-1}} \to 0 \text{ as } K \to \infty.$$

We would like $Q^* \subseteq B_S$ so we can use what we proved earlier

think of $tH_{KdV} + sH_K$ as Hamiltonian and $e^{1 \cdot J\nabla(tH_{KdV}+sH_K)}$ (time 1)

"$q(t=1)$ for $tH_{KdV}+sH_K$ flow"

(perturbation determinant α preserved,

so also get a priori bound

$\sup_{t\in\mathbb{R}} \|q(t)\|_{H^{-1}} \lesssim \|q(0)\|_{H^{-1}}$ for $q(0)\in B_S$

↑ constant should only depend on formulation for α; as long as α conserved, should have same implicit const no matter what the flow is.

$\Rightarrow$ by choosing init data even smaller, $Q^* \subseteq B_S$

$$\left| \int \left[g(x,x;q) - \frac{1}{2k} + \frac{1}{k} R_0(2k)\, q \right] f(x)\, dx \right|$$

test $f \in L^\infty$ fn to show is in L^1 (duality!)

where recall $G(x,y;k,q)$ integral kernel of $(-\partial_x^2 + q + k^2)^{-1} = \sum_{l=0}^{\infty} (-1)^l \sqrt{R_0}(\sqrt{R_0}\, q \sqrt{R_0})^l \sqrt{R_0}$

and $g(x;k,q) = G(x,x;k,q)$, we have found $g(x;k,q) = \frac{1}{2k} - \frac{1}{k} R_0(2k) q + \sum_{l=2}^{\infty} \dots$

so $= \left| \sum_{l=2}^{\infty} (-1)^l \operatorname{tr}\{ \sqrt{R_0} (\sqrt{R_0}\, q \sqrt{R_0})^l \sqrt{R_0} f \} \right|$

cycle

$$\le \sum_{l=2}^{\infty} \|R_0 f R_0\|_{op} \|\sqrt{R_0}\, q \sqrt{R_0}\|_{HS}^l \le \sum_{l=2}^{\infty} \|\sqrt{R_0}\|_{op}^2 \|f\|_\infty \left(\frac{1}{\sqrt{k}} \|q\|_{H^{-1}_k} \right)^l$$

remember $\sqrt{R_0} = (-\partial_x^2 + k^2)^{-1/2}$, on Fourier side is mult by $\frac{1}{(\xi^2+k^2)^{1/2}}$

so $\|\sqrt{R_0}\|_{op} \lesssim \frac{1}{k}$. ← Lec 6 of Fourier multipliers

$$\lesssim \frac{1}{k^2} \|f\|_\infty \frac{1}{k} \|q\|^2_{H^{-1}_k}$$

assuming $\|q\|_{H^{-1}_k} < 1$, geom series is $\sim$ leading term

$$\Rightarrow \left\| g - \frac{1}{2k} + \frac{1}{k} R_0(2k) q \right\|_{L^1} \lesssim \frac{1}{k^3} \|q\|^2_{H^{-1}_k}.$$

$H^1 \hookrightarrow L^\infty$ so $L^1 \hookrightarrow H^{-1}$, so can control

$$\Rightarrow \left\| 4k^3 \left(g - \frac{1}{2k}\right) + q \right\|_{H^{-1}} \le 4k^3 \left\| g - \frac{1}{2k} + \frac{1}{k} R_0(2k) q \right\|_{H^{-1}} + \left\| g - 4k^2 R_0(2k) q \right\|_{H^{-1}}$$

$$\lesssim 4k^3 \frac{1}{k^3} \|q\|^2_{H^{-1}_k} + \|q\|_{H^{-1}_k}$$

computation Fourier side

$$\lesssim \|q\|^2_{H^{-1}_k} + \|q\|_{H^{-1}_k}$$

Return to Goal above: $\sup_{g\in\mathcal{Q}^*}\sup_{|t|\le T}\| \underbrace{e^{tJ\nabla(H_{KdV}-H_K)}g}_{=:g(t)} - g\|_{H^{-1}} \to 0$ as $K\to\infty$.

$g(t)$ lies in $\mathcal{Q}^*$

Let $x_c \ge 1$ be fixed. Take $K \ge x_c + 1$.

By diffeomorphism $\quad g \mapsto x - \frac{1}{2g(x)}$ $\qquad g \in \mathcal{Q}^* \subseteq B_S$

$H^{-1} \supseteq B_S \longrightarrow H^1$

B_S: space on which we defined diffeomorphism.

It suffices to work in H^1 instead of H^{-1}! :

Goal': $\sup_{g\in\mathcal{Q}^*}\sup_{|t|\le T}\| \frac{1}{2g(t)} - \frac{1}{2g(0)}\|_{H^1} \xrightarrow{x_c\to\infty} 0$ where $g(t) := g(t,x,x_c) = g(x; x_c, g(t))$

$\|$ FTC

$\int_0^t \left(\frac{d}{d\tau}\frac{1}{2g(\tau)}\right)$

$$\frac{d}{dt}\frac{1}{2g(t)} = -\frac{1}{2g^2(t)}\frac{d}{dx}g(t) = \left\{\frac{1}{g(t)}\left[\partial + \frac{4K^5}{K^2-x^2}\left(g(t)-\frac{1}{2K}\right) + \frac{4x^5}{K^2-x^2}\left(\frac{1}{2x}-g(x)\right)\right]\right\}$$

$\vdots$

$$\left\|\frac{d}{dt}\frac{1}{2g(t)}\right\|_{H^{-2}} \xrightarrow{x_c\to\infty} 0.$$

we are 3 derivatives short $\frac{1}{2}$.

But actually this is fine. (using equality at $\mathcal{Q}^*$)