2/25/26 $\text{code} : X \to \omega^\omega$

$x \longmapsto (n_1 := \min n \text{ s.t. } x \in B_n \in \mathcal{B}_1,\ n_2 := \min n \text{ s.t. } x \in B_n \in \mathcal{B}_2 \ \ldots)$

"Prop 1" in my outline

**Claim:** Borel function.

inverse image of clopen $\text{code}^{-1}[n_1 \cdots n_\ell * * \ldots]$ — first check: resultion of $x$: whose closure $\subseteq B_{n_1}$; if fail checks: $n_1 \in N_1$, $n_2 \in N_2(n_1)$, $n_3 \in N_3(n_2)$, $\vdots$, $n_\ell \in N_\ell(n_{\ell-1})$ — thus immediately preimage is $\emptyset$.

assuming $n_1 \cdots n_\ell$ pass the checks

$x \in \iff x \notin (B_1 \text{ or } \ldots B_{n_1-1})$

$\wedge\ x \in B_{n_1}$

$\wedge\ \forall n \in N_2(n_1)\ (n < n_2)\ \ x \notin B_n$

$\wedge\ x \in B_{n_2}$

let $N_1$ be $\{n \in \mathbb{N} : B_n \in \mathcal{B}_1\}$

$N_2$ //

$N_2(n_1) := \{n \in N_2 : \overline{B_n} \subseteq B_{n_1}\}$

cleaner now that I developed notation

$\forall n \in N_1\ (n < n_1)\ \ x \notin B_n$

$\wedge\ x \in B_{n_1}$

$\wedge\ \forall n \in N_2(n_1)\ (n < n_2)\ \ x \notin B_n$

$\wedge\ x \in B_{n_2}$

i.e.

$$\text{code}^{-1}[n_1 \cdots n_\ell * \cdots] = \bigcap_{\substack{n \in N_1 \\ n < n_1}} \neg B_n \ \cap\ B_{n_1} \ \cap \bigcap_{\substack{n \in N_2(n_1) \\ n < n_2}} \neg B_n \ \cap\ B_{n_2} \cdots$$

in fact finite $\cap$ of basic opens $B$.

**Claim:** image $\text{code}(X) \subseteq \omega^\omega$ Borel

$x = \bigcap_i B_{n_i}$, $n_2 < n_1 \Rightarrow x \notin B_{n_2}$, $B_{n_1}$, $B_{n_2}$

If $n_1$ truly were minimal, then $\forall m_1 < n_1 \quad x \notin B_{m_1}$.

maybe fix:

$\text{code} : X \to \omega^\omega$

$x \longmapsto (n_1$ s.t. $x \in B_{n_1}$, $\forall n < n_1\ x \notin \overline{B_n}$

i.e. find set $N_1' := \{n \in N_1 : x \in B_n\}$

0 0 0 … N

hmm what if $x \in \partial B_1$ !!.

$\text{decode} : \omega^\omega \supseteq \text{Legal}_X \to X$ s.t. $!x \in \bigcap_i B_{s_i}$

$\text{Legal}_X$ vl $\text{code}(X)$

**Claim:** $\text{Legal}_X$ definitely Borel.

$N_1$, $N_2(n_1)$, $\vdots$, $\vdots$ are all fixed countably many.

$s \in \text{Legal}_X \iff s_1 \in N_1 \ \wedge\ s_2 \in N_2(s_1) \ \wedge \cdots \iff s \in [N_1 * * \ldots] := \bigcup_{n \in N_1} [n * * \ldots *]$

open — union of clopens.

— notation & relations: $R(m,n)$ true $\iff \overline{B_m} \subseteq B_n$.

e.g. $N_2(n_1) = \{n \in N_2 : nRn_1\}$.

**Claim:** $\exists$ Borel function $s \in \text{decode}^{-1}[B_n] \iff \text{decode}(s) \in B_n \iff B_{s_1} \subseteq B_n \iff \overline{B_{s_1}} \subseteq B_n \iff s$ s.t. $R(s_1, n) \iff s \in [N_1(n) * * \ldots] = \bigcup_{m \in N_1(n)} [m * \ldots]$ open

$\vee\ B_{s_2} \subseteq B_n$ $\quad \vee\ \overline{B_{s_2}} \subseteq B_n$ $\quad \vee\ s : R(s_2, n)$

$\vee\ B_{s_3} \subseteq B_n$

$\vdots$ $\quad \vdots$ $\quad \vdots$

suppose $B_n \in \mathcal{B}_L \setminus \mathcal{B}_{L+1}$

(recursion level is like size/diam, so "diam $B_n = \frac{1}{L}$")

$\text{decode}(s) \in B_n$ detectable at finite stage.

**Claim:**

$s \in \text{code}(X) \iff s \in \text{Legal}_X \ \wedge\ \text{code}(\text{decode}(s)) = s.$

4/26/26 ~9:30 am bus ride

Topological property denoted ◎

Def hard basis w/ pinpoint grading + Def+ nbhd basis prop to make decode cts

add in when we get to decode

↳ for closure of topo propy ◎ under taking closed subspaces (see Diam 2026)

ppty

ppty

ppty

I needed $\bar{B}_1 \supseteq B_1 \supseteq \bar{B}_2 \supseteq B_2 \supseteq \bar{B}_3 \supseteq \dots \to x$

$\cup\!| \quad \cup\!| \quad \cup\!|$

$F_1 \supseteq F_2 \supseteq F_3 \supseteq \dots \to x$ too.

Pf: ~~also working for boxes, same here~~ ~~$F_n$ closed $\Rightarrow \bigcap F_n$ closed~~

$x \notin \bigcap F_n$,

$\Downarrow$

$\exists N \; x \notin F_N$. $F_n$ nested

$\Downarrow$

$x \notin F_{>N}$.

open basis $\beta$, $x \in U_N \subseteq \neg F_N$.

(obviously $\bigcap F_n \subseteq \bigcap \bar{B}_n = \bigcap B_n$ but maybe empty.

not true for open $F_n$ (walking to bus, occurred to me

$\mathbb{R}^2$, open $F_n$, $\bigcap F_n = \emptyset$

Def+ $\Rightarrow$ Def++

F, F, β

But by nbhd basis prop, eventually $\bar{B}_n \subseteq \beta$, $F_n \subseteq$

$\Rightarrow \not\in$.

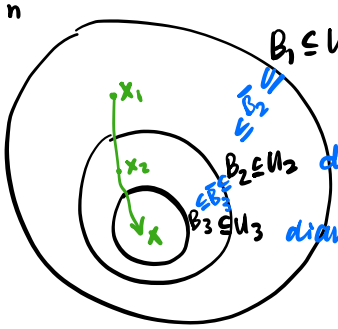
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GSO tall structure.

"1st Thm / Fund Thm of DST"

(0) Prologue BCT: $\bigcap_n$ open dense $U_n$ = dense.

Pf: in $\mathbb{R}^2$




(1) BCT argument uses structure of Def 0 (homing space) above! BCT/

Previous arg in $\mathbb{R}^2$ essentially shows complete metrizable → homing ↓ BCT.

cable basis / separable

Examples: sep cplt metric space $\mathbb{R}, \mathbb{R}^2 \dots$

Cantor, Baire space $\mathcal{B}_\ell = [n_1 \dots n_{\geq \ell} * * \dots]$ initial segment

(2) homing X, Def 1: code : $X \to \mathbb{N}^{\mathbb{N}}$ Prop 1: is Borel fn.

Def 2: $\text{LegalStrs}_X \subseteq \mathbb{N}^{\mathbb{N}}$, $\text{code}(X) \subseteq$ Prop 2: Borel subset

Def 3: decode : $\text{LegalStrs}_X \subseteq \mathbb{N}^{\mathbb{N}} \to X$ Prop 3: is Borel fn cts (edit Def 0 to Def 0+)

**Prop 4:** $code(X) \subseteq \mathbb{N}^{\mathbb{N}}$ is Borel

$s \in code(X) \iff s \in LegalStr_X \wedge code(decode(s)) = s$.

i.e.,

$code(X) = LegalStr_X \cap$ ( $s \in graph^{-1}(\text{diagonal})$ of $LegalStr_X^2$ ) ∩ ( $s \in decode^{-1}(code^{-1}(\{s\}))$ )

$h = code \circ decode : LegalStr_X \to LegalStr_X$

$graph : LegalStr_X \to LS_X^2$, $y \mapsto (y, h(y))$

Borel. (or just $\mathbb{N}^{\mathbb{N}^2}$)

$LegalStr_X^2 - \bigcup_{n \neq m} B_n \times B_m$.

$B_n \neq B_m$

or more algorithmically

$s \in decode^{-1}(code^{-1}(\{s\})) = \bigcap_{t} [s_1 \dots s_t * \dots]$

ctbl basis of $\mathbb{N}^{\mathbb{N}}$.

$\bigcap_n \left( s \in \beta_n \to s \in decode^{-1}(code^{-1}(\beta_n)) \right)$

logic fml: $code(decode(s)) = s \iff s \in decode^{-1}(code^{-1}(s))$

$s \in code(X)$

$\iff \forall n \; ( s \in \beta_n \Rightarrow s \in decode^{-1}(code^{-1}(\beta_n)) )$

$\iff s \in \bigcap_n [\neg \beta_n \vee decode^{-1}(code^{-1}(\beta_n))]$.

cheap Borel Isomorphism Thm = **Cor:** $X \hookrightarrow code(X) \subseteq_{Borel} \mathbb{N}^{\mathbb{N}}$ (Borel code)

$X \twoheadleftarrow code(X)$ (decode (Borel))

---

③ Borel subsets of $\mathbb{N}^{\mathbb{N}}$

**Thm 1** (proved later): all uncountable Borel $\subseteq \mathbb{N}^{\mathbb{N}}$ have biBorel embedding of Cantor space

assuming this thm for now.

**Prop 5:** $u : \mathbb{N}^{\mathbb{N}} \hookrightarrow 2^{\mathbb{N}}$ (u many-one) biBorel embedding

$(n_1, n_2, \dots) \mapsto \underbrace{1 \dots 1}_{n_1} 0 \underbrace{1 \dots 1}_{n_2} 0 \underbrace{1}_{n_3} 0 \dots$

**Pf:** (i) cts

"to know finite portion of output, just need finite portion of input"

$u^{-1}([\;\;\;])$ if $0, = \emptyset$.

else $[1 \dots 0 * \dots] \Rightarrow u^{-1}(\cdot) =$ decode finite portion then $*$. e.g. $u^{-1}[1100100 * \dots] = [2\,0\,1\,0 * \dots]$

or $[1 \dots 011 * \dots] \Rightarrow$ e.g. $u^{-1}[1100101 * \dots] = [2\,0\,1\,1 * \dots] \cup [2\,0\,1\,2 * \dots] \cup \vdots$

cts ∪ of open = open ✓

(ii) image is Borel.

$=$ { starts w/ 1 & ∞ many 0s } ("not fin many 0s")

$= [1 * \dots] \cap \bigcap_t [\underbrace{* * \dots * 111 \dots}_{t}]$

not findet but ∩ of findet.

$[-*-1 * \dots]$

$\cap [___11 * \dots]$

$\cap [___111 * \dots]$

$\vdots$

(iii) inverse fn is ~~Borel~~ cts

$v : \text{im}(v) \to \mathbb{N}^{\mathbb{N}}$

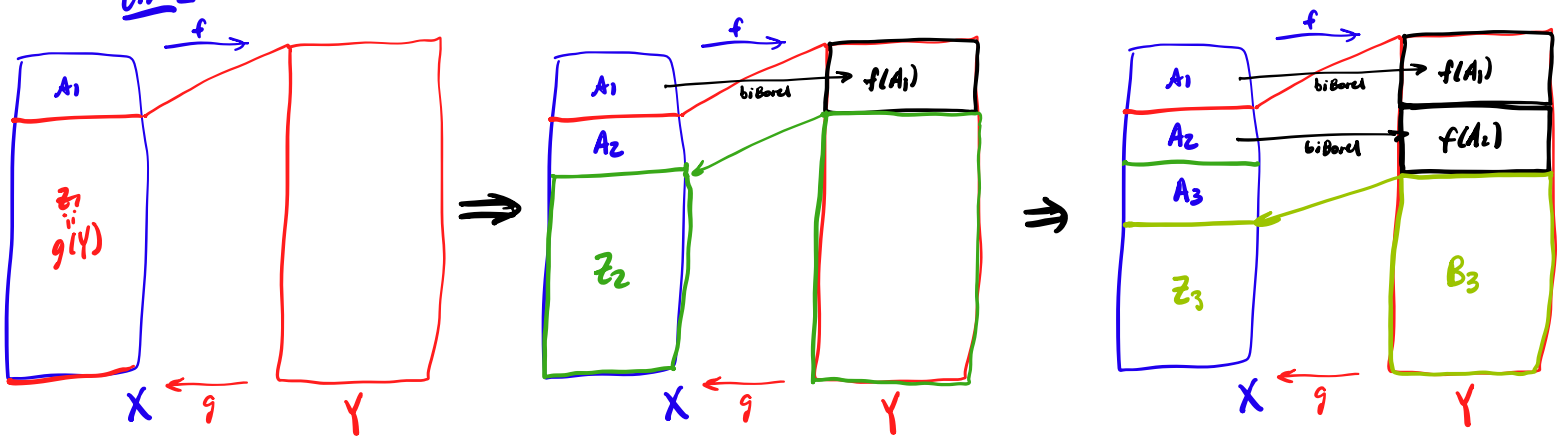
$v^{-1}[n_1 \ldots n_e * \ldots]$

$= [\underbrace{1\ 1\ 0 \ldots 0}_{n_1}\ \underbrace{1 \ldots 1\ 0}_{n_2} * \ldots]$

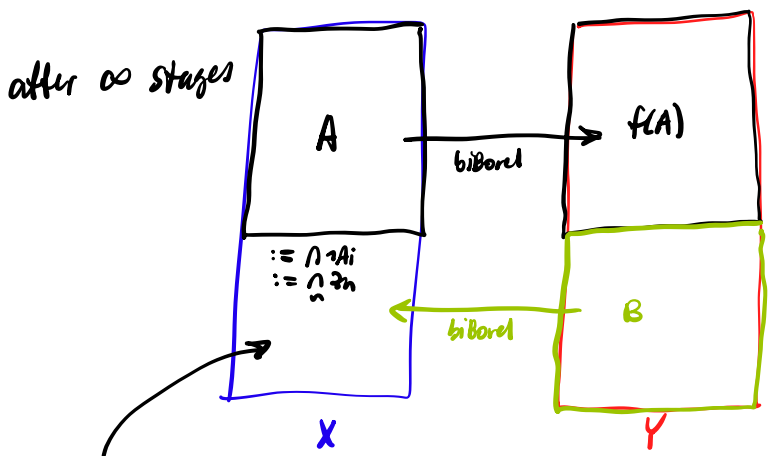
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So we have $\mathbb{N}^{\mathbb{N}} \overset{\text{biBorel}}{\hookrightarrow} 2^{\mathbb{N}} \overset{\text{biBorel}}{\hookrightarrow} E \overset{\text{Borel}}{\subseteq} \mathbb{N}^{\mathbb{N}}$

**Thm2**: Borel Cantor Schroder-Bernstein




after $\infty$ stages




$g(B)$

$= g(\bigcap_n \underbrace{f(A_1 \cup \ldots \cup A_n)}_{B_n}) = \bigcap_n \underbrace{g(B_n)}_{=z_n}$

---

④ Final Part.

**Thm3**

Let $\mathcal{O}$ be topological space property closed under
- open subspace
- closed subspace
- $\prod^{\infty}$
- $\coprod$

-possessed by open subspaces of $\mathbb{N}^{\mathbb{N}}$

Then $\forall$ Borel $E \subseteq \mathbb{N}^{\mathbb{N}}$,

$\exists$ top. space $Z$ w/ prop $\mathcal{O}$ $\xrightarrow{\text{cts biBorel bijection}}$ $E$