if I have 2 borel functions f:X\to Y, g:Y\to X, is the set {s: s=f(g(s))} Borel?

Thought for 8s

Yes — under the usual topological/Borel-space hypotheses.

**Assume** $X, Y$ are topological spaces and "Borel function" means Borel measurable (in particular this holds if $X, Y$ are Polish/metric/Hausdorff). Let

$$h := f \circ g : Y \to Y,$$

which is Borel measurable as a composition of Borel maps. Define the map

$$\Phi : Y \to Y \times Y, \qquad \Phi(y) = (y, h(y)).$$

Since the identity map $y \mapsto y$ and $h$ are Borel, $\Phi$ is Borel. The set of fixed points is

$$S = \{y \in Y : y = h(y)\} = \Phi^{-1}(\Delta),$$

where $\Delta = \{(y, y) : y \in Y\}$ is the diagonal in $Y \times Y$. If $Y$ is Hausdorff (in particular if it is a metric or Polish space) the diagonal $\Delta$ is closed, hence Borel, so $S = \Phi^{-1}(\Delta)$ is Borel.

(If you only assume arbitrary measurable spaces without any topological/Polish hypothesis, the diagonal need not be measurable, so you can't conclude in total generality.)



prove from first principles that in an uncountable closed subset F of polish space, there are at least 2 distinct points s.t. every nbhd around each contains uncoutnably many points of F

Thought for 7s ›

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# Proof

Let $X$ be a Polish space and let $F \subseteq X$ be closed and uncountable. Recall a Polish space is separable and metrizable, hence second countable: there is a countable base $\mathcal{B} = \{B_1, B_2, \dots\}$ for the topology of $X$.

Define

$$A := \{x \in F : \ \exists \text{ open } U \ni x \text{ with } U \cap F \text{ countable}\}.$$

(So $A$ is the set of points of $F$ that admit a neighborhood meeting $F$ in only countably many points.)

If $x \in A$ choose some open $U_x$ with $x \in U_x$ and $U_x \cap F$ countable. By the definition of a base, there exists some basis element $B_i \in \mathcal{B}$ with $x \in B_i \subseteq U_x$. Then $B_i \cap F$ is a countable set containing $x$. Hence

$$A \ = \bigcup_{\{i:\ B_i \cap F \text{ is countable}\}} (B_i \cap F).$$

The index set $\{i : \ B_i \cap F \text{ is countable}\}$ is a subset of $\mathbb{N}$, so it is countable; each $B_i \cap F$ appearing in the union is countable. Therefore $A$ is a countable union of countable sets, so $A$ itself is countable.

Since $F$ is uncountable and $A$ is countable, the complement $F \setminus A$ is uncountable. By definition of $A$, every point $x \in F \setminus A$ has the property that no neighborhood of $x$ meets $F$ in only countably many points; equivalently, **every neighborhood of $x$ contains uncountably many points of $F$.**

Thus $F \setminus A$ is an uncountable set of points each of which has the required property; in particular there are at least two distinct such points. ∎

*(Remark: second countability (hence the separability/countable base property of Polish spaces) is essential in this argument; without a countable base the conclusion can fail.)*

