245A Homework 3

Daniel Rui - 11/8/21

Exercise 1.3

(Generalized DCT) Suppose that we have $f_n, g_n, f, g \in L^1(\mu)$, and $f_n \to f$ and $g_n \to g$ (both μ-a.e.), and $|f_n| \leqslant g_n$ μ-a.e. and $\int g_n \, d\mu \to \int g \, d\mu$. We want to show that $\int f_n \, d\mu \to \int f \, d\mu$. First observe that $|f_n| \leqslant g_n$ a.e. $\iff -g_n \leqslant f_n \leqslant g_n \implies 0 \leqslant f_n + g_n$ a.e. and $g_n - f_n \geqslant 0$ a.e. (the sum/difference is defined a.e., because indeterminate forms $\pm\infty + \mp\infty$ only occur on a null set, since L^1 functions are only infinite on a null set). Thus, by Fatou, we have that

$$\int \liminf_{n\to\infty}(f_n + g_n) \, d\mu \leqslant \liminf_{n\to\infty}\left(\int (f_n + g_n) \, d\mu\right)$$

and the same for $(g_n - f_n)$. Because $f_n \to f$ and $g_n \to g$ a.e., $\lim_{n\to\infty}(f_n + g_n) \, d\mu = f + g$, and so the LHS of the above inequality becomes $\int f + g \, d\mu$. Now because $f_n, f, g_n, g \in L^1(\mu)$, $\int f + g \, d\mu = \int f \, d\mu + \int g \, d\mu$, and $\int f_n + g_n \, d\mu = \int f_n \, d\mu + \int g_n \, d\mu$. With all these reductions, the above inequality now says that

$$\int f \, d\mu + \int g \, d\mu \leqslant \liminf_{n\to\infty}\left(\int f_n \, d\mu + \int g_n \, d\mu\right)$$

Similarly, the inequality for $(g_n - f_n)$ yields

$$-\int f \, d\mu + \int g \, d\mu \leqslant \liminf_{n\to\infty}\left(-\int f_n \, d\mu + \int g_n \, d\mu\right).$$

Now, consider the following fact: if we have a two sequences of real numbers $\{a_n\}_1^\infty$ and $\{b_n\}_1^\infty$, where $b_n \to b$, then $\liminf(a_n + b_n) = (\liminf a_n) + b$. This is true, because for every $\epsilon > 0$, there is some N s.t. $n \geqslant N \implies b_n < b + \epsilon$, and so we get that $\liminf(a_n + b_n) \leqslant \liminf(a_n + b + \epsilon)$ (because for $n \geqslant N$, the LHS sequence is $<$ the RHS sequence). But $b + \epsilon$ is not dependent on n, so $\liminf(a_n + b_n) \leqslant (\liminf a_n) + b + \epsilon$, which implies $\liminf(a_n + b_n) \leqslant (\liminf a_n) + b$ because $\epsilon > 0$ was arbitrary. The $\geqslant$ direction is proven in the same way, but using that $n \geqslant N \implies b_n > b - \epsilon$ instead. Thus, the two big inequalities above become

$$\int f \, d\mu + \int g \, d\mu \leqslant \liminf_{n\to\infty}\left(\int f_n \, d\mu\right) + \int g \, d\mu \quad \text{and} \quad -\int f \, d\mu + \int g \, d\mu \leqslant \liminf_{n\to\infty}\left(-\int f_n \, d\mu\right) + \int g \, d\mu$$

Recall that $g \in L^1(\mu)$, so $\int g \, d\mu < \infty$, so we can subtract it from both sides. Also, notice that $\liminf(-a_n) = -\limsup(a_n)$, and so we have that

$$\int f \, d\mu \leqslant \liminf_{n\to\infty}\left(\int f_n \, d\mu\right) \quad \text{and} \quad -\int f \, d\mu \leqslant -\limsup_{n\to\infty}\left(\int f_n \, d\mu\right) \iff \int f \, d\mu \geqslant \limsup_{n\to\infty}\left(\int f_n \, d\mu\right)$$

or in other words $\limsup_{n\to\infty}\left(\int f_n \, d\mu\right) \leqslant \int f \, d\mu \leqslant \liminf_{n\to\infty}\left(\int f_n \, d\mu\right) \implies \int f \, d\mu = \lim_{n\to\infty} \int f_n \, d\mu$.

Summary: for the standard DCT, we use Fatou on $g \pm f_n$ to get $\int \liminf_{n\to\infty} g \pm f_n \, d\mu \leqslant \liminf_{n\to\infty} \int g \pm f_n \, d\mu$ (to use Fatou it is critical we have "domination" to keep integrands $\geqslant 0$), and then "pull out the g's" (using linearity of integral, which L^1 is more than strong enough to give), cancel them (critical that g have finite integral), and conclude via the observation that "liminf of negation is negation of limsup". The proof for generalized DCT is of exactly the same spirit, except the "pull out the g's" step is a bit more subtle (where we use the assumption $\int g_n \, d\mu \to \int g \, d\mu$). Alternatively (like in the proof of DCT in Evans & Gariepy) we can use Fatou on $(g_n + g) - |f_n - f| \geqslant 0$; the process of "pulling out the g's" and cancelling them is exactly the same, but we get a slightly stronger (but actually equivalent; see Ex. 1.4 below) conclusion of convergence in L^1 instead of convergence of 1st moment — one added benefit is that we only need to consider 1 inequality with this route, namely $\limsup_{n\to\infty} |f - f_n| \, d\mu \leqslant 0$, since $\liminf_{n\to\infty} \int |f - f_n| \, d\mu \geqslant 0$ is trivial.

Exercise 1.4

(Almost everywhere convergence $\implies$ L^1 convergence equivalent to 1st moment convergence) Let (Ω, μ) be an outer measure space, and suppose $f_n, f \in L^1(\mu)$ with $f_n \to f$ a.e. as $n \to \infty$. Define $||f||_1 := \int |f| \, d\mu$. We want to show that $||f_n||_1 \to ||f||_1$ and $\|f_n - f\|_1 \to 0$ are equivalent.

($\implies$): we want to show that $||f_n - f||_1 \to 0$. We **use the generalized DCT** from Ex. 1.3 above: our "f_n" will be $|f_n - f|$, our "f" will be 0, our "g_n" will be $|f_n| + |f|$, and "g" will be $2|f|$. Now the conditions: we are given that $f_n \to f$ a.e., implying that $|f_n - f| \to 0$ a.e. and $g_n := |f_n| + |f| \to 2|f| =: g$ a.e.; the triangle inequality gives $|f_n - f| \leqslant |f_n| + |f| =: g_n$ a.e.; and we also are given $\int |f_n| \, d\mu \to \int |f| \, d\mu$ (all finite) which implies that $\int g_n \, d\mu := \int |f_n| + |f| \, d\mu = \int |f_n| \, d\mu + \int |f| \, d\mu \to \int 2|f| \, d\mu =: \int g \, d\mu$ (all finite) — note the usage the linearity of the integral for L^1 functions. Moreover, it is clear that all the functions here are L^1. The conditions are satisfied, so we conclude by the generalized DCT that $\int |f_n - f| \, d\mu \to \int 0 \, d\mu = 0$, i.e. $||f_n - f||_1 \to 0$.

($\impliedby$): by the reverse triangle inequality, $|f| - |f_n| \leqslant |f - f_n|$ a.e. and so by monotonicity/linearity of the integral for L^1 functions we have $\int |f| \, d\mu - \int |f_n| \, d\mu = \int |f| - |f_n| \, d\mu \leqslant \int |f - f_n| \, d\mu \to 0$ as $n \to \infty$, implying of course that $\int |f_n| \, d\mu \to \int |f| \, d\mu$, as desired.

Remark: sometimes known as Scheffé's lemma. This exercise is subsumed by the far more general Vitali convergence theorem; in particular because convergence a.e. implies convergence in measure for $\mu(\Omega) < \infty$ (easy consequence of Egoroff/the proof is very similar to that of Egoroff) this problem is a direct corollary of the Vitali theorem as stated in the linked reference (in both cases $\mu(\Omega) < \infty$ or $= \infty$).

Exercise 1.5

Let $f : [0, 1] \to [0, \infty)$ be a λ^1-measurable function ($\lambda = \lambda_1$ denotes Lebesgue measure on $\mathbb{R}$) s.t. $\int_0^1 f(x) \, dx = \int_{[0,1]} f(x) \, d\lambda(x) < \infty$. Suppose that for all $n \in \mathbb{N}$, we know that $\int_0^1 f^n \, d\lambda = \int_0^1 f \, d\lambda$. We want to show that f must be a.e. equal to a characteristic function of a set $E \subseteq [0, 1]$.

Consider the functions $f^n \cdot 1_{[f>1]}$. These are increasing pointwise to ∞ on all of $[f > 1]$ (and of course non-negative), and so by the monotone convergence theorem $\int_{[f>1]} \infty \, d\lambda = \lim_{n\to\infty} \int_0^1 f^n \cdot 1_{[f>1]} \, d\lambda \leqslant \lim_{n\to\infty} \int_0^1 f^n \, d\lambda = \lim_{n\to\infty} \int_0^1 f \, d\lambda = \int_0^1 f \, d\lambda < \infty$, and the only way this is possible is if $\lambda([f > 1]) = 0$.

Thus f is a.e. a function $[0, 1] \to [0, 1]$. We can split $f^n = f^n \cdot 1_{[f<1]} + 1 \cdot 1_{[f=1]}$. These are non-negative, decreasing pointwise everywhere to $1_{[f=1]}$ on $[0, 1]$, and dominated by $f \in L^1([0, 1], \lambda)$ so by the DCT we have $\int_0^1 1_{[f=1]} \, d\lambda = \lim_{n\to\infty} \int_0^1 f^n \, d\lambda = \int_0^1 f \, d\lambda$. Subtracting $\int_0^1 1_{[f=1]} \, d\lambda$ from both sides yields $\int_0^1 f \cdot 1_{[f<1]} \, d\lambda = 0$, and so as $f \cdot 1_{[f<1]}$ is non-negative we must have $f \cdot 1_{[f<1]} = 0$ a.e. (if a non-negative function g has $\mu([g > 0]) > 0$, then by continuity of measure from below we must have $\mu([g > \epsilon]) > 0$ for some $\epsilon > 0$ and then the claim follows).

Thus we have that f must be 0 a.e. on $[f < 1]$ and must have $\lambda([f > 1]) = 0$, implying that of course $f = 1_{[f=1]}$ almost everywhere, as desired.

Exercise 1.6

(Egoroff but convergence to ∞) Suppose μ is a finite outer measure on X and $\{f_n\}_{n=1}^{\infty}$ is a sequeunce of μ-measurable functions $X \to [0, \infty)$ s.t. the limit $\lim_{n\to\infty} f_n(x) = \infty$ for every $x \in X$. We want to show that for every $\epsilon > 0$ there exists a μ-measurable set E s.t. $\mu(X \setminus E) < \epsilon$ and $\{f_n\}$ converges uniformly to $+\infty$ on E (in the sense that for every $t > 0$ there exists $n \in \mathbb{N}$ s.t. $n \geqslant N \implies f_n(x) > t$ for all $x \in E$).

Recall the statement of Egoroff's theorem from class: for μ a finite outer measure on X, and functions $f_n, f : X \to \mathbb{R}$ s.t. $f_n \to f$ μ-a.e. on X, for every $\epsilon > 0$ there exists a μ-measurable set E s.t. $\mu(X \setminus E) < \epsilon$ and $\{f_n\}$ converges uniformly to f on E.

We may use this theorem to prove our exercise as follows: since $\arctan(x)$ is an increasing function on $\mathbb{R}$ that has limiting value $\frac{\pi}{2}$ as $x \to \infty$, we have that $\arctan(f_n(x)) \to \frac{\pi}{2}$ as $n \to \infty$ for all $x \in X$. By Egoroff's theorem as stated above, fixing arbitrary $\epsilon > 0$, we have a μ-measurable set E s.t. $\mu(X \setminus E) < \epsilon$ and $\{\arctan(f_n)\}$ converges uniformly to $\frac{\pi}{2}$ on E. Thus for every lesser value $\ell < \frac{\pi}{2}$, there exists $N \in \mathbb{N}$ s.t. $n \geqslant N \implies \arctan(f_n) \geqslant \ell \implies f_n \geqslant \tan(\ell)$ on all of X (using that $\tan(x)$ is increasing function on $(-\frac{\pi}{2}, \frac{\pi}{2})$. But as $\tan(x) \to \infty$ as $\ell \nearrow \frac{\pi}{2}$, we have indeed shown that $\{f_n\}$ converges uniformly to $+\infty$ on E.

Exercise 1.7

Suppose $X \subseteq \mathbb{R}^d$ is a compact set and $f_n \in \mathcal{C}(X)$ is a sequence of continuous functions s.t. $f_n \leqslant f_{n+1}$ for all $n \in \mathbb{N}$. We assume that $\lim_{n\to\infty} f_n(x)$ exists and is finite for all $x \in X$, and we call this limit $f(x)$. We want to show that f is continuous if and only if f_n converges uniformly to f.

($\implies$): since the sequence is increasing in n, it is true that $f(x) \geqslant f_n(x)$ for every $x \in X$ and $n \in \mathbb{N}$. Thus if f is continuous, then $f - f_n \geqslant 0$ (and is a continuous function) on X for all $n \in \mathbb{N}$ where furthermore for fixed $x \in X$, $f(x) - f_n(x)$ is a sequence decreasing (pointwise to 0) as $n \to \infty$. Fix now arbitrary $\epsilon > 0$ and define the set $E_n := \{x \in X : [f - f_n](x) < \epsilon\}$ for every $n \in \mathbb{N}$ (notice that because $f - f_n$ is decreasing in n, E_n is increasing in n). These sets E_n are open in X (inverse image via continuous function of open set $(-\infty, \epsilon)$) and since for fixed $x \in X$, $[f - f_n](x) \to 0$, it is true that $X \subseteq \bigcup_{n=1}^{\infty} E_n$.

But by compactness there is some finite subcover $\{E_n\}_{n=1}^{N}$ that suffices to cover X, and of course as the sets E_n are increasing we have that $\bigcup_{n=1}^{N} E_n = E_N$, meaning that indeed $|f - f_N| < \epsilon$ on all of $E_N = X$. As the $f - f_n$ are decreasing in n we have furthermore that $n \geqslant N \implies |f - f_n| < \epsilon$, so indeed we have uniform convergence of $f_n \rightrightarrows f$ on X.

($\impliedby$): this is just a "triangle inequality/$\frac{\epsilon}{3}$" style argument (we don't need "increasing in n" for this part). Fix $\epsilon > 0$. Then there is $N \in \mathbb{N}$ s.t. $n \geqslant N \implies \|f - f_n\|_\infty < \frac{\epsilon}{3}$. As f_N is continuous, we have for any fixed $x_0 \in K$ that is $\delta > 0$ s.t. $|x - x_0| < \delta \implies |f_N(x) - f_N(x_0)| < \frac{\epsilon}{3}$. Hence, $|x - x_0| < \delta \implies |f(x) - f(x_0)| \leqslant |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) + f(x_0)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$, and so f is continuous at $x_0 \in K$. As $x_0 \in K$ was arbitrary, we have f is continuous on all of K, as desired.

Exercise 1.8

We define absolute continuity, notated $\mu \ll \nu$ to mean that $\nu(E) = 0 \implies \mu(E) = 0$ for all E both μ- and ν-measurable. We prove that for finite μ (so $\mu(X) < \infty$), $\mu \ll \nu \iff$ for every $\epsilon > 0$ there is $\delta > 0$ s.t. $\nu(E) < \delta \implies \mu(E) < \epsilon$ (for all E both μ- and ν-measurable). Let $\mathcal{A}$ denote all μ- and ν-measurable sets.

($\implies$): suppose not. Then, there exists $\epsilon > 0$ s.t. for all $\delta > 0$, there is $A \in \mathcal{A}$ s.t. $\mu(A) \geqslant \epsilon$ and $\nu(A) < \delta$. More specifically, for any $k \in \mathbb{N}$ we can find $A_k \in \mathcal{A}$ s.t. $\nu(A_k) < \frac{1}{2^k}$ and $\mu(A_k) \geqslant \epsilon$. Define the decreasing sequence of sets $B_k = \bigcup_{i=k}^{\infty} A_i$. From that definition, we can say that

$$\nu(B_k) \leqslant \sum_{i=k}^{\infty} \nu(A_i) < \frac{2}{2^k} = \frac{1}{2^{k-1}}.$$

Then, define $B = \lim_{k\to\infty} B_k = \bigcap_{k=1}^{\infty} B_k$. Then because limits of monotone decreasing sets and measures commute (again, as long as the measure of the sets in the sequence are not infinite), we know that

$$\nu(B) = \nu\left(\lim_{k\to\infty} B_k\right) = \lim_{k\to\infty} \nu(B_k) = \lim_{k\to\infty} \frac{1}{2^{k-1}} = 0$$

However, $\mu(B_k) \geqslant \mu(A_k) \geqslant \epsilon$, so by limit-measure commutativity (and that μ is finite), we know that

$$\mu(B) = \mu\left(\lim_{k\to\infty} B_k\right) = \lim_{k\to\infty} \mu(B_k) \geqslant \epsilon$$

But by absolute continuity, $\nu(B) = 0 \implies \mu(B) = 0$; hence, contradiction.

Summary: the method we used here was to assume the contrapositive, getting a sequence of "exceptional" objects that in a sense become "worse and worse", and then take a "limit" of some sort to get a limiting object that is so "exceptionally bad", that it creates a contradiction.

($\impliedby$): this direction is quite simple. Assume $\nu(N) = 0$ for some $N \in \mathcal{A}$. Then, we know that for all $\epsilon > 0$ there exists $\delta > 0$ s.t. $\nu(A) < \delta \implies \mu(A) < \epsilon$. Well, $\nu(N) = 0$ which is less than δ for any δ, so for all $\epsilon > 0$, $\mu(N) < \epsilon \implies \mu(N) = 0$ and so $\mu \ll \nu$.

Exercise 1.9

We say that a collection of functions $\Phi \subseteq L^1(\mu)$ is uniformly integrable precisely when for every $\epsilon > 0$ there exists $\delta > 0$ s.t. for every $f \in \Phi$, $\mu(E) < \delta \implies \int_E |f| \, d\mu < \epsilon$ (where E is taken to be μ-measurable).

(i) We want to show that any finite subset of $L^1(\mu)$ is uniformly integrable. We start with a single function $f \in L^1(\mu)$. Fix $\epsilon > 0$. First, $\lambda(E) := \int_E |f| \, d\mu$ is a measure on $\mathcal{M}(\mu)$ — check the two properties (measure of null set is 0, and countably additive):

- $\lambda(\varnothing) = \int_\varnothing f \, d\mu = \int f \cdot 1_\varnothing \, d\mu = \int 0 \, d\mu = 0$
- Let $\{A_k\}_{k=1}^\infty$ be a sequence of disjoint measurable sets, and let A be their union. Then,

$$\int_A f \, d\mu = \int f \cdot 1_A \, d\mu = \int f \cdot \sum_{k=1}^\infty 1_{A_k} \, d\mu = \int \sum_{k=1}^\infty f \cdot 1_{A_k} \, d\mu = \sum_{k=1}^\infty \int f \cdot 1_{A_k} \, d\mu = \sum_{k=1}^\infty \int_{A_k} f \, d\mu$$

  where the second-to-last equality comes from the monotone convergence theorem. Thus, we've shown that for disjoint measurable sets $\{A_k\}_{k=1}^\infty$, $\lambda(\bigcup_{k=1}^\infty A_k) = \sum_{k=1}^\infty \lambda(A_k)$.

Using this, we obviously have $\mu(E) = 0 \implies \lambda(E) = 0$ and so by absolute continuity (see Ex. 18 above) there exists $\delta > 0$ s.t. $\mu(E) < \delta \implies \int_E |f| \, d\mu < \epsilon$. Now for finitely many functions we have $\delta_1, \ldots, \delta_n$ s.t. $\mu(E) < \delta_i \implies \int_E |f_i| \, d\mu < \epsilon$ for $i \in [n]$, and so taking $\delta := \min\{\delta_1, \ldots, \delta_n\} > 0$ suffices.

(ii) We want to show that if $\Phi := \{f_n\}_{n=1}^\infty$ is a sequence that converges in $L^1(\mu)$ to some $f = f_0$ then Φ is uniformly integrable. Let us fix $\epsilon > 0$. By convergence in L^1, there is $N \in \mathbb{N}$ s.t. $n \geqslant N \implies \int_\Omega |f - f_n| \, d\mu < \frac{\epsilon}{2}$. Since $\{f_0, f_1, \ldots, f_N\}$ is a finite subset of $L^1(\mu)$, we know by part (i) that it is uniformly integrable, so we have some δ s.t. $\mu(E) < \delta \implies \int_E |f_i| \, d\mu < \frac{\epsilon}{2}$ for $i \in \{0, \ldots, N\}$. Then, for any $n \geqslant N$ and μ-measurable E with $\mu(E) < \delta$, we see that $\int_E |f_n| \, d\mu \leqslant \int_E |f| + |f_n - f| \, d\mu \leqslant \int_E |f| \, d\mu + \int_E |f_n - f| \, d\mu < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$; and for any $n \in \{1, \ldots, N\}$, we have $\mu(E) < \delta \implies \int_E |f_n| \, d\mu < \frac{\epsilon}{2} < \epsilon$, so indeed Φ is uniformly integrable.

Remark: this exercise is subsumed by the far more general Vitali convergence theorem; in particular because L^1-convergence easily implies convergence in measure (by Markov's inequality) this problem is a direct corollary of the Vitali theorem as stated in the linked reference.