In today's GSO, Daniel defined:

For a topological space X, a "hard basis" is a collection U of open sets such that $\forall x \in X, u \ni x$ open, $\exists u' \in U$ with $x \in u' \subseteq \overline{u'} \subseteq u$. A "honing structure" on X is a sequence $B_0 \supseteq B_1 \supseteq \dots$ of *countable* hard bases of X such that for any $b_0 \in B_0, b_1 \in B_1, \dots$, if $b_0 \supseteq \overline{b_1} \supseteq b_1 \supseteq \overline{b_2} \supseteq b_2 \supseteq \dots$ then $\exists x \in X : \bigcap_n b_n = \{x\}$.

1 Item 1

Daniel asked: If X has a honing structure, then for any $b_0, b_1, \dots$ as above with $\bigcap_n b_n = \{x\}$, must every open $u \ni x$ contain some b_n?

No: Let $X := \{0, 1, \dots, \infty\}$ with the discrete topology, $u_x := \{z : z \geq x\}$ for $x \in X$, and honing structure $B_n := \{u_x : x \geq n\} \cup \{\{x\} : x \in X\}$. Then $u_0 \in B_0, u_1 \in B_1, \dots$ satisfies the conditions with $\bigcap_n u_n = \{\infty\}$, but the open set $\{\infty\}$ contains no u_n.

Once we show (in Item 2) every space with a honing structure is metrizable, it's easy [assuming dependent choice] that, in general, a counterexample $(X', \{B'_n\}, \{b'_n\}, x', u')$ is "the same data" as a topological embedding $\varphi : X \hookrightarrow X'$ with compatible honing structures, meaning $\exists h : \mathbf{Z}_{\geq 0} \to \mathbf{Z}_{\geq 0}$ strictly increasing with $(\{B_n\}, \{b_n\}, \infty, \{\infty\}) = \varphi^*((\{B'_{h(n)}\}, \{b'_{h(n)}\}, x', u'))$ (pullback).

2 Item 2

Someone asked: Is every topological space with a honing structure Polish [separable + completely metrizable]?

Yes: Let X have a honing structure.

Step 1: X is T_1: If $x_1 \neq x_2 \in X$, pick $b_0, b_1, \dots$ with $\bigcap_n b_n = \{x_1\}$ (since B_n is a hard basis, we can iteratively pick b_n so $x \in b_n \subseteq \overline{b_n} \subseteq b_{n-1}$). One of them doesn't contain x_2.

Step 2: X is regular: In fact, "X has a hard basis" implies X is regular (but not necessarily T_1). Indeed, let $x \in X, C \subseteq X \setminus \{x\}$ closed. By the hard basis, $\exists u$ open with $x \in u \subseteq \overline{u} \subseteq \complement C$. For each $c \in C$, by the hard basis there's some open u_c with $c \in u_c \subseteq \overline{u_c} \subseteq \complement \overline{u}$. Then $u \ni x$ is disjoint from $\bigcup_{c \in C} u_c \supseteq C$.

Step 3: X is metrizable: Follows from Urysohn metrization [2nd countable + T_1 + regular $\implies$ metrizable]. (X is 2nd countable because B_0 is a countable basis.)

Step 4: X is completely metrizable: It's well-known that a metrizable space is completely metrizable iff Player II wins the strong Choquet game. Player II wins by, on turn n, picking an element of B_n.

Step 5: X is Polish: All that's left is to show X is separable. This follows from X having a countable hard basis.

3 Item 3

Daniel showed: For every complete metric space with countable dense set S and every $r_0 \geq r_1 \geq \dots \to 0$, there's a honing structure $B_n = \{B_{r_m}(s) : m \in \mathbf{Z}_{\geq 0}, m \geq n, s \in S\}$.

While Item 2 shows every topological space with a honing structure is Polish, there might not be a metric making the honing structure of this form.

Indeed, for any honing structure of this form, for any $b_0, b_1, \dots$ with $\bigcap_n b_n = \{x\}$, any $x_0, x_1, \dots \in X$ with $x_n \in b_n$ converges to x in X. But Item 1's example doesn't satisfy this: $0, 1, 2, \dots \not\to \infty$.

(Open question: Is this the only obstruction?)

4 Item 4

Someone asked: Does "hard basis" = "basis in a T_3 space"? Not quite: Item 2 Step 2 shows regularity (without T_1) suffices.

But every basis of a T_3 (or even regular) space is a hard basis: Let X be regular with basis B, $x \in X, u \ni x$ open. We'll construct $b \in B$ with $\overline{b} \subseteq u$. Regularity gives u_1, u_2 disjoint open with $x \in u_1, \complement u \subseteq u_2$. Since B is a basis, $\exists b \in B : b \subseteq u_1$. This works.

Conclusion: X is regular (not necessarily T_1) $\iff$ "there exists a hard basis" $\iff$ "every basis is a hard basis".