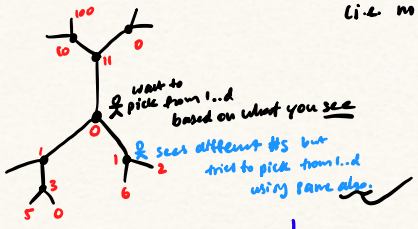
Thm (Marks): $\forall d \in \mathbb{N}$ ($d=1,2$ done) so $d \geq 3$

$\exists$ Borel d-regular forest G s.t. $\chi_B(G) \geq d+1$.

(i.e. no way to color w/ $\leq d$ colors)

Construction:



algo: label you see $\mapsto$ color 1..d s.t. proper: 2 labels same after 1 shift $\mapsto$ different colors.

Marks construct G s.t. every coloring of it corresponds to $\Rightarrow$ $\chi_B(G)$ can not be $\leq d$.

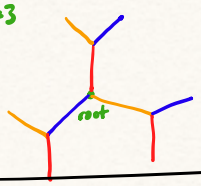
we show this also must fail.

Let $\mathbb{T}_d$ be
rooted d-regular tree
whose edges are properly colored w/ colors $1..d$

e.g. $d=2$



$d=3$



Observe: $\forall$ vtx $v \in \mathbb{T}_d$,
$\exists!$ color preserving automorphism of $\mathbb{T}_d$ sending root $\mapsto v$

we work in the space $\mathrm{Col}(\mathbb{T}_d, \mathbb{N})$ of all proper $\mathbb{N}$-colorings of $\mathbb{T}_d$

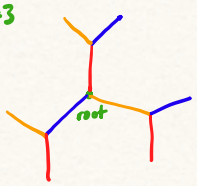
6/4/25 **Thm (Meeks)** $\forall d \in \mathbb{N}$
$\exists$ acyclic Borel graph G, max deg $\leq d$
w/ $\chi_B(G) = d+1$.

e.g. $d=2$



Pf: Def $\mathbb{T}_d$ rooted d-regular tree w/ edges properly colored using $\{1 \dots d\}$ colors

$d=3$



Rmk: up to isomorphism, only 1 coloring possible?

(also $\mathbb{T}_d$ = Cayley graph of $(\mathbb{Z}/2\mathbb{Z})^{*d}$ ($*$ free product; colors = the generators)

Observe: $\forall$ vtx $v \in \mathbb{T}_d$,
$\exists!$ color preserving automorphism $F_v : \mathbb{T}_d \to \mathbb{T}_d$
root $\mapsto v$

Let v_i be nbr of root connected via edge w/ color i



$F_i := F_{v_i}$ "shift" root to v_i
i.e. move along edge w/ color i.

Note: $F_i^2 = \mathrm{id}$ involution

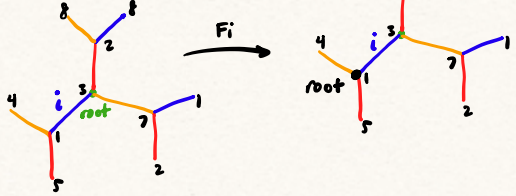
Def $\mathrm{Col}(\mathbb{T}_d, \mathbb{N})$ ($\{1..d+1\}$ also works, but $\mathbb{N}$ easier to write)
be space (Polish) of all proper $\mathbb{N}$-colorings $x : \mathbb{T}_d \to \mathbb{N}$
(closed subspace of $\mathbb{N}^{\mathbb{N}}$)

Of course, can "lift" automorphism to act on colorings, by attaching colors to vertices
setting permuted

$F_i[x] \in \mathrm{Col}(\mathbb{T}_d, \mathbb{N}) \quad \forall x \in \mathrm{Col}(\mathbb{T}_d, \mathbb{N})$

$F_i[x] := x \circ F_i$

e.g.



$G := G_{F_1 \dots F_d}$, i.e.
$V(G) = \mathrm{Col}(\mathbb{T}_d, \mathbb{N})$
$E(G) = \{\{x, F_i[x]\} : x \in \mathrm{Col}(\mathbb{T}_d, \mathbb{N}), i \in \{1..d\}\}$
(diff bc color of root diff by preserves)

the adjacent vertices in graph

Fact: G has max deg $\leq d$.

But... G not forest bc could $\exists$ pattern in labelling
(s.t. walking around, see same pattern again.)
Intuition: for most/generic X, don't have this issue

Fix: $G' :=$ subgraph of G induced by set of vtxs x s.t. the comp. of x in G is tree
Fact: $V(G')$ is meagre in $Col(T_d, N)$ so $G' \neq \emptyset$.
Borel by Lusin–Novikov

$\Rightarrow$ G' Borel forest max deg $\leq d$.

---

we show $\chi_B(G') > d$.

Actually, we show $\forall$ Borel $f: V(G') \to \{1..d\}$,
$\exists\ x \in V(G')$; $i \in \{1..d\}$ s.t.
$f(x) = f(F_i(x)) = i$
"f not proper" — the twist "edge bt them prevents color of endpts"

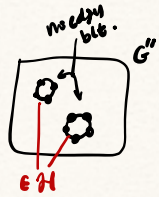
Actually can forget about G' bc enough to prove for G in place of G'

Pf sketch "uses previous lec's tricks, good review"

Def: function f failing to have above prop
a ~~coo~~ coloring.
"no ii" "no iii" "no iii"
WTS: G has no coot Borel colony $\Rightarrow$ G' has no coal Borel colony

Change perspective: Write $G = G' \sqcup G''$ where every comp of G'' has cycle.

Claim G'' does have a Borel "no iii" coloring.
Why? "Similar to Gallai tree arguments"



① find Borel maximal fam $\mathcal{H}$ of disjoint nonadj cycles in G''

② $A := \{v \in V(G'') : \exists C \in \mathcal{H} : v \in C\}$
i.e. vertices in cycles in $\mathcal{H}$.

$\Rightarrow$ A is Borel set
meeting $\forall$ comp of G'',
and comps of $G''[A]$ are
exactly cycles $\in \mathcal{H}$.

③ find Borel proper coloring of complement $G'' - A$ using colorings $\{1..d\}$

④ $\forall$ vtx $x \in A$,
has ≥ 2 available color $\Rightarrow$ $\exists$ extension of to yield a no iii coloring of G''
combinatorics
(perhaps nonproper tho e.g. if C odd length and only 2 colors available)

⑤ Bc $G''[A]$ is component-finite, can be Borel extension.

Return to our G: $V(G) = Col(T_d, N)$
$E(G) := \{\{x, F_i(x)\}\}$

Given Borel $f: V(G) \to \{1..d\}$,
want to find $x \in Col(T_d, N)$, $i \in \{1..d\}$ s.t. $f(x) = f(F_i(x)) = i$

we know mble, BM approach fail.

Our last new idea: **Infinite Games** (ω many rounds)

Framework:

"payoff set" $A \subseteq \mathbb{N}^{\mathbb{N}}$ ← in fact any other set works

| e.g. Go
- finite set
- moves you aren't allowed
- game ends

} can still fit in our framework by adding ∞ dummy moves and restricting A.

can define game $\Gamma(A)$ defined as

Alice (1)	n_0		n_2		...
Bob (2)		n_1		n_3	

Alice wins $\Leftrightarrow (n_0 \dots) \in A$

Def: strategy for Alice is alg (/oracle/function)

$$\sigma : (\text{position Alice may find herself facing}) \to \mathbb{N}$$

i.e. $\mathbb{N}^* \to \mathbb{N}$

A play of game $(n_0 \dots)$ is consistent w/ σ means $n_0 = \sigma(\phi)$, $n_2 = \sigma(n_0, n_1)$, ⋮ ("Alice played according to strategy σ")

and σ is winning strat (for Alice) means ∀ play $(n_0 \dots)$ consistent w/ σ ends up $\in A$.

analogous def for Bob

Observe: obviously impossible that a game $\Gamma(A)$ both winning strat for Alice $\exists \sigma_A, \sigma_B, \dots$

Punchline: **Thm (Martin)** (, UCLA!): If $A \subseteq \mathbb{N}^{\mathbb{N}}$ is Borel (aka pointset), then either Alice or Bob (exactly one of the other) has a winning strat. (i.e. $\Gamma(A)$ is determined)

(Martin first proved analytic determinacy assuming Large Cardinals)

reduce to closed set

but closed set ... ∎

6/6/25 Finish Mark's proof using ∞ games

Recall graph we've been working w/.

T_d: v_d — d — root — 1 — v_1, 2 — v_2, 3 — ... (1 = color of edge)

$F_i : T_d \to T_d$ unique (edge) color preserving automorphism sending root $\mapsto v_i$

which lets us fix labelings of T_d:

$x : T_d \to \mathbb{N} \rightsquigarrow F_i[x] := x \circ F_i$

i.e. $F_i[x]$ is move the "root" label to v_i

G $V(G) = \mathrm{Col}(T_d, \mathbb{N}) := \{\text{proper } \mathbb{N}\text{-colorings } x \text{ of } T_d\} \subseteq \mathbb{N}^{T_d} \cong \mathbb{N}^{\mathbb{N}}$

$E(G) = \{x, F_i[x]\}$ $\forall x, \forall i$ (only reason do this is to guarantee no loops) (simple reason)

"proper colorings vs arbitrary/general $\mathbb{N}$-labelings

Now, fix Borel fn $f: V(G) \to \{1..d\}$
$= Col(T_d, \mathbb{N})$

Use Martin's Borel determinacy thm to find a pt $x \in Col(T_d, \mathbb{N})$
and an index $i \in \{1..d\}$ s.t. $f(x) = f(F_i(x)) = i$

f not proper

What Game?

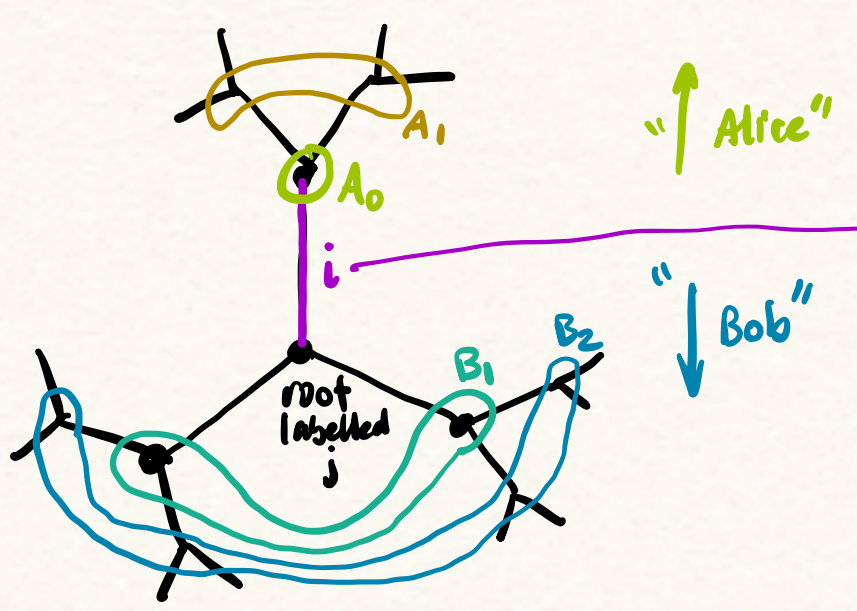
"The genius of all this is to cook up up the right game"

For $i \in \{1..d\}$, $j \in \mathbb{N}$ define game
(the color we want a "triple" of)

Γ_{ij} : game board $= T_d$
Alice, Bob take turns labelling vtxs of T_d using $\mathbb{N}$
ultimately producing a pt $x \in Col(T_d, \mathbb{N})$

On kth move, Alice / Bob color vertices in set A_k / B_k (resp.) as shown

(e.g $d=3$)



Both players are required to maintain that the coloring is proper.

Alice wins $\iff f(X) \neq i$

Bob wins $\iff f(X) = i$ ("Bob win $\iff$ "achieve half of our goal")

only place we use f Borel. in whole proof!

B/c f Borel,
each game Γ_{ij} is determined (by Martin)

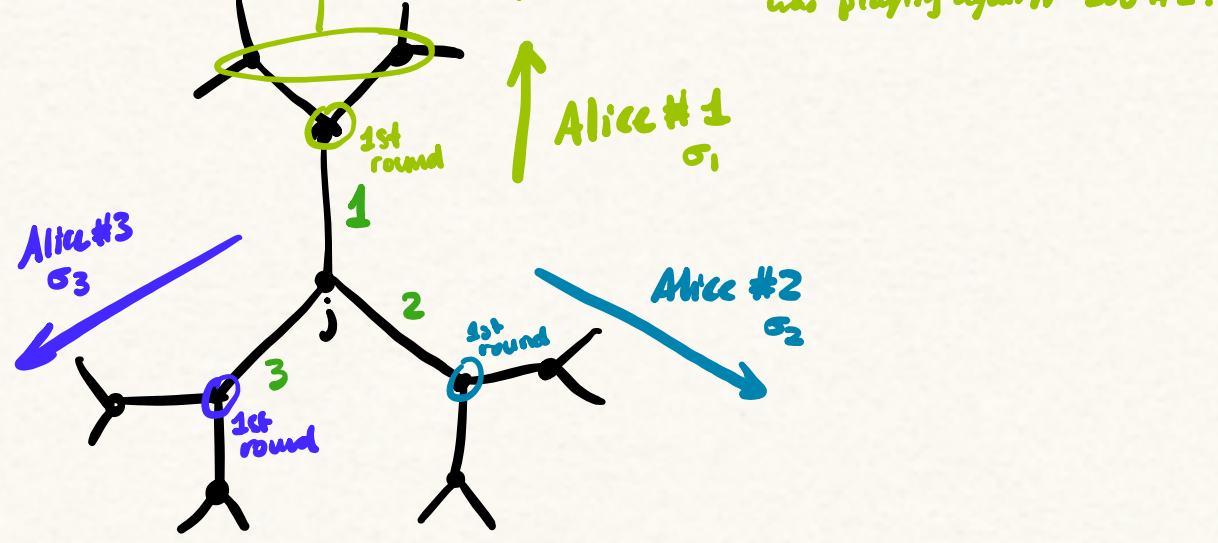
Claim: $\forall j \in \mathbb{N}$,
$\exists i \in \{1..d\}$ s.t. Bob wins $\Gamma_{i,j}$

(For fixed j, you have d different games $\Gamma_{1,j} \dots \Gamma_{d,j}$)
we're saying must have ≥ 1 of ↑, Bob has winning strat.

Pf. suppose fsoc not.
Then Alice has winning strat σ_i for $\Gamma_{i,j}$ $\forall i \in \{1..d\}$

Idea: Have d Alices ("team effort") play in parallel σ_i against each other

upon seeing other Alices' moves, it's as if Alice #1 was playing against imagining Bob #1.



σ_i winning $\Rightarrow f(x) \neq i$ $\forall i \in \{1..d\}$ ✗.

all Alices winning at same time "Alice democracy"

↓

"Bob's job is hard (has to = i)
but he can win!"

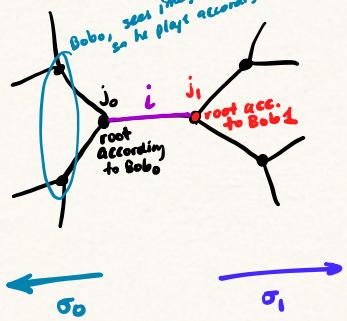
Next idea: pigeonhole (that's why can replace N by $\{1..d+1\}$)

$\exists j_0 \neq j_1$ and same $i \in \{1..d\}$ s.t.

Bob wins both Γ_{i,j_0} and Γ_{i,j_1}

Final idea: 2 Bobs play against each other.

Bob$_0$ sees imaginary Alice play j_1 "across the board", so he plays according to his winning strat σ_0.



Then Bob$_1$ will see imag. Alice 1 "across the board" play j_0, ...

So, both Bobs win!

But then σ_0 winning for Bob$_0$ $\Rightarrow f(x) = i$

σ_1 . . . . . $\Rightarrow f(F_i(x)) = i$

QED

Try to gen to not trees.

What else is out there??

* matching theory (Banach Tarski)

Thm (Hall): G loc. fin. bipartite graph
if $\forall$ finite $S \subseteq V(G)$, it $|N_G(S)| \geq |S|$
$\Rightarrow$ G has perfect matching.

As stated, fail w/ μ, BM. But

Thm (Marks & Unger): G Borel loc fin. graph on Polish space

s.t. $\exists \varepsilon > 0$:
$\forall S \subseteq V(G)$ finite,
$|N_G(S)| \geq (1+\varepsilon)|S|$

then G has BM perfect matching (μ fails)

$\Rightarrow$ BM version of Banach Tarski

* Application in ergodic theory

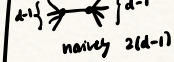
* Applications / connections w/ CS.
(distributed computing)

* connections to metric geometry / geom. group theory
often allow improving BM/μ to Borel

Leave w/ open questions ("I had many to choose from, but I'll give a particularly interesting one")

normal graph theory

Thm (Vizing 64): G has max deg $\leq d \in \mathbb{N}$,
then $\chi'(G) \leq d+1$
index = coloring of edges



naively $2(d-1)+1 = 2d-1$

Borel? $\chi'_B(G) \leq 2d-1$. ✓ Optimal (Marks again, by Gamer)

UCLA 2023
Thm (Grebik / Grebik-Pikhurko): $\chi'_\mu(G) \leq d+1$

BM? **Thm (Bean-Weilacher):** $\chi'_{BM}(G) \leq d+2$. Anton: 😠!

"one of few natural problems that BM harder than μ"