Corfits Over Well of Arboreal Bored Lawns.
" Lone
UCLA
From bowl to lawn, comb to sorrel

Prologue:

Consider the following graph: undirected

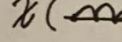
Ex1
$V = \mathbb{R}$
$E = \{\{x, x+1\} : \forall x \in \mathbb{R}\}$



What is each connected component? (comp)

(Bamboo) Tree continuum many disjoint conns.

Chromatic # = min # color needed for proper coloring

$\chi($  $) = 2.$

Algorithm? $[0,1)$ color red, determines everything.

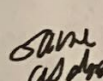
Algo: $c(x) = \begin{cases} R & \lfloor x \rfloor \text{ even} \\ B & \lfloor x \rfloor \text{ odd.} \end{cases}$ Easy

---

Ex2
$V = S^1 = \mathbb{R}/\mathbb{Z}$
" $\pi \in \mathbb{C}$
$E = \{\{x, e^{2\pi i \alpha} x\} : x \in \pi\}$ for fixed $\alpha \notin \mathbb{Q}$



What is each comp?

same as above $\Rightarrow \chi($  $) = 2.$

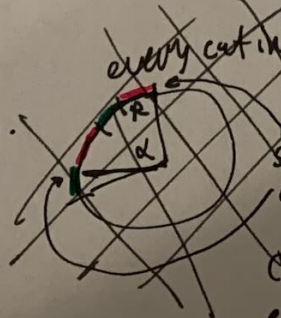


but algorithm? Axiom of Choice (continuum many choices)

How to formalize this notion of no algorithm?

every cut into R RANG!

Note: if allow 3 colors, have algo.



3rd color used "$< \varepsilon$" amount

Start w/ Red end at beginning of Green.

Can thus decide color on each interval. where $\nexists$, use color

Chapter 1: Formalizing "Algorithmic" (technical) not needed for talk/Marko thm, but important to justify philosophy of "Borel = algorithmic"

> Have printed document w/ def of [cktl Boolean pseudo circuit.]

Say outloud def.
On board draw example

& def of [an eval of C on input X]

cktl circuit (1) output input 1 1 1 ..... 2 evaluations top is 1

(2) 0 evaluations or top 0.

Ex: from ws notes 2220 of well-fdd

(1) (2)

$\exists \vec{x}$: $x_0 = 1$, $\exists \geq 1$ other 1 }

(3) countable height tree. x_0 x_1 x_2

~~$\exists \vec{x}$: $\forall i$ $x_i \neq i \bmod 2$~~ $\vec{x} = 1010\ldots$ }

x_0 x_1 x_2 x_3 . . .

$\exists \vec{x}$: no two consecutive 1's in $\vec{x}$ }.

ASSERTION 1: well-fdd pseudo circuit C, $\forall$ input x
$\exists!$ evaluation $eval_x : V \to 2$. $C(x) := eval_x(0)$.
of C on x — output node.

Def [set $A \subseteq 2^{\mathbb{N}} = \{0,1\}^{\mathbb{N}}$ = set of all ∞ 0/1 sequences
called CCC "cktl circuit computable" if $\exists$ C circuit s.t. $C(x) = 1_A(x)$.]

ASSERTION 2: $A \subseteq \{0,1\}^{\mathbb{N}}$ Cantor space is CCC $\Leftrightarrow$ Borel. (if you know topo def. If not, take this as def of Borel.)
for those who missed Borel. — think "computable".

↑ 8 min checkpoint

But this is just about Cantor space / How to feed in input $x \in X$ for general X?
Here is general strategy that works for a large class of X, but I'll just do 2 examples:

Ex 1: $X = \mathbb{R}$. cbly many $\{B(q, \frac{1}{n}) : q \in \mathbb{Q}, n \in \mathbb{N}^+\}$.

enumerate as $B_1, B_2 \ldots$

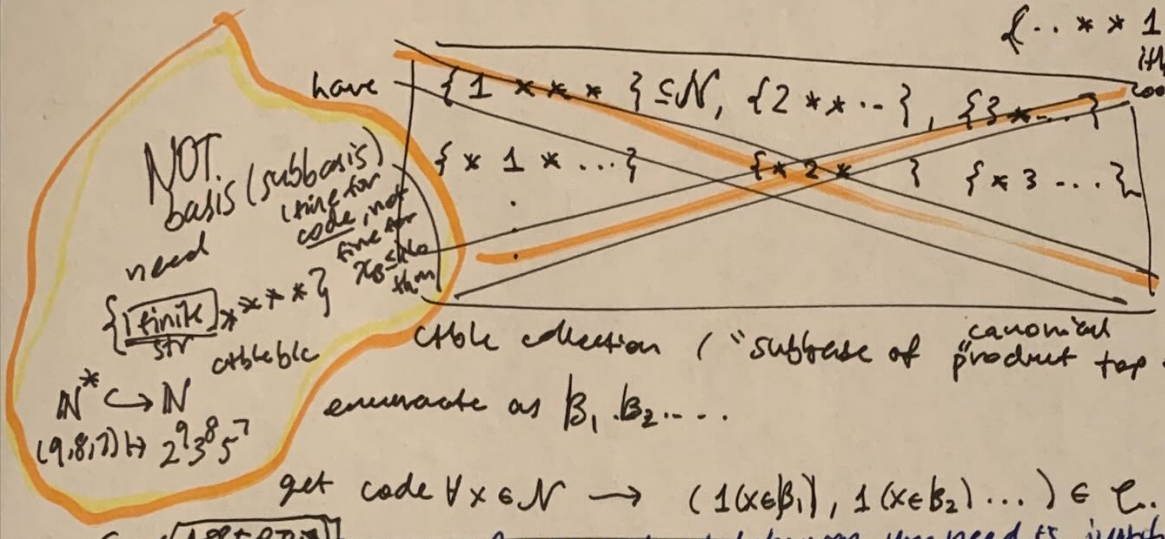
Then $\forall x \in \mathbb{R}$, get code $(1_{(x \in B_1)}, 1_{(x \in B_2)} \ldots) \in \zeta$. which then goes through cktl Boole circuit

Assert: Borel subsets of $\mathbb{R}$ are exactly those $\{x \in \mathbb{R} : C(\text{code}(x)) = 1\}$ computed by some circuit C with Bool gates.

Ex2: $X = \mathbb{N}^{\mathbb{N}}$ (Baire space).

much like (x_i) gate in Cantor circuit is $1(x_i \in B_i)$ when B_i is basic subbasic

$\{\ldots * * 1 * * \ldots\} \subseteq \tau$.



get code $\forall x \in \mathbb{N} \to (1(x \in B_1), 1(x \in B_2) \ldots) \in \tau$.

~~Sanity ASSERTION~~

Again, only takeaway you need is justification/intuition for "Borel ≈ algorithmic" (or at least not Borel ⇒ not algorithmic)

Returning to Prologue, I claim $\gamma_B(\text{🐭}) = 3$, i.e. > 2.

i.e. suppose A, B is proper coding of 🐭.
Then A, B can't be Borel.

we prove this using MST theory

State on printout

Black Box: $\exists$ "fiction of mean size" μ^*: $\text{Pow}(\mathbb{R}) \to [0,\infty]$ s.t. ~~reflecting~~
- $\mu^*(\emptyset) = 0$ - nonotonicity $E \subseteq F \Rightarrow \mu^*(E) \le \mu^*(F)$; countable subadd $\mu^*(\bigcup_1^\infty A_i) \le \sum_1^\infty \mu^*(A_i)$.
- ~~reflect~~ translation inv.
- we can define "nullset" $A \subseteq \mathbb{R} \iff \mu(A) = 0$.
- outer regular
- can define $\mu\text{Sbl}(\mathbb{R}) \subseteq \text{Pow}(\mathbb{R})$ by $:= \{A : \exists \text{Borel } B \text{ s.t. } \mu(A \triangle B) = 0\}$ (Borel(R))

i.e. Borel but can ~~edit~~ (add/delete) a nullset

good restriction μ: $\text{Leb}_{\mu\text{Sbl}}$: $\mu\text{Sbl}(\mathbb{R}) \to [0,\infty]$, μ is countably additive: $A_i \ldots$ disjoint all $\in \mu\text{Sbl}(\mathbb{R})$, $\Rightarrow \mu(\bigsqcup_1^\infty A_i) = \sum_{i=1}^\infty \mu(A_i)$.

coroot outer reg: 99% lem (all blobs in 2231)
$\forall A \in \text{Pow}(\mathbb{R}), \forall \varepsilon > 0, \exists U \supseteq A$ s.t. $\frac{\mu^*(A)}{\mu(U)} > 1 - \varepsilon$. A fills up $>99\%$ of U.

By identifying S^1 w/ $\mathbb{R}/2\pi\mathbb{Z}$,
get rotation inv countably additive msr on Msble sets.

prove that $\chi_\mu(\mathbb{S}^1) > 2$.

Suppose $S^1 = R \sqcup B$ (red, blue), where $R, B \in \text{Mble}(\mathbb{R})$.

Observe $R = B + \alpha$ (rotation by α). (blc. $B+\alpha \subseteq R$, $R-\alpha \subseteq B$)

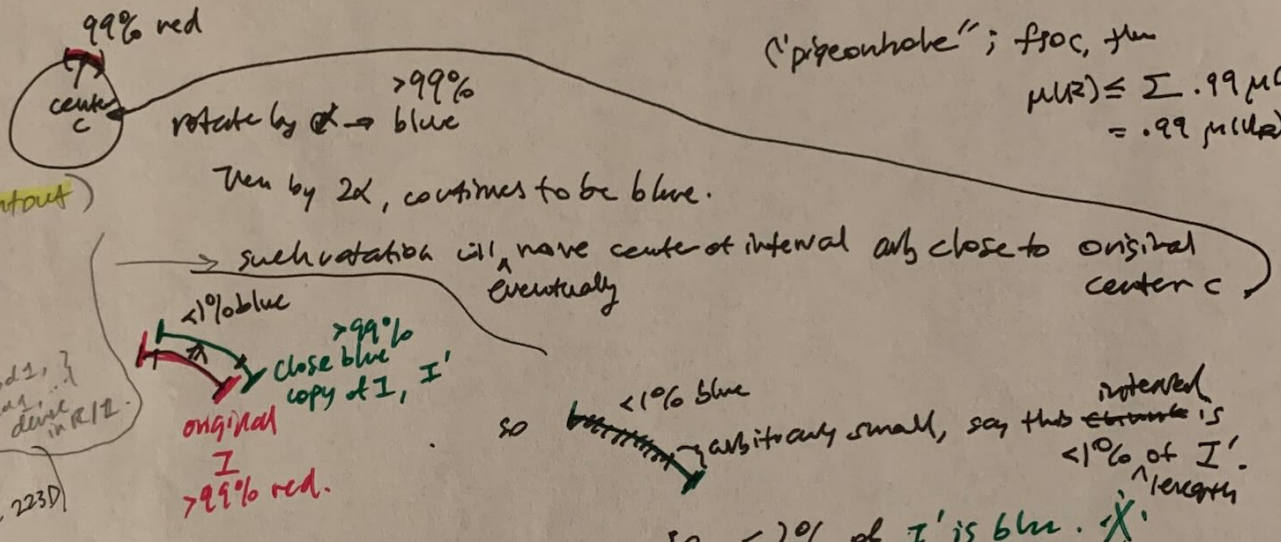
By finite! additivity, $\text{Leb}(R) = \text{Leb}(B) = \frac{1}{2}$. (normalizing $\text{Leb}(S^1) = 1$ say.)
rotation invariance.

99% Leb. says $\exists$ open $U_R \supseteq R$, ~~$U_R \subseteq B$ / of mass $\leq \frac{1}{2}+\varepsilon$~~

R is $> 99\%$ of U_R.

$U_R = \bigsqcup_1^\infty$ open intervals, must $\exists$ 1 such interval I s.t.
R fills up $> 99\%$ of I.

("pigeonhole"; proc, then $\mu(R) \leq \sum .99\,\mu(I) = .99\,\mu(U_R)$ ✗)



(oh, I just found, 5/16/25 Lec 23(D))

Conclusion Chp 1 & Prologue.

So using powerful tools of our theory, found $\chi_\mu(\mathbb{S}^1) > 2$ $(= 3)$.

Borel nicer than μ-mble, so $\chi_B \geq \chi_\mu$ always.
~~so Borel coloring harder than mble color~~

Thus lower bd on χ_B via χ_μ is common strategy.

The final half of this talk will be about a problem in which this strategy doesn't work.

↑ 7min ~~thereof~~ 2

(Blackbox 1) I tracked down source — my memory still working!
Lebesgue density thm says Toto 524 11/30/20
A mble $\Rightarrow$ $\forall^{a.e.}$ $x \in A$, $\exists$ interval $I \ni x$ s.t. $\forall\, x \in J \subseteq I$ (interval containing x)
A is $> 99\%$ of J.

Chapter 2: d-reg forest

First, I must state some black-boxes. or point out?

(5/12/25 Lec 19 2230 Notes)

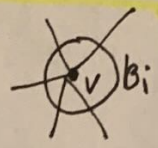
Thm: If G has loc finite Borel graph — think vertex set V a Borel subset of one of the spaces I brought up previously: $\mathbb{C}, \mathbb{N}, \mathbb{R}$ and edges as a countable ctbl. union of Borel subsets of $V \times V$

then $\chi_B(G) \le \aleph_0$.

Pf: have ctbly many Borels $B_1, B_2, \dots$ (I introduced exactly as before in Chap 1)

Define $c(v) := \min\{i \in \mathbb{N} : v \in B_i \setminus N_G(B_i)\}$.

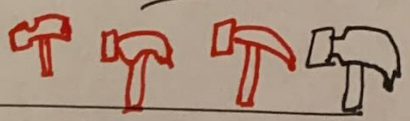
"$v \in B_i$ & no nbr of v is in B_i."



$\Leftrightarrow \{v \in B_i,\ N_G(v) \cap B_i = \emptyset\}$

well-defined b/c of loc fin! proper ✓

Borel b/c finding your nbrs is a "Borel operation", NOT trivial, instead a deep thm by Lusin–Novikov (Blackbox 2)



(5/16/25 Lec 21 2230)

Prop "Borel Zorn Lemma"

loc ctbl Borel graph w/ $\chi_B(G) \le \aleph_0$.

Then G has maximal (i.e. can not add any vtx) anticlique / indep / 1-colorable subset

Pf: $\le \aleph_0$-coloring means $V(G) = I_0 \sqcup I_1 \sqcup \dots$

Try GREED: $I_0' := I_0$

$I_n' := I_n - N_G(I_0' \cup \dots \cup I_{n-1}')$.

(ctbly) — Lusin Novikov

i.e. start w/ I_0. Then add to I everyone from I_1 you can w/o creating induced edge.

Assorts

(KST 19)

Thm: G Borel graph, max deg $d \in \mathbb{N}$

Then $\chi_B(G) \le d+1$.

Pf: induct! (very finite)

$d = 0$ ✓ (no edges)

Use previous prop $\to$ $I \subseteq V(G)$ Borel max'l indep / 1-colorable set.

Then induced subgraph $G - I$ (i.e. $(V - I, G^{\text{induced edges}})$) has max deg $\le d-1$ b/c any vtx $\in G - I$ has nbr in I

"Literature Review" $d \geq 3$.

Prev: $\chi_B(G) \leq d+1$ sharp of course, if G has components $\cong K_{d+1}$,

then $\chi_B(G) \geq \chi(G) = d+1$.

($d=2$, every component is cycle or path; odd cycle $\Rightarrow \chi = 3$.)

(3/19/25 notes)

Assert b

printout

UCLA!

Fact (Conley, Marks, Tucker-Drob '16) for $d \geq 3$

"measurable Brooks"

G has no $K_{d+1} \Rightarrow \chi_\mu(G) \leq d$.

$\forall$ Borel ~~prob~~ finite msr on $V(G)$

(i.e. holds trivially unless $\chi_\mu = d+1$)

7 mar Check p 3

enough to make my point that χ_μ can't help w/ χ_B; below just fun results.

(Thm (A. Bernshteyn '23) $d \geq 10^{600}$, G has no $K_d \Rightarrow \chi_\mu(G) \leq d-1$.

write on board for drama

Thm: $d \geq 10^{600}$, then $\chi_\mu(G) = \chi(G)$ as long as you know one is $\geq d-a$

for $a \leq \sqrt{d - \frac{1}{4}} - \frac{5}{2}$. gap ≤ 2.

~~our statement~~ ~~for~~ false for any $a \geq \sqrt{d - \frac{3}{4}} + \frac{1}{2}$.

also write on board for drama.

Thm (A-B '23): on other extreme, G has no triangles (ΔK_3) (so no $K_{\geq 3}$) (spoiler / a fortiori),

& no 4-cycles, then $\chi_\mu(G) \leq (1+o(1)) \frac{d}{\log d}$.

also this condition, proved $\chi_\mu(G) \leq (4+o(1)) \frac{d}{\log d}$. $(\frac{1}{4}+o(1))$ conj, but OPEN.

And for G a special d-regular Borel forest (the one we will discuss until the end of the talk),

matching lower bd prior

Thm (Rahman–Virág '17) prob. paper in Annals of Probability

$\chi_\mu(G) \geq (1-o(1)) \frac{d}{\log d}$.

In the remaining time, I prove Marks-116 thm of a special Borel

(Notes 6/4/25)

Define $\mathcal{A}_d$ as rooted d-reg tree w/ edges properly labelled using $\{1..d\}$ colors. and notational

consistently vertices w/ $\mathbb{N}$. root = 0, dist 1 nodes in lex order

dist 2 . . . .

Ex: $d=3$, ~~R=1~~ ~~B=2~~ ~~Y=3~~. no colors, use labels. draw (or predraw?)

Def $X := \mathrm{Col}(\mathcal{A}_d, \mathbb{N})$, closed subspace of proper $\mathbb{N}$-colorings of $\mathcal{A}_d$

a coloring $x \in X$ you should think of as what person standing on root sees as he looks around

"view"

Def "adjacent nbr view"

$F_i : X \to X$ - another view

$x \mapsto$ what person sees after walking road i.

"I'm standing on color 13, up road 1, see color 7, road 2, . . . 2.

picture

$G = (\mathrm{Col}(\mathcal{A}_d, \mathbb{N}),$ edge = nbring view$)$ has max deg $\le d$,

elts x called "view"

but NOT forest. Why?

draw vtces of G as "view"

could $\exists$ pattern (periodicity in labelling). dibe a periodic tiling

7mm

checkpt

Slightly annoying $G = G' \sqcup G''$ all comps are trees, all comps have cycle

ASSERT Borel partition

~~G' subgraph after removing all "periodic"~~

~~$\Leftrightarrow$ keeping only x s.t. $\mathrm{comp}_G(x)$ is tree.~~

G' is Borel d-reg forest. WTS $\chi_B(G') > d$.

think: ∀ fixed have ckt circuit that takes in coloring x of $\mathcal{A}_d$ "$\in \mathbb{N}$" returns a score $1..d$

"view-preserving"

i.e. WTS ∀ Borel labeling $f : V(G') \to \{1..d\}$.

$x \mapsto f(x)$ in view of landscape

think of guy standing at root of $\mathcal{A}_d$, looking around, runs & holds up the label/score $1..d$

score card

$\exists x \in V(G')$ s.t.

$\exists i \in 1..d$ $f(x) = f(F_i(x))$.

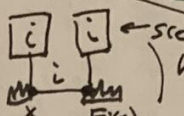
will we actually prove a stronger statement, "every Borel $f : V(G') \to 1..d$ can find "i"."

This stronger "$\forall$ Borel view-scoring f, $\exists$ iii" claim is more convenient ~~to work with~~, b/c it allows us to go back/retreat to thinking about G, instead of G'.

I called "slight annoyance"

Blc: ASSERT: G'' actually has a Borel view-scoring $\ddot{f}_\star$ that has no "iii"!

Thus, proving that $\forall f: V(G) \to 1..d$ has "iii", implies that $\forall \dot{f}: V(\dot{G}) \to 1..d$, can form $f := \dot{f} \sqcup \ddot{f}_\star$, implies f has "iii", which must indeed be in the $\dot{f}$ portion.

So. Final GOAL: $\forall$ Borel "view scoring" $f: V(G) \to 1..d$, $\exists$ "iii" (i.e. $\exists x \in \mathrm{Col}(\star, \mathbb{N})$, $i \in 1..d$ s.t.  ← scorecards held up by guy.

17 mins

As said previously, μ can't help. Need something special for Borel.

Infinite Games

"payoff set" $\forall A \subseteq \mathbb{N}^{\mathbb{N}}$, game $\Gamma(A)$ defined by 2 players playing natural #

Alice	n_0		n_2	
Bob		n_1		n_3

Alice wins $\Leftrightarrow$ $\underbrace{(n_0, n_1, \ldots)}_{\text{play of game}} \in A$.

On printout: Def: strategy for Alice is recipe

σ: (position Alice sees on board) $\to \mathbb{N}$

$\subseteq \mathbb{N}^*$

↕ Subtree of tree on $\mathbb{N}$

Def: play of game compatible w/ strat σ means $n_0 = \sigma(\emptyset)$, $n_2 = \sigma(n_0, n_1)$, ⋮

Blackboard: AMA!

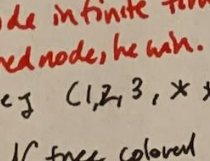
Thm (Morton, 1975): $A \subseteq \mathbb{N}^{\mathbb{N}}$ Borel $\Rightarrow$ exactly one of Alice or Bob has winning strat

Borel determinacy

Example pf: Open determinacy. $A = \bigcup$ initial segments, e.g. $(1, 2, 3, \star, \star, \star)$

seq $x \in A \Leftrightarrow$ hit red node in finite time. (COG)

if Bob never hit red node, he win.



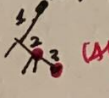
← reach this node ⇒ Alice wins.

⇓ start'l nodes in $\mathbb{N}$-tree colored red for Alice win.

Color red all nodes s.t.

~~Bleed upwards~~ if Alice can force a win. e.g.

~~Do this infinitely~~. Alice winstrat $\Leftrightarrow$ root node is red



(Alice can force win, by playing "3" if game reaches node (12).)

our magic tool/hammer.

Okay so we have Borel determ (and I gave a little idea of some ideas behind the pf).

8 min checkpt 6 Last stretch!

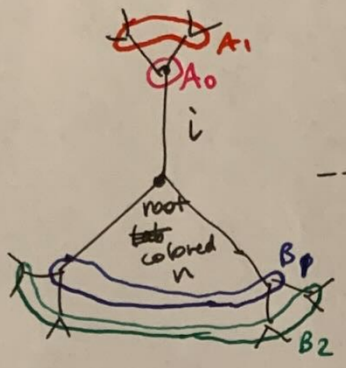
Fix Borel "nice scoring" $f: V(G) \to \{1..d\}$.
"Col($\mathcal{A}$, $\mathbb{N}$)"

We define the following ingenious game!

For $i=1..d$, $\forall n \in \mathbb{N}$, define game Γ_n^i : game board $\mathcal{A}$
"the color we want the triple iii" of

Alice & Bob take turns coloring vtxs of $\mathcal{A}$ using $\mathbb{N}$, ultimately producing coloring $x \in Col(\mathcal{A}, \mathbb{N})$.

$\Gamma[\mathcal{A}_n^i]$



↑ "Alice"
↓ "Bob"

Alice win $\Leftrightarrow$ $f(\vec{x}) \neq i$ score is not i. $(x \notin f^{-1}(i))$ Borel

Bob wins $\Leftrightarrow$ $f(x) = i$. ("half of our GOAL").

ONLY use of Borel assumption in whole pf!!!

Blc f Borel, $f^{-1}(i)$ Borel $\xrightarrow{\text{Martin}}$ $\Gamma_{i,n}$ determined!

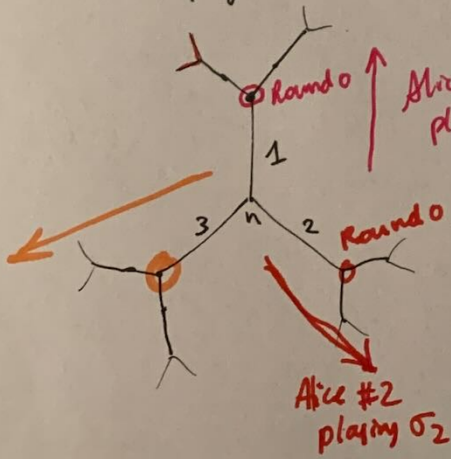
Bob's job is hard (has to $=i$ out of d possibilities), but he can force wins sometimes:

Claim-1: $\forall n$ = color of root $\rightsquigarrow$ d different games $\Gamma_{1,n} \dots \Gamma_{d,n}$. $\Gamma_n^1 \dots \Gamma_n^d$, Bob can force win ≥ 1 of them

$\exists i \in 1..d$ s.t. Bob has winning strat for Γ_n^i

Suppose not/soc. Then Alice has winning strat σ_i for each Γ_n^i.

Pf: d Alices play simultaneously against each other.



Start of Round 0,
each Alice sees $\emptyset$.
They each calculate $\sigma_i(\emptyset)$,
then fill in their Round 0 region.
end of Round 0.

Now, each Alice looks around, looks as if "Bob" Round 1 ended!

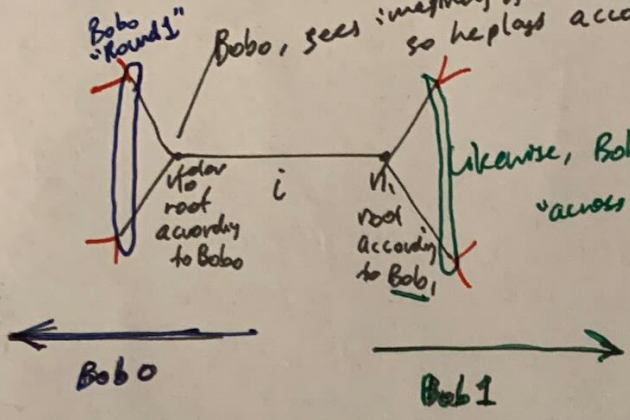
But then $f(\vec{x}) \neq i$ $\forall i = 1..d$, ↯

Remember idea: pigeonhole

$\exists n_0 \neq n_1 \in \mathbb{N}$ w/ same $i \in 1..d$ s.t. Bob can force win at games $\Gamma^i_{n_0}, \Gamma^i_{n_1}$

→ can replace $\mathbb{N}$ by $\{1..d+1\}$ for call $(A, d+1)$).

Last idea: 2 Bobs play each other.



Both Bobs win! But σ_0 winning for Bob_0 ⇒ $f(\vec{x}) = i$ view from Bobo root

σ_1 . . . . . ⇒ $f(F_i(x)) = i$ view from Bob_1 root

QED.

10 min
Check-in / Finish Line

Math 3 C Final Exam Spring 2024

Name : _____

UCLA I D number Put one number in each box

Section A B *(Circle correct section)*

Problem number	**Points**	You earned :
1	**6**	
2	**6**	
3	**9**	
4	**12**	
5	**12**	
6	**5**	
Your score	**50 points**	

Scientific Calculator allowed ----------NO Graphing calculator . You must show all of your work to receive full credit . Every step.