9/7/25 HK

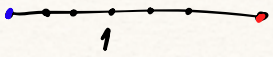
Spencer Lemma

Ky Fan Lemma

"molerat"

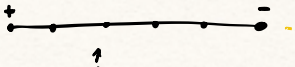
"same pf"

Spencer base case



1 in LHS must have odd # of bichromatic edges.

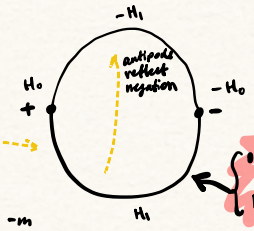
Fan base case:



↑ in blk odd # of edges w/ alternating signs

↳ either complimentary $+m, -m$

↳ or "good" $+m, -n$ $m \neq n$. wlog $m < n$



odd # of alternating simplices in this half "interval"

some "positive" (smallest # is +)

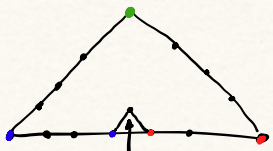
some "negative"

but each's antipodal reflection is opposite,

∴ total odd # of positive alt edges & neg alt edges

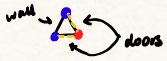
assuming NO complimentary edges

Next case Spencer

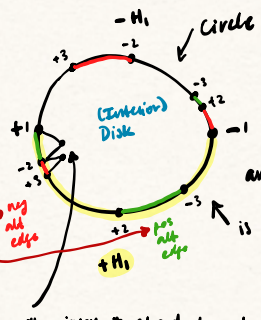


↳ either green = win and hallway ends

↳ or blue (say wlog), then door next



Next case Fan



and its antipodal reflection is 3 hemisphere of a 2D sphere

↳ either insert # at end of ordered abs val list = win, hallway ends

e.g. $\ldots, -2, +3, \ldots$

beginning of list changes signum of simplex

end of list does not change signum of simplex

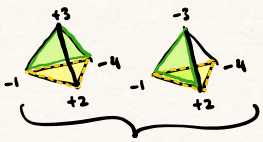
↳ or insert # in blk or on top of (abs val) element in list.

must be same exact # as below (in pictr. same signal) blk

e.g. $-1, +2, \square, -4, +5$ or $-1, +2, -4, +5$ (+2 on top of)

in blk

↳ next door! exs:



2 doors (highlighted)

2 walls

max'lly colored simplices

=

odd # doors on exterior,

walk thru each door, keep walking, either deadend = win or exit (out another door)

witnessed by door pair exterior

and odd # deadend rooms blk spencer in deadend room, walk, either in deadend ↳ witnessed by traproom pair

or exit out an exterior door!

(and w previous shaded: odd # exterior doors lead to trap rooms!)

odd # pos alt "doors" = simplices

(and odd # neg alt "doors")

Need to ensure entering pos door can't exit out a neg door!

This is when signum of simplices saves us

Maximal alt simplex (e.g. not since these abs val ordered lists $\{-1,+2,+3,-4\}$, $\{-1,+2,-3,-4\}$ are not alternating)



e.g. has 2 doors = remove 1st or last ent of abs val ordered list

one negative (signum, starts w −1)

one positive

So, if start at pos (alt) exterior door, can only walk thru positive (alt) doors!

neg . . . . . neg . . . . .

⇒ [odd # pos alt exterior doors (by induction) (on circle in above pic) ⇒ odd # of maximal alt. simplices reachable from positive (alt) exterior doors.]

↳ what if not reachable? Then by walking thru pos. (alt.) door, end at another maximal alt. simplex ↳ witnessed by traproom pair!

⇒ in total, odd # of maximal alt-simplices in Disk QED!

So to go from Sperner ⇝ Fan,

color ⟶ # (standard "trick")

more specifically, $n+1$ color for n-dim simplex ⟶ many #s ($\geq n+1$) for n-dim simplex

pure-maximal simplex ⇓ 2 doors if repeat color → how to maintain idea of 2 doors?

Genius ans: signs & require alternation of signs

(maybe only see this after Alg Top incarnation of these results)

I called "inscribed antipod" above in Fan exposition

introduces problem, maximal (alt.) simplex has 2 doors.

Fix: native signum of 2 doors differ.

Introduce idea of pos. & neg. door.

Issue: total # doors is odd ⇏ # pos doors odd

wait... but can guarantee exactly 1 of pos doors or neg doors is odd... !!!

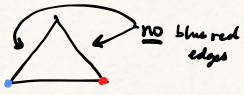
Fix: antipodal negation symmetry.

to end at negative signum maximal alt simplex. traproom

So: [have triangulation of M & labeling s.t. each max-dim simplex of has ≤ 2 doors & odd # of such doors on boundary of M] ⇒ odd # of trap rooms!

Sperner:

odd # of blue red doors on interval ⇒



satisfies condition

labeling = color by $n+1$ colors

doors = n-colored $(n-1)$-simplex

traproom = $n+1$ colored n-simplex

Fan:



odd # of alt sign doors on interval



Issue2: Need odd # of pos or odd # of neg.

A Fix: antipodal negation sym on boundary. ⇒ odd # of pos & odd # of neg.

labeling = label by #s $\{-m, \ldots -1, +1, \ldots +m\}$

doors = only ordered-ordered positive (or neg) alternating sign $(n-1)$-simplex

traprooms = maximally alt n-simplex

Issue1: but this traproom has 2 door, 😱

Fix in purple.